"$\inf \leq \alpha(G)$":

First we show that

$$\inf_{\substack{Y - J \in COP_V \\ Y_{ii} = t \quad i \in V \\ Y_{ij} = 0 \quad \{i,j\} \notin E}} t \quad = \quad \inf_{\substack{tI + zA_G - J \in COP_V \\ t, z \in \mathbb{R}}} t$$

holds.

"$\leq$": is clear

"$\geq$": Let Y be feasible for the LHS.

Define $Y' = tI + zA_G - J$ where $z = \max\{Y_{ij} : i \neq j\}$. Then $Y' \in COP_V$ because

$tI + zA_G - J \geq Y - J$ because $tI + zA_G - Y$ is component wise nonnegative (and $\mathbb{R}^{n(n+1)/2}_{\geq 0} \subseteq COP_V$).

Now: $\inf_{\substack{tI + zA_G - J \in COP_V \\ t, z}} t \leq \alpha(G)$.

By Theorem 1 we know for $x \in \mathbb{R}^V_{\geq 0}$, $\sum_{i \in V} x_i = 1$

$x^T(\alpha(G)(A_G + I))x \geq 1 = x^T J x. \Rightarrow x^T(\alpha(G)(A_G + I - J))x \geq 0$.

In particular: $\alpha(G)(A_G + I) - J \in COP_V$. Set $t = z = \alpha(G)$. $\boxtimes$

Chapter IX Polynomial Optimization

§1 Nonnegative polynomials and sum of squares

SOS = sum of squares

Notation Polynomials in n variables.

- monomials $x_1^{\alpha_1} \cdots x_n^{\alpha_n} = x^\alpha$, $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$.
- degree of x^α: $\deg x^\alpha = |\alpha| = \sum_{i=1}^n \alpha_i$.
- $p = \sum_\alpha p_\alpha x^\alpha$ finite linear combination of monomials x^α with coefficients $p_\alpha \in \mathbb{R}$ is called polynomial $\boxed{p \in \mathbb{R}[x_1, \dots, x_n]}$
- $\deg p = \max\{|\alpha| : p_\alpha \neq 0\}$ degree of p.
- $\mathbb{R}[x]_{=d} = \{p \in \mathbb{R}[x] : \deg p = d,\ p \text{ homogeneous}\} \cup \{0\}$

  p homogeneous: $p_\alpha \neq 0 \Rightarrow |\alpha| = d$.

  (or $p(\beta x) = \beta^d p(x)$ for all $\beta \in \mathbb{R}$, $x \in \mathbb{R}^n$)

monomials of degree d form basis of $\mathbb{R}[x]_{=d}$.

$$\dim \mathbb{R}[x]_{=d} = \binom{n+d-1}{n-1} = \binom{n+d-1}{d},$$

e.g. $n = 3$, $d = 4$: $\binom{n+d-1}{n-1} = \binom{6}{2}$

$$\underset{x_1}{\circ}\ \underset{x_1}{\circ}\ *\ \underset{x_2}{\circ}\ *\ \underset{x_3}{\circ} \longrightarrow x_1^2 x_2 x_3$$

- $\mathbb{R}[x]_d = \{ p \in \mathbb{R}[x] : p \text{ polynomial}, \deg p \le d \}$

$$= \mathbb{R}[x]_{=0} \oplus \mathbb{R}[x]_{=1} \oplus \dots \oplus \mathbb{R}[x]_{=d}$$

$$\dim \mathbb{R}[x]_d = \sum_{k=0}^{d} \binom{n+k-1}{k} = \binom{n+d}{d}$$

<u>Def. 1</u> (a) $p \in \mathbb{R}[x]$ is <u>SOS</u>, if there exist $q_1, \dots, q_m \in \mathbb{R}[x]$ s.t. $p = q_1^2 + \dots + q_m^2$.

(b) $\Sigma_{n,d} = \{ p \in \mathbb{R}[x]_d : p \text{ is SOS} \}$ <u>SOS cone</u>

(c) $P_{n,d} = \{ p \in \mathbb{R}[x]_d : p(x) \ge 0 \text{ for all } x \in \mathbb{R}^n \}$

cone of <u>nonnegative polynomials</u>.

<u>Clear:</u> $\Sigma_{n,2d+1} = \Sigma_{n,2d}$, $P_{n,2d+1} = P_{n,2d}$

$\Sigma_{n,d} \subseteq P_{n,d}$

<u>Thm. 2</u> ($\rightsquigarrow$ see Ex. 6.1)

$$\Sigma_{n,2d} = \{ p \in \mathbb{R}[x]_{2d} : \exists Q \in S_{\succeq 0}^{\binom{n+d}{d}} : p = [x]_d^T Q [x]_d \},$$

where $[x]_d \in \mathbb{R}[x]^{\binom{n+d}{d}}$ is the vector containing all monomials in $\mathbb{R}[x]_d$.

Example: $n=2$, $d=2$, $[x]_2 = (1, x_1, x_2, x_1x_2, x_1^2, x_2^2)^T$.

Proof (Thm. 2)

"$\subseteq$": Let $p = q_1^2 + \dots + q_m^2$ and write $q_i = \sum_\alpha q_{i\alpha} x^\alpha$
$$= \underline{q_i}^T [x]_d,$$
with $\underline{q_i} = (q_{i\alpha}) \in \mathbb{R}^{\binom{n+d}{d}}$. Then
$$q_i^2 = (\underline{q_i}^T [x]_d)^2 = [x]_d^T \underline{q_i}\,\underline{q_i}^T [x]_d, \text{ and}$$
$$p = \sum_{i=1}^m [x]_d^T \underline{q_i}\,\underline{q_i}^T [x]_d = [x]_d^T \underbrace{\left(\sum_{i=1}^m \underline{q_i}\,\underline{q_i}^T\right)}_{=Q \in S_{\succeq 0}^{\binom{n+d}{d}}} [x]_d.$$

"$\supseteq$": Given $p = [x]_d^T Q [x]_d$ with $Q \succeq 0$. Use spectral decomposition $Q = \sum_i \lambda_i u_i u_i^T$. Then
$$p = \sum_{i=1}^{\binom{n+d}{d}} \Big(\underbrace{(\sqrt{\lambda_i}\, u_i)^T [x]_d}_{=q_i}\Big)^2.$$
$\boxtimes$

Remarks
- Decision problem "$p \in \Sigma_{n,2d}^2$" can be reduced to SDP feasibility.
- Rank of Q determine the number of summands in the representation of p.