<u>Question</u>: What is the relation between $\Sigma_{n,2d}$ and $P_{n,2d}$?

<u>Theorem 3</u> $\Sigma_{1,2d} = P_{1,2d}$.

<u>Proof</u> Let $p \in P_{1,2d}$ with $\deg p = 2d$. By the fundamental theorem of algebra

$$p(x) = c \prod_{i=1}^{r} (x-\alpha_i)^2 \prod_{i=1}^{s} (x-\beta_i)(x-\overline{\beta_i}),$$

where : α_i real root

$\beta_i, \overline{\beta_i}$ pairs of complex roots

$c > 0$ coefficient of monomial x^{2d}.

We have
$$\begin{aligned}(x-\beta_i)(x-\overline{\beta_i}) &= x^2 - (\beta_i + \overline{\beta_i})x + \beta_i\overline{\beta_i} \\ &= x^2 - 2\operatorname{Re}\beta_i \, x + (\operatorname{Re}\beta_i)^2 + (\operatorname{Im}\beta_i)^2 \\ &= (x - \operatorname{Re}\beta_i)^2 + (\operatorname{Im}\beta_i)^2.\end{aligned}$$

Therefore, $\prod_{i=1}^{s} (x-\beta_i)(x-\overline{\beta_i}) = \sum_{j=1}^{m} (q_j(x))^2$ for some $q_j \in \mathbb{R}[x]$.

Hence,
$$p(x) = \sum_{j=1}^{m} \left(\sqrt{c} \prod_{i=1}^{r} (x-\alpha_i)\, q_j \right)^2. \qquad \boxtimes$$

Example 4 Motzkin polynomial

$$M(x,y) = x^2y^2(x^2+y^2-3)+1 \in P_{2,6} \setminus \Sigma_{2,6}.$$

Proof $M \in P_{2,6}$: If $x^2+y^2-3 \geq 0$, then $M(x,y) \geqslant 0$.

If $x^2+y^2-3 < 0$, then define $z^2 = 3-x^2-y^2$. (>0)

By the AM-GM-inequality:

$$\underset{\left(-M(x,y)+1\right)^{1/3}}{\overset{\|}{\left(x^2y^2z^2\right)^{1/3}}} \leq \frac{x^2+y^2+z^2}{3} = 1 \quad \Rightarrow M(x,y) \geq 0.$$

$M \notin \Sigma_{2,6}$: Suppose $M = \sum_{k=1}^{m} q_k^2$. Then $\deg q_k \leq 3$.

Coefficients of q_k:

x^3 : 0

y^3 : 0

x^2 : 0

y^2 : 0

x : 0

y : 0

So: $q_k = a_k xy^2 + b_k x^2y + c_k xy + d_k$

Coefficient of x^2y^2 in M is -3, i.e. $-3 = \sum_{k=1}^{m} c_k^2$. $\lightning$ $\boxtimes$

Theorem 5 (Hilbert, 1888)

$$\Sigma_{n,2d} = P_{n,2d} \iff \begin{array}{l} n=1 \text{ or} \\ d=1 \text{ or} \\ n=2,\ d=2. \end{array}$$

Proof: • $n=1$: Thm. 3

• $d=1$: Cholesky factorisation

• $n=2$, $d=2$: not so easy; not here.

• all other cases: use variants of Motzkin polynomial. ☒

Remarks • Decision problem "$p \in P_{n,4}$" is NP-hard.

• Blekherman (2006): $P_{n,2d}$ is much bigger than $\Sigma_{n,2d}$.

§2 Global optimisation with polynomials

polynomial optimisation problem (POP)

<u>given</u>: $f, g_1, \dots, g_m \in \mathbb{R}[x]$

<u>find</u>: $$p_{min} = \inf_{x \in K} f(x)$$

where $K = \{x \in \mathbb{R}^n : g_1(x) \geq 0, \dots, g_m(x) \geq 0\}$.

POP is very general and very difficult.

<u>Dual approach</u>: $\mathcal{P}(K) = \{p \in \mathbb{R}[x] : p(x) \geq 0 \ \forall x \in K\}$

is convex cone

Then $$p_{min} = \sup\{\lambda : f - \lambda \in \mathcal{P}(K)\}.$$

<u>But</u> Optimisation over $\mathcal{P}(K)$ is difficult.

<u>SOS relaxation</u> (Lasserre; Parrilo 2001)

- Find sequence of cones $C_1 \subseteq C_2 \subseteq \dots \subseteq \mathcal{P}(K)$ for which it is easier to determine $\sup\{\lambda : f - \lambda \in C_i\}$.
- Gives sequence $p_1 \leq p_2 \leq \dots \leq p_{min}$ of lower bounds.

- optimization over C_i is easier than optimization over C_{i+1}

<u>SOS hierarchy</u> Define

$$p_{sos,i} = \sup\{\lambda : p-\lambda \in \Sigma_{n,2i} + g_1\Sigma_{n,2i} + \cdots + g_m\Sigma_{n,2i}\}$$

Then $p_{sos,1} \leq p_{sos,2} \leq \cdots \leq p_{min}$.

<u>Theorem</u> (Putinar, 1993)

If K is compact (and K is well represented by $g_1,\dots,g_m$; "the quadratic module is Archimedian").

Then $p_{min} = \lim_{i\to\infty} p_{sos,i}$.

→ important theoretical result; useful in practice.

→ sometimes: $p_{sos,i} = p_{min}$ for small value of i.

→ starting point of ongoing research.