Theorem 6 $\mathcal{CP}_n \subsetneq \mathcal{S}^n_{\geq 0} \cap \mathbb{R}^{n(n+1)}_{\geq 0}$ for $n \geq 5$.

Lemma 7 Let $A \in \mathcal{CP}_n$. Let

> Gray-Wilson (1980): $\mathcal{CP}_n = \mathcal{S}^n_{\geq 0} \cap \mathbb{R}^{n(n+1)}_{\geq 0}$ for $n \leq 4$

$$G(A) = ([n], \{\{i,j\} : i,j \in [n], i \neq j, A_{ij} \neq 0\})$$

be the support graph of A. If $G(A)$ is triangle free, then the matrix $M(A) \in \mathbb{R}^{n \times n}$ defined by

$$(M(A))_{ij} = \begin{cases} |A_{ij}| & \text{if } i = j \\ -|A_{ij}| & \text{if } i \neq j \end{cases}$$

is positive semidefinite.

Proof Consider the decomposition $A = \sum_{i=1}^{n} b_i b_i^T$, $b_i \in \mathbb{R}^n_{\geq 0}$.
For every i, $\operatorname{supp} b_i = \{j \in [n] : (b_i)_j \neq 0\}$, is a clique in $G(A)$. Since $G(A)$ does not contain triangles we know $|\operatorname{supp} b_i| \leq 2$. Define $d_i \in \mathbb{R}^n$ by

$$d_i = \begin{cases} b_i & \text{if } |\operatorname{supp} b_i| = 1 \\ (0, \dots, 0, +(b_i)_j, 0, \dots, 0, -(b_i)_k, 0, \dots, 0)^T & \text{if } |\operatorname{supp} b_i| = 2 \\ & \text{and } (b_i)_j, (b_i)_k \neq 0. \end{cases}$$

Then $M(A) = \sum_{i=1}^{n} d_i d_i^T \succeq 0$. $\boxtimes$

Proof (of Theorem 6)

Define matrix $B \in \mathbb{R}^{k \times (k-1)}$ by

$$B = \begin{bmatrix} 1 & 0 & 0 & & & 0 \\ 1 & 1 & 0 & \cdots & & 0 \\ 0 & 1 & 1 & & & 0 \\ 0 & 0 & 1 & & & 0 \\ \vdots & \vdots & \vdots & \cdots & & \\ 0 & 0 & 0 & & 1 & 1 \\ 1 & -1 & 1 & \cdots & (-1)^{k-1} & 0 \end{bmatrix}$$

$B_{ii} = 1, \quad i = 1, \dots, k-1$

$B_{i+1,i} = 1, \quad i = 1, \dots, k-2$

$B_{ki} = (-1)^{i+1}, \quad i = 1, \dots, k-2$

$B_{ij} = 0$ otherwise

and $A = BB^T$ which is

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & \cdots & & 1 \\ 1 & 2 & 1 & 0 & & & 0 \\ 0 & 1 & 2 & 1 & & & 0 \\ & & \vdots & & & & \vdots \\ 0 & \cdots & & & 1 & 2 & (-1)^{k-1} \\ 1 & 0 & \cdots & 0 & 0 & (-1)^{k-1} & k-2 \end{bmatrix} \in S^k_{\geq 0} \cap \mathbb{R}^{k(k+1)/2}_{\geq 0}$$

if k is odd.

$G(A)$ is a k-cycle C_k. So for $k \geq 4$ it does not contain a triangle. We have

$$M(A) = \begin{bmatrix} 1 & -1 & 0 & & 0 & -1 \\ -1 & 2 & \ddots & & & 0 \\ & & \ddots & \ddots & & 0 \\ 0 & & & & 2 & -1 \\ -1 & 0 & \cdots & 0 & -1 & k-2 \end{bmatrix}$$

But $\det M(A) = (k-2) - 2 - k = -4 < 0$

$\xRightarrow{\text{Lemma 7}} A \notin CP_k$. $\boxtimes$

§ 2 A copositive reformulation of the independence number of a graph

Theorem 1 (Motzkin, Straus, 1965)
Let $G = (V, E)$ be a graph. Then

$$\frac{1}{\alpha(G)} = \min\left\{ x^T (A_G + I) x : x \in \mathbb{R}^V_{\geq 0}, \sum_{i \in V} x_i = 1 \right\},$$

where $A_G \in \mathbb{R}^{V \times V}$ is the adjacency matrix of G.

Proof "$\geq$": Let $S \subseteq V$ be an independent set in G with $\alpha(G) = |S|$. Define $x \in \mathbb{R}^V_{\geq 0}$ by $x_i = \begin{cases} \frac{1}{\alpha(G)}, & \text{if } i \in S \\ 0, & \text{otherwise.} \end{cases}$

Then $\sum_{i \in V} x_i = 1$ and

$$x^T (A_G + I) x = 2 \sum_{\{i,j\} \in E} x_i x_j + \sum_{i \in V} x_i^2 = 0 + \frac{\alpha(G)}{\alpha(G)^2} = \frac{1}{\alpha(G)}.$$

"$\leq$": Let $x^* \in \mathbb{R}^V_{\geq 0}$ be a minimizer of the RHS (which exists because of continuity and compactness).

1st case: $S = \{ i \in V : x_i^* > 0 \}$ is an independent set.

Then

$$(x^*)^T (A_G + I) x^* = 2 \sum_{\{i,j\} \in E} x_i^* x_j^* + \sum_{i \in S} (x_i^*)^2 \geq \frac{1}{|S|} \geq \frac{1}{\alpha(G)},$$

because

$$\left(\sum_{i\in S}(x_i^*)^2\right)\cdot |S| \geq \sum_{i\in S} x_i = 1$$

by Cauchy-Schwarz.

<u>2nd case</u>: $S = \{i \in V : x_i^* > 0\}$ is not an independent set.

Then $F_{x^*} = \{\{i,j\} \in E : x_i^* x_j^* > 0\} \neq \emptyset$.

<u>Goal</u>: Construct minimiser y^* so that $F_{y^*} \subsetneq F_{x^*}$.

[Then repeat until $F = \emptyset$ so that we are in the 1st case.]

Choose edge $\{i,j\} \in F_{x^*}$. Define $z \in \mathbb{R}^V$ by

$$z_k = \begin{cases} x_k^* + \varepsilon & \text{for } k = i \\ x_k^* - \varepsilon & \text{for } k = j \\ x_k^* & \text{otherwise.} \end{cases}$$

for some $\varepsilon \in \mathbb{R}$.

Then

$$z^T(A_G + I)z = (x^*)^T(A_G + I)x^* + \ell(\varepsilon),$$

where ℓ is a <u>linear</u> function in ε. Since z is feasible for $|\varepsilon|$ sufficiently small and since x^* is a minimizer, we must have $\ell(\varepsilon) = 0$. Define $\varepsilon = x_j^*$. Then $z_j = 0$. and $y^* = z$ is a new minimizer with $F_{y^*} \subsetneq F_{x^*}$ because $\{i,j\} \notin F_{y^*}$. ☒

Theorem 2 Let $G = (V, E)$ be a graph. Then

$$\alpha(G) = \sup_{\substack{X \in CP_V,\ Tr(X) = 1 \\ X_{ij} = 0 \text{ if } \{i,j\} \in E}} \langle J, X \rangle = \inf_{\substack{Y - J \in COP_V, \\ Y_{ii} = t \text{ for } i \in V \\ Y_{ij} = 0 \text{ if } \{i,j\} \notin E}} t$$

Remark If we replace CP_V or COP_V by $S^V_{\succeq 0}$, then we get the Lovász ϑ-number of G, for which only $\alpha(G) \leq \vartheta(G)$ holds.

Proof "$\alpha(G) \leq \sup$": Let $S \subseteq V$ be independent. Define again $x \in \mathbb{R}^V_{\geq 0}$ by $x_i = \begin{cases} \frac{1}{\sqrt{|S|}} & \text{if } i \in S \\ 0 & \text{otherwise} \end{cases}$. Then $X = xx^T$ is feasible for the maximization problem and $\langle J, X \rangle = |S|$ holds.

"$\sup \leq \inf$": weak duality