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A breakthrough in sphere packing: the search for magic functions

This paper by David de Laat and Frank Vallentin is an exposition about the two recent breakthrough results in the theory of sphere packings. It includes an interview with Henry Cohn, Abhinav Kumar, Stephen D. Miller and Maryna Viazovska.

The sphere packing problem asks for a densest packing of congruent solid spheres in n-dimensional space $\mathbb{R}^n$. In a packing the (solid) spheres are allowed to touch on their boundaries, but their interiors should not intersect.

While the case of the real line, $n=1$, is trivial, the case $n=2$ of packing circles in the plane was first solved in 1892 by the Norwegian mathematician Thue (1863–1922). He showed that the honeycomb hexagonal lattice gives an optimal packing; see Figure 1.

The first rigorous proof is due to the Hungarian mathematician Fejes Tóth (1915–2005) in 1940. He also proved that this packing is unique (up to rotations, translations, and uniform scaling) among periodic packings. For $n=3$, the sphere packing problem is known as the Kepler conjecture. It was solved by the American mathematician Hales in 1998 following an approach by Fejes Tóth. Hales' proof is extremely complex, takes more than 300 pages, and makes heavy use of computers. One of the difficulties of the sphere packing problem in three dimensions is the fact that there are uncountably many inequivalent optimal packings. In 2014 a fully computer verified version of Hales' proof was completed; it was a result of the collaborative Flyspeck project, also directed by Hales [7].

Recently, Maryna Viazovska, a postdoctoral researcher from Ukraine working at the Humboldt University of Berlin, solved the eight-dimensional case. On 14 March 2016 she announced her spectacular result in the paper titled 'The sphere packing problem in dimension 8' [10] on the arXiv-preprint server. Only one week later, on 21 March 2016, Henry Cohn, Abhinav Kumar, Stephen D. Miller, Danylo Radchenko and Maryna Viazovska announced a proof for the $n=24$ case [3], building on Viazovska's work.

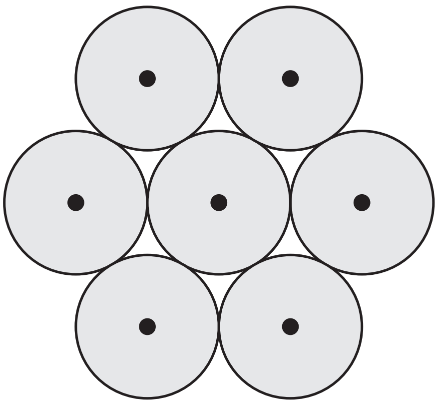


Figure 1 The hexagonal lattice and the corresponding circle packing.

Here we want to illustrate that the optimal sphere packings in dimensions 8 and 24 are very special (in the next section we give constructions of the E_8 lattice and of the Leech lattice Λ_{24}, which provide the optimal sphere packings in their dimensions), and we aim to explain the main ideas of the recent breakthrough results in sphere packing:

Theorem 1. *The lattice* E_8 *is the densest packing in* $\mathbb{R}^8$. *The Leech lattice* Λ_{24} *is the densest packing in* $\mathbb{R}^{24}$. *Moreover, no other periodic packing achieves the same density in the corresponding dimension.*

In the last section we will see that the beautiful proofs of these theorems use ideas from analytic number theory. Viazovska found a 'magic' function for dimension 8, which together with the linear programming bound of Cohn and Elkies, as explained later on, gives a proof for the optimality of the E_8 lattice. Her method gave a hint how to find a magic function for dimension 24. Although the proof is relatively easy to understand, and basically no computer assistance is needed for its verification, computer assistance was crucial to conjecture the existence of, and to find, these magic functions.

Optimal lattices

In this section we introduce the two exceptional sphere packings in dimension 8 and 24. The book *Sphere Packings, Lattices, and Groups* of Conway and Sloane [5] is the definitive reference on this topic; the Italian-American combinatorialist Rota (1932–1999) reviewed the book saying:

> "This is the best survey of the best work in one of the best fields of combinatorics, written by the best people. It will make the best reading by the best students interested in the best mathematics that is now going on."

Lattice packings

How does one define a packing of unit spheres in n-dimensional space? In general, such a packing is defined by the set of centers L of the spheres in the packing.

We talk about *lattice packings* when L forms a *lattice*. Then there are n linearly independent vectors $b_1,\dots,b_n \in L$, called a *lattice basis* of L, so that L is the set of integral linear combinations of $b_1,\dots,b_n$. For instance, the three lattices

$$\mathbb{Z},\quad \mathbb{Z}\begin{pmatrix}2\\0\end{pmatrix}+\mathbb{Z}\begin{pmatrix}-1\\\sqrt{3}\end{pmatrix},$$
$$\mathbb{Z}\begin{pmatrix}-\sqrt{2}\\-\sqrt{2}\\0\end{pmatrix}+\mathbb{Z}\begin{pmatrix}\sqrt{2}\\-\sqrt{2}\\0\end{pmatrix}+\mathbb{Z}\begin{pmatrix}0\\\sqrt{2}\\-\sqrt{2}\end{pmatrix}$$

define densest sphere packings in dimensions 1, 2, and 3.

We should also define what we mean when we talk about *density*. Intuitively, the density of a sphere packing is the fraction of space covered by the spheres of the packing. When the sphere packing is a lattice, this intuition is easy to make precise: The density of the sphere packing determined by L is the volume of one sphere divided by the *volume of* L, that is, the volume of a fundamental domain of L. One possible fundamental domain of L is given by the parallelepiped spanned by the lattice basis $b_1,\dots,b_n$, so that we have

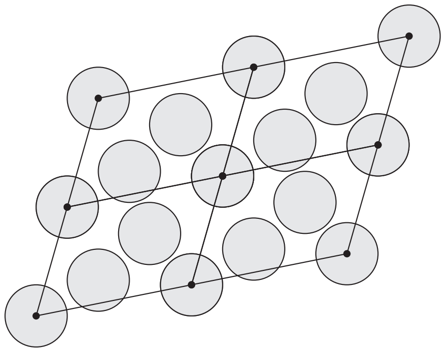


Figure 2 A periodic packing that is not a lattice packing.

$$\operatorname{vol}(\mathbb{R}^n/L) = |\det(b_1,\dots,b_n)|.$$

The density of L is then given by

$$\Delta(L) = \frac{\operatorname{vol}(B_n(r_1/2))}{\operatorname{vol}(\mathbb{R}^n/L)},$$

where r_1 is the shortest nonzero vector length in L. Here, $B_n(r_1/2)$ is the solid sphere of radius $r_1/2$, whose volume is

$$\operatorname{vol}(B_n(r_1/2)) = (r_1/2)^n\frac{\pi^{n/2}}{\Gamma(n/2+1)},$$

where Γ is the gamma function; it satisfies the equation $\Gamma(x+1)=x\Gamma(x)$, and two particularly useful values are $\Gamma(1)=1$ and $\Gamma(1/2)=\sqrt{\pi}$.

The optimal sphere packing density can be approached arbitrarily well by the density of a periodic packing. In a *periodic packing* the set of centers is the union of a finite number m of translates of a lattice L:

$$\{x_1+v : v\in L\}\cup\{x_2+v : v\in L\} \cup\cdots\cup\{x_m+v : v\in L\};$$

see Figure 2 for an example where $m=3$. The density of a periodic packing is $m\cdot\operatorname{vol}(B_n(r))/\operatorname{vol}(\mathbb{R}^n/L)$, where r is the radius of the spheres in the packing. It is possible that in some dimensions the optimal sphere packing is not a lattice packing; for example, the best known sphere packing in $\mathbb{R}^{10}$ is a periodic packing but it is not a lattice packing.

A few last definitions: From a lattice L we can construct its *dual lattice* by

$$L^* = \{y\in\mathbb{R}^n : x\cdot y\in\mathbb{Z} \text{ for all } x\in L\}.$$

It is not difficult to see that the volume of a lattice and its dual are reciprocal values, so that $\operatorname{vol}(\mathbb{R}^n/L)\cdot\operatorname{vol}(\mathbb{R}^n/L^*)=1$ holds. When a lattice equals its dual ($L=L^*$) and when the square of every occurring vector length is an even integer, then L is called an *even and unimodular* lattice.

The E_8 *lattice*

The nicest lattices are those which are even and unimodular. However, they only occur in higher dimensions: one can show that the first appearance of such an even and unimodular lattice is in dimension 8. It is the E_8 lattice, which was first explicitly constructed by the Russian mathematicians Korkine (1837–1908) and Zolotareff (1847–1878) in 1873.

Here we give a construction of the E_8 lattice which is based on lifting binary error correcting codes. For this we define the (extended) *Hamming code* $\mathcal{H}_8$ via a regular three-dimensional tetrahedron: consider the binary linear code $\mathcal{H}_8$, which is the vector space over the finite field $\mathbb{F}_2$ (consisting of the elements 0 and 1) spanned by the rows of the matrix

$$G=(I\,|\,A)=\left(\begin{array}{c|c}1000&0111\\0100&1011\\0010&1101\\0001&1110\end{array}\right)\in\mathbb{F}_2^{4\times 8},$$

where I is the identity matrix and where A is the adjacency matrix of the vertex-edge graph of a three-dimensional tetrahedron with vertices v_1,v_2,v_3,v_4. Hence, the Hamming code is a 4-dimensional subspace of the vector space $\mathbb{F}_2^8$. It consists of $2^4=16$ code words:

$$\begin{array}{ccccc}
0000\,|\,0000 & 1000\,|\,0111 & 1100\,|\,1100 & 0111\,|\,1000 & 1111\,|\,1111\\
 & 0100\,|\,1011 & 1010\,|\,1010 & 1011\,|\,0100 & \\
 & 0010\,|\,1101 & 1001\,|\,1001 & 1101\,|\,0010 & \\
 & 0001\,|\,1110 & 0110\,|\,0110 & 1110\,|\,0001 & \\
 & & 0101\,|\,0101 & & \\
 & & 0011\,|\,0011 & &
\end{array}$$

It is interesting to look at the occurring *Hamming weights* (the number of non-zero entries) of code words. In $\mathcal{H}_8$, one code word has Hamming weight 0, 14 code words have Hamming weight 4, and one code word has Hamming weight 8. Since all occurring Hamming weights are divisible by four, and four is two times two, the Hamming code $\mathcal{H}_8$ is called *doubly even*.

Let us compute the *dual code*

$$\mathcal{H}_8^{\perp} = \left\{y\in\mathbb{F}_2^8 : \sum_{i=1}^{8} x_i y_i = 0 \pmod 2 \quad \text{for all } x\in\mathcal{H}_8\right\}.$$

Squaring the matrix A yields

$$\begin{aligned}A_{ij}^2 &= \sum_{k=1}^{4} A_{ik}A_{kj}\\ &= |\{k : v_i\sim v_k \text{ and } v_k\sim v_j\}|\\ &= \begin{cases}3 & \text{if } i=j,\\ 2 & \text{if } i\neq j.\end{cases}\end{aligned}$$

Hence, $A^2 = I \bmod 2$. From this, $GG^T = I + A^2 = 0 \bmod 2$ follows. Hence, we have the inclusion $\mathcal{H}_8 \subseteq \mathcal{H}_8^\perp$ and by considering dimensions we see that $\mathcal{H}_8$ is a *self-dual code*; that is, $\mathcal{H}_8^\perp = \mathcal{H}_8$ holds.

We can define the lattice E_8 by the following lifting construction (which is usually called *Construction A*):

$$\mathsf{E}_8 = \left\{ \frac{1}{\sqrt{2}} x : x \in \mathbb{Z}^8,\ x \bmod 2 \in \mathcal{H}_8 \right\}.$$

Now it is immediate to see that E_8 has 240 shortest (nonzero) vectors:

$16 = 2^4$ vectors:

$$\pm \sqrt{2}\, e_i,\ i = 1, \ldots, 8$$

$224 = 2^4 \cdot 14$ vectors:

$$\frac{1}{\sqrt{2}} \sum_{i=1}^{8} (\pm x_i) e_i,\ x \in \mathcal{H}_8 \text{ and } \mathrm{wt}(x) = 4,$$

where $e_1, \ldots, e_8$ are the standard basis vectors of $\mathbb{R}^8$ and where $\mathrm{wt}(x) = |\{i : x_i \neq 0\}|$ denotes the Hamming weight of x. The shortest nonzero vectors of E_8 have length $\sqrt{2}$. The occurring vector lengths in E_8 are $0, \sqrt{2}, \sqrt{4}, \sqrt{6}, \ldots$, so that E_8 is an even lattice.

From the lifting construction it follows that the density of E_8 is $|\mathcal{H}_4| = 16$ times the density of the lattice $\sqrt{2}\,\mathbb{Z}^8$ which is spanned by $\sqrt{2}\, e_1, \ldots, \sqrt{2}\, e_8$. Thus,

$$\begin{aligned} \mathrm{vol}(\mathbb{R}^8/\mathsf{E}_8) &= \frac{1}{16} \cdot \mathrm{vol}(\mathbb{R}^8/\sqrt{2}\,\mathbb{Z}^8) \\ &= \frac{1}{16} \cdot (\sqrt{2})^8 = 1, \end{aligned}$$

so that E_8 is unimodular. One can show that E_8 is the *only* even unimodular lattice in dimension 8. In general, the lifting construction always yields an even unimodular lattice when we start with a binary code which is doubly even and self-dual.

Next to this exceptional number theoretical property, E_8 also has exceptional geometric properties: In 1979, Odlyzko and Sloane, and independently Levenshtein, proved that one cannot arrange more vectors on a sphere in dimension 8 of radius $\sqrt{2}$ so that the distance between any two distinct vectors is also at least $\sqrt{2}$; the 240 vectors give the unique solution of the kissing number problem in dimension 8 as was shown in by Bannai and Sloane in 1981. Blichfeldt (1873–1945) showed in 1935 that E_8 gives the densest sphere packing among lattice packings. For a long time it has been conjectured that E_8 also gives the unique densest sphere packing in dimension 8, without imposing the (severe) restriction to lattice packings. Now this conjecture has been proved in the breakthrough work of Maryna Viazovska.

The Leech lattice

We turn to 24 dimensions and to the Leech lattice. In 1965 Leech (1926–1992) realized that he constructed a surprisingly dense sphere packing in dimension 24. For his construction, he used the (extended binary) Golay code which is an exceptional error correcting code found by Golay (1902–1989) in 1949. To define the Leech lattice we modify the lifting construction of the E_8 lattice. We replace the Hamming code by the Golay code and apply two extra twists.

For defining the Golay code we replace the regular tetrahedron in the construction of the Hamming code by the regular icosahedron and we apply the first twist. Consider the binary code $\mathcal{G}_{24}$ spanned by the rows of the matrix

$$\begin{aligned} G &= (I \,|\, J - A) \\ &= \left(\begin{array}{c|c} 100000000000 & 100000111111 \\ 010000000000 & 010110001111 \\ 001000000000 & 001011100111 \\ 000100000000 & 010101110011 \\ 000010000000 & 011010111001 \\ 000001000000 & 001101011101 \\ 000000100000 & 101110101100 \\ 000000010000 & 100111010110 \\ 000000001000 & 110011101010 \\ 000000000100 & 111001110100 \\ 000000000010 & 111100011010 \\ 000000000001 & 111111000001 \end{array}\right) \in \mathbb{F}_2^{12 \times 24}, \end{aligned}$$

where we use $J - A$ instead of A, with J the all-ones matrix and A the adjacency matrix of the vertex-edge graph of a three-dimensional icosahedron. This code, the *extended binary Golay code* $\mathcal{G}_{24}$, is a 12-dimensional subspace in $\mathbb{F}_2^{24}$. It contains one vector of Hamming weight 0, 759 vectors of Hamming weight 8, 2576 vectors of Hamming weight 16, 759 vectors of Hamming weight 20, and one vector of Hamming weight 24; $\mathcal{G}_{24}$ is a doubly even and self-dual code.

We define the even unimodular lattice

$$L_{24} = \left\{ \frac{1}{\sqrt{2}} x : x \in \mathbb{Z}^{24}, x \bmod 2 \in \mathcal{G}_{24} \right\}.$$

Since the minimal non-zero Hamming weight occurring in the Golay code is 8, this lattice has 48 shortest vectors $\pm\sqrt{2}\, e_i$, with $i = 1, \ldots, 24$, of length $\sqrt{2}$. To eliminate them we make the second twist, we define

$$\begin{aligned} \Lambda_{24} = &\left\{ x \in L_{24} : \sqrt{2} \sum_{i=1}^{24} x_i = 0 \bmod 4 \right\} \\ \cup &\left\{ (1, \ldots, 1) + x : x \in L_{24},\ \sqrt{2} \sum_{i=1}^{24} x_i = 2 \bmod 4 \right\}. \end{aligned}$$

This is the Leech lattice. It is an even unimodular lattice. In Λ_{24} there are 196560 shortest vectors which have length $\sqrt{4}$. The occurring vector lengths in Λ_{24} are $0, \sqrt{4}, \sqrt{6}, \sqrt{8}, \ldots$

In 1969 Conway showed that the Leech lattice again has a remarkable number theoretical property: It is the only even unimodular lattice in dimension 24 which does not have vectors of length $\sqrt{2}$. He used this result to determine the automorphism group (the group of orthogonal transformation which leave Λ_{24} invariant) of the Leech lattice and it turned out the number of automorphisms equals

$$\begin{aligned} |\mathrm{Aut}(\Lambda_{24})| &= 2^{22} \cdot 3^9 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 23 \\ &= 8315553613086720000, \end{aligned}$$

and that this group contained three new sporadic simple groups $\mathrm{Co}_1, \mathrm{Co}_2 . \mathrm{Co}_3$. The classification theorem of finite simple groups, which was announced in 1980, says that there are only 26 finite simple sporadic groups. They are sporadic in the sense that they are not contained in the infinite families of cyclic groups of prime order, alternating groups and groups of Lie type.

Similar to the eight-dimensional case, by results of Odlyzko, Sloane, Levenshtein, Bannai and Sloane, the 196560 shortest vectors of Λ_{24} give the unique solution of the kissing number problem in dimension 24. In 2004 Cohn and Kumar proved the optimality of the sphere packing of the Leech lattice among lattice packings by a computer assisted proof, see [2] and section 'Producing numerical evidence' further on. However, despite all the similarities of E_8 and Λ_{24}, there is a puzzling difference between E_8 and Λ_{24} when it comes to sphere coverings: Schürmann and Vallentin [9] showed in 2006 that Λ_{24} provides at least a locally thinnest sphere covering in the space of 24-dimensional lattices, whereas Dutour-Sikirić, Schürmann and Vallentin [9] showed in 2012 that one can improve the sphere covering of the E_8 lattice when picking a generic direction in the space of eight-dimensional lattices.

Theta series and modular forms

As already indicated, the class of even unimodular lattices is restrictive, at least in small dimensions. One can show that they only exist in dimensions that are divisible by 8, furthermore for every such dimension n there are only finitely many even uni-

$h_8 = 1$	Mordell, 1938
$h_{16} = 2$	Witt, 1941
$h_{24} = 24$	Niemeier, 1973
$h_{32} \geq 1162109024$	King, 2003

Table 1

modular lattices, this number is denoted by h_n. In Table 1 we summarize the known values of h_n.

A major tool for studying even unimodular lattice are their theta series (first studied by Jacobi (1804–1851)): The *theta series* of a lattice L is

$$\vartheta_L = \sum_{r=0}^{\infty} n_L(r) q^r$$
$$\text{with } n_L(r) = \{x \in L : x \cdot x = 2r\},$$

the generating function of the number of lattice vectors of length $\sqrt{2r}$. In order to work with them analytically we set $q = e^{2\pi i z}$ where z lies in the complex upper half plane $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$, so that ϑ_L is a function of z. The theta function is periodic mod $\mathbb{Z}$: we have $\vartheta_L(z) = \vartheta_L(z+1)$. The *Poisson summation formula* states

$$\sum_{x \in L} f(x+v) = \frac{1}{\operatorname{vol}(\mathbb{R}^n/L)} \sum_{y \in L^*} e^{2\pi i x \cdot y} \widehat{f}(y),$$

with $v \in \mathbb{R}^n$, where

$$\widehat{f}(y) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i y \cdot x} dx$$

is the n-dimensional *Fourier transform*. Using the Poisson summation formula one can show that ϑ_L satisfies the transformation law

$$\vartheta_L(-1/z) = (z/i)^{n/2} \frac{1}{\operatorname{vol}(\mathbb{R}^n/L)} \vartheta_{L^*}(z),$$

which in particular shows that ϑ_L is a modular form of weight $n/2$. From this it is not difficult to derive that an even unimodular lattice can only exist when n is a multiple of 8.

What is a modular form? The group

$$\mathrm{SL}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\},$$

which is generated by the matrices

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

acts on upper half plane $\mathbb{H}$ by fractional linear transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}.$$

The action of the generator S corresponds to the involution $z \mapsto -1/z$ and the action of the generator T corresponds to the translation $z \mapsto z+1$.

A *modular form of weight* k is a holomorphic function $f : \mathbb{H} \to \mathbb{C}$ that satisfies the transformation law

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$$
$$\text{for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}), z \in \mathbb{H},$$

and which has a power series expansion in $q = e^{2\pi i z}$. Next to theta series, *Eisenstein series*, due to Eisenstein (1823–1852), form an important class of modular forms. For an integer $k \geq 3$ define

$$E_k(z) = \frac{1}{2\zeta(k)} \sum_{(c,d) \in \mathbb{Z}^2 \setminus \{0\}} \frac{1}{(cz+d)^k}, \qquad (1)$$

where ζ is the Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$. For even integers $k \geq 3$, the Eisenstein series E_k is a modular form of weight k.

Curiously, a theorem of Siegel (1896–1981) gives a relation between the theta series of even unimodular lattices and Eisenstein series. Let $L_1, \ldots, L_{h_n}$ be the set of even unimodular lattices in dimension n. Define

$$M(n) = \frac{1}{|\operatorname{Aut}(L_1)|} + \cdots + \frac{1}{|\operatorname{Aut}(L_{h_n})|}.$$

Then

$$E_{n/2}(z) = \frac{1}{M(n)} \sum_{j=1}^{h_n} \frac{1}{|\operatorname{Aut}(L_j)|} \vartheta_{L_j}(z).$$

Another striking fact is that one can show that the modular forms form an algebra which is isomorphic to the polynomial algebra $\mathbb{C}[E_4, E_6]$.

When k is even, the Eisenstein series has the Fourier expansion

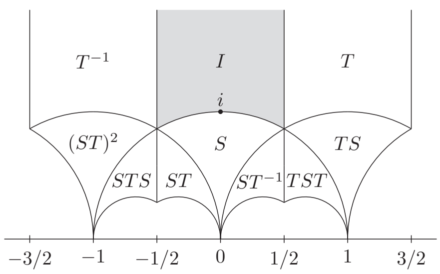


Figure 3 A fundamental domain of the action of $\mathrm{SL}_2(\mathbb{Z})$ on the upper half plane $\mathbb{H}$.

$$E_k(z) = 1 + \frac{2}{\zeta(1-k)} \sum_{n=1}^{\infty} \sigma_{k-1}(n) e^{2\pi i n z},$$

where $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$ is the divisor function. We can express the theta series of the E_8 lattice and the Leech lattice Λ_{24} by E_4 and E_6. For the E_8 lattice we have

$$\begin{aligned} \vartheta_{\mathsf{E}_8}(z) &= E_4(z) = 1 + 240 \sum_{r=1}^{\infty} \sigma_3(r) q^r \\ &= 1 + 240 \cdot 1 \cdot q + 240 \cdot 9 \cdot q^2 + 240 \cdot 28 \cdot q^3 + \cdots \\ &= 1 + 240 \cdot q + 2160 \cdot q^2 + 6720 \cdot q^3 + \cdots. \end{aligned}$$

For the Leech lattice we have

$$\begin{aligned} \vartheta_{\Lambda_{24}}(z) &= E_4^3(z) - 720\Delta(z) \\ &= 1 + 196560 \cdot q^2 + 16773120 \cdot q^3 + \cdots \end{aligned}$$

where Δ is defined as

$$\Delta(z) = \frac{E_4(z)^3 - E_6(z)^2}{1728}.$$

When $k = 2$, we can still write down the series as in (1), but then we loose some pleasant properties. For example, the series no longer converges absolutely, so the ordering of summating matters. Then

$$\begin{aligned} E_2(z) &= \frac{1}{2\zeta(2)} \sum_{c \in \mathbb{Z}} \sum_{d \in \mathbb{Z}} \frac{1}{(c+dz)^2} \\ &= 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) e^{2\pi i n z}, \end{aligned}$$

where we of course omit the pair $(c, d) = (0, 0)$ in the first sum. This *forbidden Eisenstein series* is not a modular form; instead it satisfies the following transformation law

$$E_2(-1/z) = z^2 E_2(z) - \frac{6iz}{\pi}.$$

It is a *quasi-modular form*.

The LP Bound of Cohn and Elkies

We can use optimization techniques, in particular linear and semidefinite programming, to obtain upper bounds on the optimal sphere packing density.

Let us recall some facts about linear programming. In a *linear program* we want to maximize a linear functional over a polyhedron. For example, we maximize the functional $a \mapsto c \cdot a$ over all (entrywise) nonnegative vectors $a \in \mathbb{R}^d$ satisfying the linear system $Aa = b$, or we maximize the functional over all (nonnegative) vectors a satisfying the inequality $Aa \leq b$. Linear programs can be solved efficiently in practice by a simplex algorithm (which traverses a path along vertices of the polyhedron) or by Karmakar's interior point method, where the latter runs in polynomial time.

The following theorem, the *linear programming bound of Cohn and Elkies* from 2003, can be used to obtain upper bounds on the sphere packing density. In the statement of this theorem we restrict to Schwartz functions because the proof, which we give here as it is simple and insightful, uses the Poisson summation formula. A function $f:\mathbb{R}^n\to\mathbb{R}$ is a *Schwartz function* if all its partial derivatives exist and tend to zero faster than any inverse power of x. There are alternative proofs that do not use Poisson summation and for which the Schwartz condition can be weakened.

Theorem 2. *If $f:\mathbb{R}^n\to\mathbb{R}$ is a Schwartz function and r_1 is a positive number with $\widehat{f}(0)=1$, $\widehat{f}(u)\geq 0$ for all u, and $f(x)\leq 0$ for $\|x\|\geq r_1$, then the density of a sphere packing in $\mathbb{R}^n$ is at most $f(0)\cdot\operatorname{vol}(B_n(r_1/2))$.*

Proof. Let P be a periodic packing of solid spheres of radius $r_1/2$. This means there is a lattice L and points $x_1,\ldots,x_m$ in $\mathbb{R}^n$ such that

$$P=\bigcup_{v\in L}\bigcup_{i=1}^{m}(v+x_i+B_n(r_1/2)).$$

The density of P is $m\cdot\operatorname{vol}(B_n(r_1/2))/\operatorname{vol}(\mathbb{R}^n/L)$. By Poisson summation we have

$$\sum_{v\in L}\sum_{i,j=1}^{m}f(v+x_j-x_i)$$
$$=\frac{1}{\operatorname{vol}(\mathbb{R}^n/L)}\sum_{u\in L^*}\widehat{f}(u)\sum_{i,j=1}^{m}e^{2\pi i u\cdot(x_j-x_i)},$$

and because $\widehat{f}(0)=1$ and $\widehat{f}(u)\geq 0$ for all u, this is at least $m^2/\operatorname{vol}(\mathbb{R}^n/L)$. On the other hand, by the condition $f(x)\leq 0$ for $\|x\|\geq r_1$, we have

$$\sum_{v\in L}\sum_{i,j=1}^{m}f(v+x_j-x_i)\leq mf(0).$$

Hence, the density of P is at most $f(0)\cdot\operatorname{vol}(B_n(r_1/2))$. The density of any packing can be approximated arbitrarily well by the density of a periodic packing, so this completes the proof. □

We can additionally require either $f(0)=1$ or $r_1=1$, without weakening the theorem. Moreover, we can restrict to *radial functions*, for if a function f satisfies the conditions of the theorem for some r_1, then the function

$$x\mapsto\int_{S^{n-1}}f(\|x\|\xi)d\omega(\xi),$$

where ω is the normalized invariant measure on S^{n-1}, also satisfies these conditions. For $f:\mathbb{R}^n\to\mathbb{R}$ radial, we (ab)use the notation $f(r)$ for the common value of f on the vectors of length r. The (inverse) Fourier transform maps radial functions to radial functions. Moreover, the Gaussian $x\mapsto e^{-\pi\|x\|^2}$ is fixed under the Fourier transform, and, more generally, the sets

$$P_d=\{x\mapsto p(\|x\|^2)e^{-\pi\|x\|^2}:$$
$$p\text{ is a polynomial of degree at most d}\}.$$

are invariant under the Fourier transform. A *computer-assisted* approach to find good functions for the above theorem is to restrict to functions from P_d for some fixed value of d. Any function from this set that satisfies $f(0)=1$ can be written as

$$f_a(x)=\left(1+\sum_{k=1}^{d}a_k k!\pi^{-k}L_k^{n/2-1}(\pi\|x\|^2)\right)\times e^{-\pi\|x\|^2}\qquad(2)$$

for some $a\in\mathbb{R}^d$, where $L_k^{n/2-1}$ is the Laguerre polynomial of degree k with parameter $n/2-1$ (Laguerre polynomials are a family of orthogonal polynomials). We choose this form for f_a, so that its Fourier transform is

$$\widehat{f_a}(u)=\left(1+\sum_{k=1}^{d}a_k\|u\|^{2k}\right)e^{-\pi\|u\|^2},\qquad(3)$$

which means that $a\geq 0$ immediately implies $\widehat{f_a}(u)\geq 0$ for all u. Setting $r_1=1$ in the above theorem, we see that the optimal sphere packing density is upper bounded by the maximum of the linear functional $a\mapsto f_a(0)$ over all nonnegative vectors $a\in\mathbb{R}^d$ for which the linear inequalities $f_a(r)\leq 0$ for $r>1$ are satisfied. For every fixed value of d this gives a semi-infinite linear program, which is a linear program with finitely many variables and infinitely many linear constraints.

One approach to solving these semi-infinite programs is to select a finite sample $S\subseteq[1,\infty)$ and only enforce the constraints $f_a(r)\leq 0$ for $r\in S$. For each S this yields a linear program whose optimal solution a can be computed using a linear programming solver. Then we verify that $f_a(r)\leq 0$ for all $r>1$, or if this is not (almost) true, we run the problem again with a different (typically bigger) sample S. This approach works well in practice for the *spherical code problem*, which is a compact analogue of the sphere packing problem. Here, given a scalar $t\in(-1,1)$, we seek a largest subset of the sphere $S^{n-1}\subseteq\mathbb{R}^n$ such that the inner product between any two distinct points is at least t. A spherical code corresponds to a spherical cap packing where we center spherical caps of angle $\arccos(t/2)$ about the points in the code. The Cohn–Elkies bound can be seen as a noncompact analogue of a similar bound for the spherical code problem known as the *Delsarte linear programming bound*. However, because of noncompactness this sampling approach does not work well for the sphere packing problem.

Another approach, based on semidefinite programming, does work well for noncompact problems such as the sphere packing problem. In [8] this approach is used to compute upper bounds for packings of spheres and spherical caps of several radii. *Semidefinite programming* is a powerful generalization of linear programming, where we maximize a linear functional over a spectrahedron instead of a polyhedron. That is, we maximize a functional $X\mapsto\langle X,C\rangle$ over all positive semidefinite $n\times n$ matrices X that satisfy the linear constraints $\langle X,A_i\rangle=b_i$ for $i=1,\ldots,m$. Here $\langle A,B\rangle=\operatorname{trace}(B^{\top}A)$ denotes the trace inner product. As for linear programs, semidefinite programs can be solved efficiently by using interior point methods.

The usefulness of semidefinite programming in solving the above semi-infinite linear programs stems from the following two observations: Firstly, Pólya and Szegö showed that a polynomial is nonnegative on the interval $[1,\infty)$ if and only if it can be written as $s_1(r)+(r-1)s_2(r)$, where s_1 and s_2 are sum of squares polynomials. Secondly, a sum of squares polynomial of degree $2d$ can be written as $b(r)^{\top}Qb(r)$, where $b(r)=(1,r,\ldots,r^d)$, and where Q is a positive semidefinite matrix. (To see that a polynomial of this form is a sum of squares polynomial one can use a Cholesky factorization $Q=R^{\top}R$.) Using these observations we can introduce two positive semidefinite matrix variables Q_1 and Q_2, and replace the infinite set of linear constraints $f_a(r)\leq 0$ for $r>1$, by a set of $2d+2$ linear constraints that enforce the identity

$$f_a(r)=(1-r)b(r)^{\top}Q_2b(r)-b(r)^{\top}Q_1b(r).$$

In this way we obtain a semidefinite program, which can be solved with a semidefinite programming solver, and whose optimal value upper bounds the sphere packing density. For each d this finds the optimal function f_a.

Producing numerical evidence

Now we want to understand what has to happen when the Cohn–Elkies bound can be used to *prove the optimality* of the sphere packing given by an even and unimodular lattice L.

Then there exists a *magic function* $f : \mathbb{R}^n \to \mathbb{R}$ and a scalar r_1 that satisfy the conditions of Theorem 2, such that the density of the sphere packing equals $f(0) \cdot \mathrm{vol}(B_n(r_1/2))$. As observed before, we may assume $f(0) = 1$, so that r_1 is the shortest nonzero vector length in L. Under these assumptions on f we can derive extra properties that the function f and its Fourier transform must satisfy. Since $f(x) \le 0$ and $\widehat{f}(x) \ge 0$ for all $\|x\| \ge r_1$, and $f(0) = (0) = 1$, we have *equality* in the following chain of inequalities

$$1 \le \sum_{x \in L} \widehat{f}(x) = \sum_{x \in L} f(x) \le 1.$$

This says that we have to have $f(x) = (x) = 0$ for all $x \in L \setminus \{0\}$. In fact, we can apply this argument to any rotation of L, so that $\widehat{f}(x) = f(x) = 0$ for all x where $\|x\|$ is a nonzero vector length in L. As noted before, we may take f to be radial, and then we have (again abusing notation)

$$f(r) = \widehat{f}(r) = 0$$

for all nonzero vector lengths r in L. This also tells us something about the *orders of the roots*. We have $f(0) = 1$ and $f(r) \le 0$ for $r \in [r_1, \infty)$, so the roots at the vector lengths that are strictly larger than r_1 must have even order. We have $\widehat{f}(0) = 1$ and $\widehat{f}$ is nonnegative on $[0, \infty)$, so the roots at the nonzero vector lengths must have even order.

If f does not have additional roots, then in [1] it is shown that there is no other periodic packing achieving the same density as L. To apply this it is important that E_8 is the only even unimodular lattice in $\mathbb{R}^8$ and Λ_{24} is the only even unimodular lattice in $\mathbb{R}^{24}$ that does not contain vectors of length $\sqrt{2}$.

The Cohn–Elkies paper

In [1] Cohn and Elkies used this insight about the potential locations of the roots and double roots to derive a numerical scheme to find functions that are *close* to magic functions. They parametrized the function f_a as in (2). Then they required that f_a and $\widehat{f}_a$ have as many roots and double roots as possible, depending on the degree d. Afterwards they applied Newton's method to perturb the roots and double roots in order to optimize the value of the bound. In dimension 8 and 24 they obtained bounds which were too high only by factors of 1.000001 and 1.0007071. This provided the first strong evidence that magic functions exist for these two dimensions. Since the magic functions f have to have infinitely many roots, the degree d has to go to infinity. Could this method, in the limit, actually give the exact sphere packing upper bounds?

The Cohn–Kumar paper

The next step was taken by Cohn and Kumar in [2]. They improved the numerical scheme and by using degree $d = 803$ (with 3000-digit coefficients) they showed that in dimension 24 there is no sphere packing which is $1 + 1.65 \cdot 10^{-30}$ times denser than the Leech lattice. The actual aim of their paper was to show that the Leech lattice is the unique densest lattice in its dimension. For this they used the numerical data together with the known fact that the Leech lattice is a strict local optimum. The proof of Cohn and Kumar is a beautiful example of the symbiotic relationship between human and machine reasoning in mathematics.

The Cohn–Miller paper

For a long time Cohn and Miller were fascinated by the properties of these conjecturally existing magic functions. In their paper [4], submitted 15 March 2016 to the arXiv-preprint server, they gave a 'construction' of the magic functions using determinants of Laguerre polynomials. However, they could not prove that this construction indeed worked. With the use of high precision numerics they experimented with their construction. This resulted in improved bounds, optimality of Λ_{24} within a factor of $1 + 10^{-51}$. Even more importantly, they detected some unexpected rationalities: For instance using their numerical data they conjectured that for $n = 8$, the magic function f and $\widehat{f}$ have quadratic Taylor coefficients $-27/10$ and $-3/2$, respectively. For $n = 24$, the corresponding coefficients should be $-14347/5460$ and $-205/156$.

Viazovska's breakthrough

Viazovska made the spectacular discovery that a magic function indeed exists for dimension $n = 8$. Building on this, Cohn, Kumar, Miller, Radchenko and Viazovska found a magic function for $n = 24$. Viazovska's construction is based on a couple of new ideas, which we want to explain briefly.

Each radial Schwartz function $f : \mathbb{R}^n \to \mathbb{R}$ can be written as a linear combination of radial eigenfunctions of the Fourier transform in $\mathbb{R}^n$ with eigenvalues $+1$ and -1. Viazovska wrote the magic function as a linear combination $f = \alpha f_+ + \beta f_-$, where f_+ is a radial eigenfunction of the Fourier transform with eigenvalue $+1$ and f_- is a radial eigenfunction with eigenvalue -1. The coefficients α and β are determined later on.

She makes the Ansatz that for $r > r_1$, we can write these functions f_+ and f_- as a squared sine function times the Laplace transform of a (quasi)-modular form. That is, she proposes that

$$f_+(r) = -4 \sin(\pi r^2/2)^2 \times \int_0^{i\infty} \psi_+(-1/z) z^{n/2-2} e^{\pi i r^2 z} dz$$

and

$$f_-(r) = -4 \sin(\pi r^2/2)^2 \int_0^{i\infty} \psi_-(z) e^{\pi i r^2 z} dz,$$

where ψ_+ is a quasi-modular form and ψ_- is a modular form.

The $\sin(\pi r^2/2)^2$ factor insures (assuming the above integrals do not have cusps) that the resulting function f (as well as its Fourier transform) have double roots at all but the first occurring vector lengths.

Viazovska noticed that an analytic extension of f_- exists and that it is an eigenfunction of the Fourier transform having eigenvalue -1 when the following modularity relation holds:

$$\psi_-\left(\frac{az+b}{cz+d}\right) = (cz+d)^{2-n/2} \psi_-(z)$$
$$\text{for all} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}), a, d \text{ odd}, b, c \text{ even}.$$

For the explicit definition of f_- we need the theta functions

$$\Theta_{01}(z) = \sum_{n \in \mathbb{Z}} (-1)^n e^{\pi i n^2 z}$$

and

$$\Theta_{10}(z) = \sum_{n \in \mathbb{Z}} e^{\pi i (n+1/2)^2 z}.$$

Similarly, but technically much more involved, the analytic extension of f_+ is an eigenfunction for the eigenvalue $+1$ when ψ_+ is a quasi-modular form. The forbidden Eisenstein series E_2 becomes important here.

Now it needs to be shown that there exists a linear combination $f = \alpha f_+ + \beta f_-$ so that $\widehat{f}(0) = f(0) = 1$ holds, and that the sign conditions $f(r) \leq 0$ for $r \geq r_1$ and $\widehat{f}(u) \geq 0$ for all $u \geq 0$ are fulfilled. For this, and more, we refer to the beautiful original papers. Here we want to end by taking a look at the magic functions: see Table 2.

E_8	Λ_{24}
$\psi_+ = \dfrac{(E_2E_4 - E_6)^2}{\Delta}$	$\psi_+ = \dfrac{25E_4^4 - 49E_6^2E_4 + 48E_6E_4^2E_2 + 25E_6^2E_2^2 - 49E_4^3E_2^2}{\Delta^2}$
$\psi_- = \dfrac{5\Theta_{01}^{12}\Theta_{10}^{8} + 5\Theta_{01}^{16}\Theta_{10}^{4} + 2\Theta_{01}^{20}}{\Delta}$	$\psi_- = \dfrac{7\Theta_{01}^{20}\Theta_{10}^{8} + 7\Theta_{01}^{24}\Theta_{10}^{4} + 2\Theta_{01}^{28}}{\Delta^2}$
$f(x) = \dfrac{\pi i}{8640} f_+(x) + \dfrac{i}{240\pi} f_-(x)$	$f(x) = -\dfrac{\pi i}{113218560} f_+(x) - \dfrac{i}{262080\pi} f_-(x)$

Table 2 The magic functions.

Interview with Henry Cohn, Abhinav Kumar, Stephen D. Miller and Maryna Viazovska

The interview was conducted by Frank Vallentin using the online communication platform Google Hangouts between 20 May and 14 June 2016.

Computer Assistance

Dear Henry, Abhinav, Steve, and Maryna, First of all let me congratulate you to your breakthrough papers. This issue of the 'Nieuw Archief voor Wiskunde' is a special issue focussing on computer-assisted mathematics. In it we have two articles about sphere packings: One about the formal proof of the Kepler conjecture and one about your recent breakthrough on sphere packings in dimensions 8 and 24. At the moment a proof of the Kepler conjecture without computer-assistance is not in sight, but your proofs in dimension 8 and 24 require almost no computers. How were computers helpful to you when finding the proofs?

Abhinav: "The Cohn–Elkies paper and later the Cohn-Kumar and the Cohn–Miller papers were certainly useful as indicators that the solution was out there waiting for the right functions. In our new Cohn–Kumar–Miller–Radchenko–Viazovska paper at least, the numerical data was quite useful because once we had figured out the right finite dimensional space of modular or quasi-modular forms, the numerics helped us pin down the exact form (up to scaling) of the function. In particular, we matched values and derivatives of f and $\widehat{f}$ at lattice vector lengths to cut down the space by imposing linear conditions. Steve has a Mathematica code to do some of this but we also used PARI/GP and occasionally Maple.

A couple more things — in both the proofs the final inequalities needed for the functions and Fourier transforms are done with computer assistance. There might be more elegant hand proofs, but so far we haven't found them. And we did quite a lot of messing around with q-expansions et cetera, which would have been very painful outside of a computer algebra system."

Steve: "Though I think there will eventually be a slick proof that can be written by hand, computers were completely essential in this story. To me the best example is the appearance of rational numbers that Henry Cohn and I discovered.

I can only speak for myself, but I was completely fascinated by the (then proposed) existence of these 'magic' functions which completely solve sphere packing in special dimensions. To satisfy this curiosity, Henry and I began computing their features to see if we could learn more about them. We found — using some serendipity with the online 'inverse symbolic calculator' website — that their quadratic Taylor coefficients were rational, and furthermore related to Bernoulli numbers. This was a strong hint that modular forms were connected, though we never understood why until we saw Maryna's paper. We found other rationalities (such as the derivatives at certain points) using some theoretical motivation, combined with good numerical approximations. It was a type of 'moonshine', with a fascinating sequence of numbers and an amazing structure we could not otherwise access.

Once Maryna's paper appeared, it took just a few days to combine her insight with our previous numerics in an exact way. It's important to stress that at this point we could derive the magic function for 24 dimensions without using floating point calculations, since we had already extrapolated exact expressions from previous numerics. Maybe we will later understand a way to derive the 24-dimensional functions without such information, but at the time it was highly convenient to leverage them."

Maryna, did you also use computer assistance when you found the function for dimension 8? In your paper you mention in passing that one compute the first hundred terms of Fourier expansions of modular forms in a few second using PARI/P or Mathematica.

Maryna: "Numerical evidence was crucial to believe in the existence and uniqueness of 'magic' functions. I used computer calculations to verify that approximations to the magic function computed from linear equations (similar to the equations considered in the Cohn–Miller paper) converge to the function computed as an integral transform of a modular form. I used Mathematica and PARI/GP for this purpose."

Checking the positivity conditions is the only part of the proof which depends on the use of computers. How do you make sure that your computer proof is indeed mathematically rigorous?

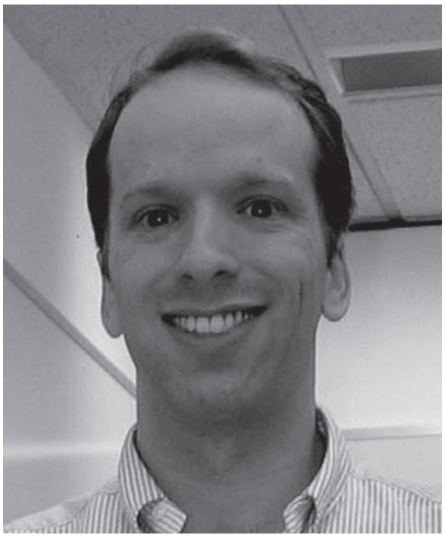
Henry: "In her 8-dimensional proof, Maryna used interval arithmetic. In the 24-dimensional case, we used exact rational arithmetic. Either way, it's not a big obstacle. The inequalities you need have a little slack, which means you can bound everything

Photo: Archives of the Mathematisches Forschungsinstitut Oberwolfach

Henry Cohn

in any number of different ways, without needing to do anything too delicate.”

Steve: “To elaborate: Our positivity check involves showing certain power series $f(q)$ in a parameter q are positive, where $q < 1$. It is not difficult to bound the coefficients and deduce this positivity for $q < c$ (where c is an effective constant), so the problem reduces to showing positivity for q in the interval $[c, 1]$. Numerically one can plot this directly, of course. From such a graph we see $f(q) > b$ for some explicit constant b. Write $f(q) = p(q) + t(q)$, where $p(q)$ is a polynomial consisting of the first several terms in the power series and $t(q)$ the tail, with the number of terms chosen so that $t(q)$ is provably less than $b/2$ for q in $[c, 1]$. We are now reduced to showing $p(q) - b/2 > 0$ for $q \in [c, 1]$, and such an inequality can be rigorously established using Sturm’s theorem.”



Stephen D. Miller

Modular forms

For a long time it has been known that the E_8 lattice and the Leech lattice have strong connections to modular forms, simply by their theta series. However, one thing which puzzles us (we mainly work in optimization) is that you solve an optimization problem, an infinite-dimensional linear program, with tools from analytic number theory, especially using modular forms. How surprising was it to you that using modular forms was one key to the proof?

Henry: “Modular forms are by far the most important class of special functions related to lattices, so in that sense it’s not so surprising that they come up. Over the years many people had suggested using them, but it wasn’t clear how. For example, many years ago I had tried the Laplace transform of a modular form, but without Maryna’s $\sin^2$-factor. Without that, it seemed impossible to get anything like the right roots, and therefore the approach was completely useless. What I find beautiful about her proof is how ingeniously it puts everything together (when I know from personal experience that thinking ‘I’d better use modular forms’ will not just lead you to this proof).

Before Maryna’s proof, I could imagine two possibilities. One was that the right approach would be to solve the LP problem in general, getting the exceptional dimensions just as special cases. This approach might not have involved modular forms at all. The other was that there would be particular special functions in those dimensions. I always hoped for something special in 8 and 24 dimensions (e.g., based on the numerical experiments Steve and I worked on), but I was a little worried that maybe the difficulty of writing these functions down indicated that this might be the wrong approach (while solving the problem in general seemed even harder). It was great to see that everything was as beautiful as we had always hoped.”

Steve: “The use of Poisson summation in the Cohn–Elkies paper (and an earlier technique of Siegel) had already brought methods of analytic number theory into the subject. It was clear relatively early on that modular forms must be somehow involved in the final answer. This is both because of the appearance of special Bernoulli numbers (that prominently arise in modular forms) as well as the overall structure of the summation formulas. For example, Henry and I derived a relevant Fourier eigenfunction with simple zeros using Voronoi summation formulas and derivatives of modular forms.

However, while these ingredients had been on the table for a long time, we didn’t know how to combine them until Maryna’s paper appeared. At that point many of the various pieces of evidence we had suddenly fit together. Quasi-modular forms (which are derivatives of modular forms) are very crucial to this story.”

Abhinav Kumar

Collaboration

Maryna’s breakthrough paper which solved the sphere packing problem in dimension 8 *was submitted on March 14, 2016 to the arXiv-preprint server. Then it took only one week until you submitted the solution of the sphere packing problem in dimension*

Photo: Archives of the Mathematisches Forschungsinstitut Oberwolfach

Maryna Viazovska

24 to the arXiv. Working at five different, distant places, how did you collaborate? What were the difficulties when going from 8 to 24 dimensions?
Steve: "That was certainly an exciting and memorable week. Once we had assembled our team, things moved extremely quickly. This is mainly because Maryna's methods are so powerful, but it was also important that certain pairs of us (Henry–myself, Abhinav–Henry, and Danylo–Maryna) were established collaborators that had already worked together well.

In addition to phone calls and email, we used Skype and Dropbox to communicate our ideas. I particularly like drawing mathematics on a tablet PC and sharing the screen on Skype — this allows the others to watch as if I'm writing on a blackboard. As soon as we saw Maryna's paper on Tuesday morning, we tried to make concrete bridges with the numerical observations Henry and I made in our Cohn-Miller paper. After some reformulation of her modular forms as quotients, by Wednesday it was then clear what properties of the q-expansions would be needed to obtain the 24-dimensional functions. We also set up computer programs to match potential candidate functions with the numerical values that we could compute separately.

On Thursday we had the right space of modular forms and a program ready to search in them. By matching the conjectured rationalities, we found the even eigenfunction on Thursday night and the odd eigenfunction on Friday morning. We used Mathematica and PARI/GP for this. We also checked using graphs and q-expansions that the necessary positivity conditions indeed hold, but did not have a completely rigorous proof of this remaining point.

By Friday afternoon I was completely exhausted (and in any event do not work on the Jewish Sabbath). It's important to note that the positivity analysis was a little different in 24 dimensions than in 8 because of an extra pole that occurs."

Going further

Now the sphere packing problem has been solved in 1, 2, 3, 8 and 24 dimensions and, coming close to the end of the interview, it is time to make speculations: Are there candidate dimensions where a solution is in sight? Do you think that your method will be useful for this (or for other problems)?
Henry: "It's hard to say for sure whether there might be further sharp cases in higher dimensions, but it seems unlikely that they would have remained undetected. (At the very least it's not plausible that they could have the same widespread occurrences in mathematics as E_8 or the Leech lattice, since they would presumably have been discovered in the process of classifying the finite simple groups, for example.) Two dimensions, $n=2$, remains open, although of course a solution is known by elementary geometry; the LP bounds behave rather differently in two dimensions compared with 8 or 24."
Steve: "Yes, dimension two seems very different because of the delicate arithmetic nature of the root lengths."
Henry: "I'm sure Maryna's wonderful approach to constructing these functions is just the tip of an iceberg, and I'm optimistic that humanity will learn more about how LP bounds and related topics work.

One mystery I find particularly intriguing is what's so special about 8 and 24 dimensions. Maryna's methods give a beautiful proof, but I still don't really know a conceptual explanation as to what's different in, say, 16 dimensions (beyond just the fact that the Barnes-Wall lattice isn't as nice as E_8 or the Leech lattice). On a slightly different topic, I hope someone solves the four-dimensional sphere packing problem, but it will require different techniques. Unlike the cases where the LP bounds are sharp, these pair correlation inequalities will not suffice by themselves. But the D_4 lattice is awfully beautiful, and the world deserves a proof of optimality. The only question is how...

Henry, Abhinav, Steve, and Maryna, thank you very much for this interview.

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