The Last Delaunay Triangulation

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1 Introduction

During the DocCourse 2009 in Barcelona we were presented with the following problem: we know that the Delaunay graph of a set of n points (n pair) in the plane in general position always contains a perfect matching [2]. The proof, by Michael Dillencourt, relies heavily on the properties of the Euclidean metric, so the question remains open for other popular metrics.

Last year Ábrego et al. proved that for the metrics L_1 and L_∞ the Delaunay graph of points in the plane not only contains a perfect matching; the set of points always contains a Hamiltonian path [1].

The problem presented in the DocCourse was: Is it true that the Delaunay graph admits a perfect matching for all the L_p metrics?

Although interested in the problem, we faced a little difficulty with it: we were not able to even visualize how the Delaunay graph looked like in metrics different to L_1, L_2 and L_∞. So we first started to try and understand what happens to the Delaunay graph in arbitrary metrics L_p, and this led us to a different problem, which we called (by a suggestion of Ferran Hurtado) *The Last Delaunay Triangulation*.

2 Definitions

The Euclidean metric is defined by the usual distance function

$$|(x_1, y_1), (x_2, y_2)|_2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = (|x_2 - x_1|^2 + |y_2 - y_1|^2)^{\frac{1}{2}}.$$

If we define this distance function as L_2, we can generalize it to a family of metrics L_p:

$$|(x_1, y_1), (x_2, y_2)|_p = (|x_2 - x_1|^p + |y_2 - y_1|^p)^{\frac{1}{p}}.$$

This family of metrics are sometimes called Minkowski metrics. We define L_∞ in the obvious way as

$$|(x_1, y_1), (x_2, y_2)|_\infty = \max(|x_2 - x_1|, |y_2 - y_1|).$$

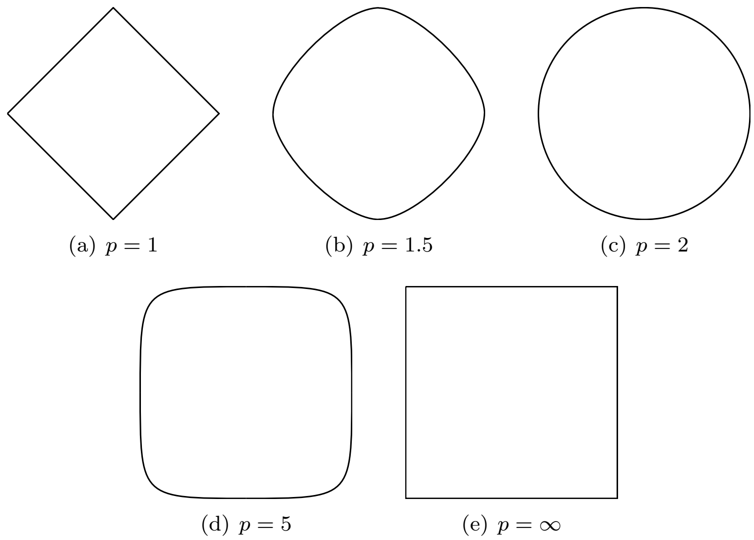


Figure 1: Spheres in different L_p metrics.

We can think of how the different L_p metrics behave by looking at the unit sphere in each of them. For L_2 it's a circle, obviously, and it starts to look more and more like a diamond when p goes to 1, and more and more like a square when p goes to ∞ (Figure 1).

For $1 < p < \infty$ every point in a sphere has exactly one support line, which is not true for L_1 nor L_∞ because they have sharp corners; the spheres in L_p with $1 < p < \infty$ are a smooth curve in all of their points. Also, if two spheres in L_p intersect each other, they do it in one or two points (or they are the same ball); again, this is not true for L_1 nor L_∞ because they contain segments of lines (Figure 2).

Because of these properties, three non-collinear points determine exactly one sphere that passes through them in L_p, for $1 < p < \infty$. Once more, this is not true for neither L_1 nor L_∞; In fact, in those metrics there are many sets of three non-collinear points, for which there is no sphere passing through all of them.

Given a set P of n points in the plane in general position, we define the Delaunay graph $DG(P)$ of P as the graph with P as its vertex set, and where two points are connected by an edge if there is a ball (given a fixed metric) that covers them but no other point in P.

Obviously it is equivalent to consider empty spheres instead of balls, where a sphere S is called empty if there are no elements of P in the bounded region of S.

For L_2 the Delaunay graph is a triangulation, and in fact it's a triangulation for all L_p with $1 < p < \infty$. For L_1 and L_∞ it is not a triangulation, and even more, it's possible that it does not contains the convex hull of P (Figure 3).

Also we observe that for every edge in a Delaunay triangulation we find a

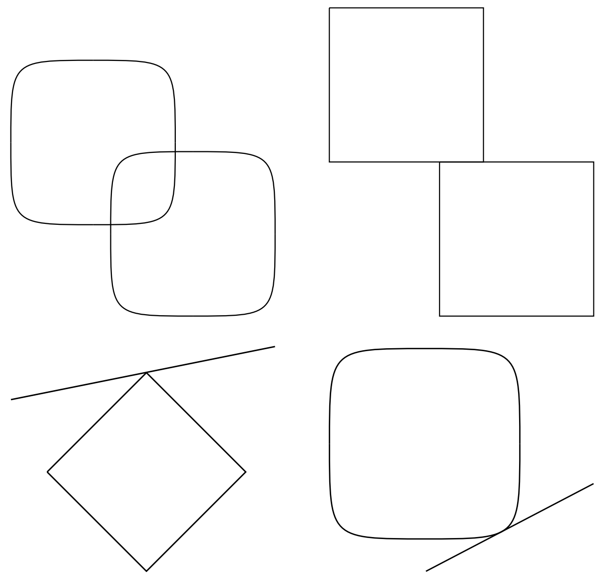


Figure 2: Differences between L_1, L_∞ and L_p with $1 < p < \infty$.

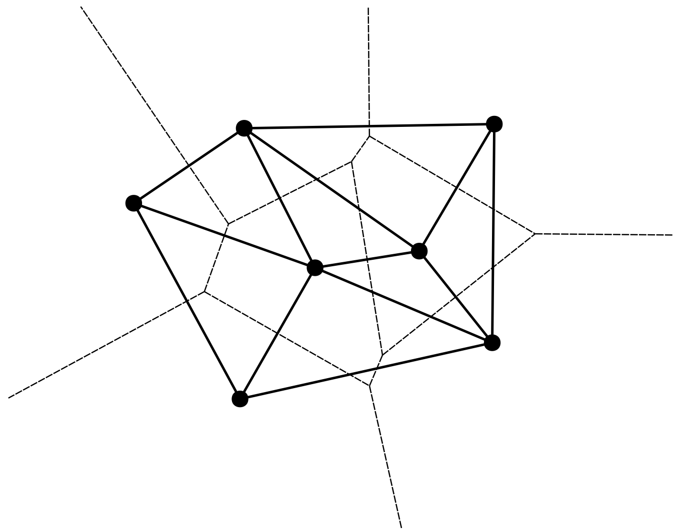
triangle containing this edge, such that there is an empty sphere through the vertices of the triangle. Therefore the Delaunay triangulation can be expressed by the collection of empty spheres passing through three points of P.

3 Delaunay triangulations in L_p

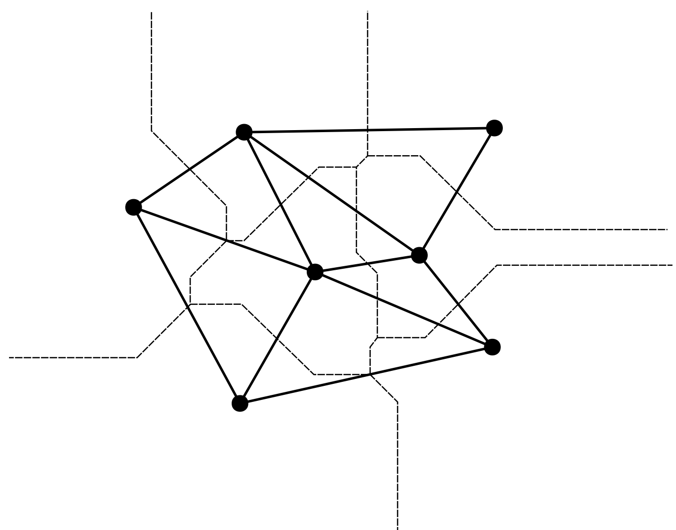
Playing with different examples of sets of points and what happens when we increase the p in L_p starting from 2, we observed the following: A triangulation remains unchanged until one of the empty spheres that passes through three points of P suddenly touches a fourth point. In this moment technically we loose the general position because this sphere passes through four points, but we ignore that and keep growing p. What happens then is that we loose one of the edges of the triangulation, and we win the crossing edge that appears with the fourth point (Figure 4).

The interesting moment is when the sphere (actually two spheres converging into one) passes through these four points. Technically we have a crossing or an empty quadrangle there, but we take only one of the diagonal edges of the quadrangle so to get only triangulations for all $1 < p < \infty$. The two original spheres converge to a unique sphere that passes through the four points, and then we have our flip (and consequently two other spheres: we loose the other two because they now enclose other points of P). This keeps happening for other quadrangles in P as we grow p: we have flips in our Delaunay triangulations.

An interesting but erroneous idea that we had was that maybe when we loose an edge with a flip, that edge does not appear again in any other metric.



(a) Delaunay triangulation in L_2, with Voronoi diagram in dashed lines.



(b) Non convex Delaunay graph in L_1 with the same set.

Figure 3: Delaunay graphs in the same set of points with different metrics.

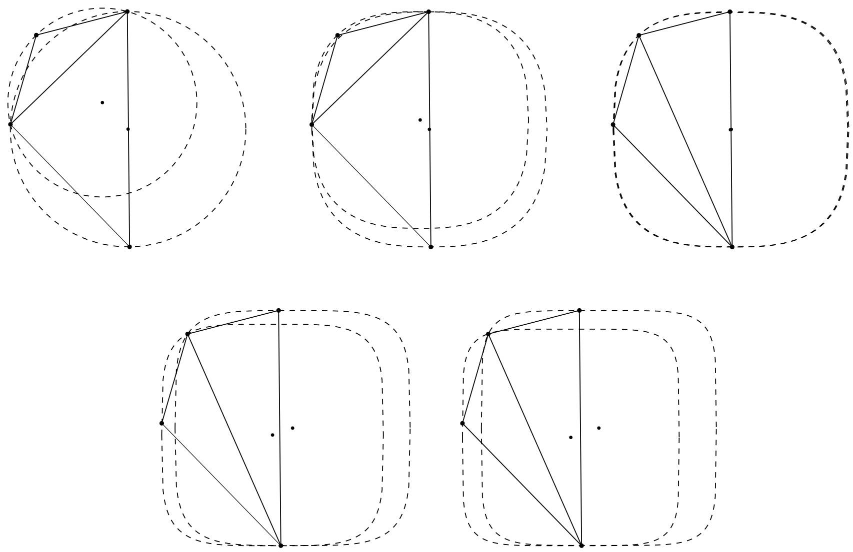


Figure 4: A flip between two different Delaunay triangulations of the same point set.

Unfortunately this is not true: we have an example where we have a flip and then for a larger p we have another flip (back to the first configuration) in the same quadrangle (Figure 5).

But what does happen is that there comes a time when the spheres in our metric L_p are so close to a square that no matter how much we keep growing p, another flip is not possible.

And then what we have is The Last Delaunay Triangulation.

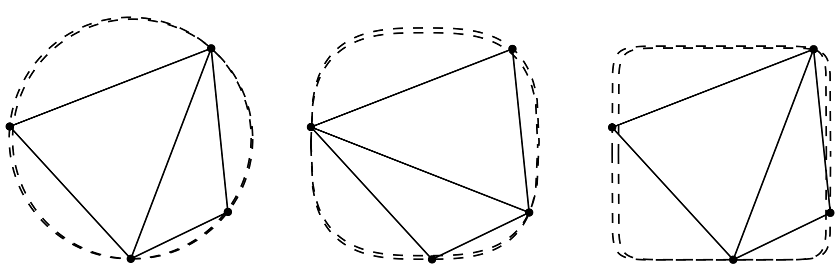


Figure 5: An edge that reappears when we grow p in L_p.

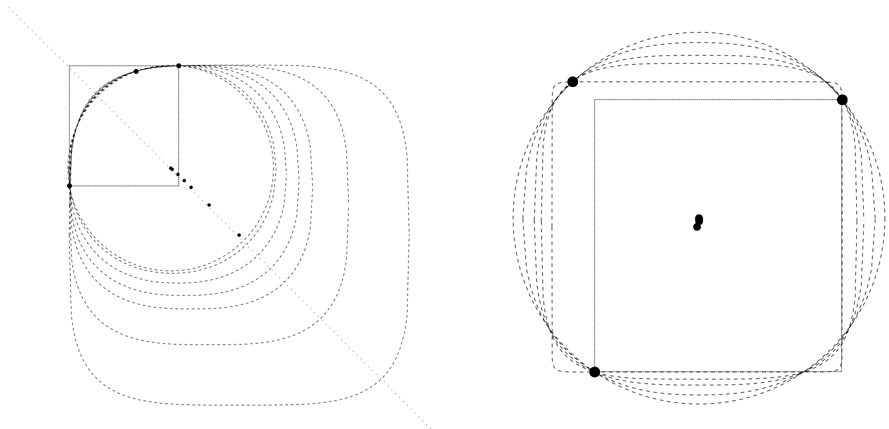


Figure 6: How the centes diverges or converges when p grows.

4 The Last Delaunay Triangulation

Instead of focusing on the empty spheres of our Delaunay triangulations, we are going to look at the centers of those spheres. As we grow p, the centers of those spheres start to move as curves; we have a flip every time these curves meet (simultaneously for the same p).

It is obvious that a curve described by a center does not intersect itself, and we haven't proved but we are convinced that the curves have the following behavior (up to symmetry): Take two of the three points that determine a sphere (without loss of generality lets assume that they do not lie on the same horizontal line), and consider the rectangle with these two points in its corners. If the third point is inside this rectangle, the curve described by the spheres passing through the three points will diverge asymptotically to a line with slope 1 or -1. If the third point is outside this rectangle, the curve will be bounded (Figure 6).

So when p is sufficiently large, the centers of the spheres will converge to a fixed point, or they will move away from the set in a stable way. There is no way in which they could intersect each other, and so the Delaunay triangulation will remain the same until p "reaches" ∞.

That is The Last Delaunay Triangulation, and the name fits because in ∞ (as we already saw) there is the possibility that the Delaunay graph is not a triangulation. Although we have sketches of proofs for all of our claims, we will need to write them down carefully and formally before we present them.

5 Conjectures and future work

We believe that the Delaunay graph in L_∞ is a subgraph of The Last Delaunay Triangulation, and we are trying to prove it. This would give us an efficient way to obtain The Last Delaunay Triangulation, because we will only need to calculate the Delaunay graph in L_∞ and then complete the triangulation (in the right way, of course).

As we saw, it's possible that a triangulation repeats itself, for growing p. How many times can this happen? How many times can an edge disappear and reappear later?

We also would like to know what is the maximum number of triangulations that a set of points can realize when we grow p starting in 2; we believe that this number of triangulations is polynomial in the number of points in the set, and that it will be n^2 or n^3.

Much of the initial insight in this problem came after we programmed a little algorithm to calculate the circumcenter of three points in L_p for arbitrary p. This allowed us to actually see how a given Delaunay triangulation changes when p is growing; unfortunately our algorithm is not very fast, and we could only work with small point sets. Although not necessary for the main result and the conjectures we are investigating, it would be nice to come up with a better algorithm that would allow us to see how larger sets of points behave when growing the p.

Acknowledgments

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