Sets with Small Neighborhood in the Integer Lattice

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Abstract

For a given cardinality we want to find lattice point configurations such that the number of lattice points with distance 1 to the set is small. Sets with the smallest number possible are called optimal. It is known that sets of points with coordinate sum less or equal to some integer k are optimal. We show that they are unique for their cardinalities. Also we will discuss the question of how to characterize optimal sets in general, and if by adding the points of distance 1 to an optimal set we will always get an optimal set. In both cases the answer is positive for the plane.

Acknowledgements

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1 Introduction

Consider the d-dimensional lattice $\mathbb{Z}^d$ with the distance $d(x, y) = \sum_i |x_i - y_i|$, where $x = (x_1, \ldots, x_d)$ and $y = (y_1, \ldots, y_d)$. Given a finite set $X \subset \mathbb{Z}^d$, the *neighborhood*, $N(X)$, of X is

$$N(X) = \left\{ y \in \mathbb{Z}^d \backslash X : d(y, X) = 1 \right\},$$

and the the size of this neighborhood is $n(X) = |N(X)|$. Further, the *boundary*, ∂X, of X are the points of X that are next to some point of $N(X)$, and the *interior*, $\operatorname{int} X$, of X are the points of X for which all neighbors are in X, i.e.,

$$\partial X = \{x \in X : d(x, N(X)) = 1\},$$

$$\operatorname{int} X = \{x \in X : d(x, N(X)) > 1\} = X \setminus \partial X.$$

For any $r \in \mathbb{N}$, the set $B_r^d = \{x \in \mathbb{Z}^d : \sum_i x_i < r\}$ is called the *ball* (of radius r). Note that the neighborhood $N(B_r^d)$ of a ball B_r^d contains exactly the points of $\mathbb{Z}^d$ with coordinate sum r.

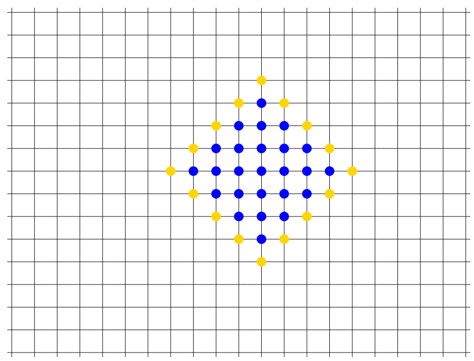

Figure 1: The ball B_4^2 (blue) and its neighborhood (yellow).

A finite set $X \subset \mathbb{Z}^d$ is called *optimal* if the size of its neighborhood $n(X)$ is minimal among all sets $Y \in \mathbb{Z}^d$ with $|Y| = |X|$.

Background. In this paper we consider the problem of characterizing point sets with minimum neighborhood size among all sets of fixed cardinality. Isoperimetric problems of this kind have arisen in a number of different contexts, with several definitions of neighborhood, and several different underlying finite and infinite lattices. One approach to a solution is providing an ordering of the lattice points such that the first j of them form an optimal set of their cardinality for every j. For the Boolean lattice (chains of length two) this is the celebrated theorem of Harper [Har66]. Kruskal-Katona, and Clements and Lindström [CL69] solve this for chains of arbitrary length l. Beruzkov and Serra [BS02] consider the problem for cartesian powers of graphs.

Macaulay [Mac27] presents an ordering of the nonnegative d-dimensional integer points $\mathbb{Z}_+^d$ having coordinate sum $\leq k$ such that the first j of them have a minimum number of neighbors with coordinate sum $k+1$ among all sets of k-sum points. Wang and Wang [WW77a] present, as a similar (and equivalent) result, sets in $\mathbb{Z}_+^d$ that minimize the number of neighbors in $\mathbb{Z}_+^d$, and extended this to an ordering of the points of $\mathbb{Z}^d$ such that the first j of them minimize the number of neighbors in $\mathbb{Z}^d$. They call these optimal sets *standard spheres*, and we adopt this terminology. Basically, a standard sphere is a ball plus possibly some points of the neighborhood of the ball.

As it happens, the sequence of standard spheres enjoys the property that it is closed under the operation of adding the neighborhood. In particular, balls B_r^d are optimal sets in $\mathbb{Z}^d$, which is the analogous result to the classical isoperimetric problem in Euclidean space. In contrast with its Euclidean counterpart, standard spheres are not, in general, the only optimal sets in lattices.

In the case of the Boolean lattice, all optimal sets have been characterized by Beruzkov [Bez89], while for general lattices the complete characterization is still an open problem.

For a survey about different types of isoperimetric problems, as well as a thorough list of references, see [Bez94].

In this paper we address the following questions.

Problem 1.1. *Let $X \subset \mathbb{Z}^d$ be an optimal set. Is it true that $N(X)$ is also an optimal set?*

The answer is yes in the Boolean lattice (see [WW77b]). In Section 3 we give a proof for $\mathbb{Z}^2$.

Problem 1.2. *Is it true that balls are the only optimal sets of their cardinality in the d-dimensional lattice?*

The answer is again yes for the Boolean lattice (see [Bez94] and references therein). We prove that it is also true for the d-dimensional lattice (see Section 2).

Problem 1.3. *What are necessary and sufficient conditions for sets $X \subset \mathbb{Z}^d$ to be optimal?*

In Section 3 we explain some necessary conditions for optimality in $\mathbb{Z}^2$.

1.1 Some Basic Observations

We call a set $X \subset \mathbb{Z}^d$ *connected* if, for any $x, y \in X$, there exists a path $(v_j)_{1\leq j\leq k}$ from x to y in X such that any two consecutive points in the path differ by at most one in each coordinate, i.e. $v_j - v_{j+1} = \sum_{i\leq d} c_i e_i$ with $c_i \in \{-1, 0, 1\}$.

Proposition 1.4. *A necessary condition for a finite set $X \subset \mathbb{Z}^d$ to be optimal is that X is connected.*

Proof. Consider a finite set X consisting of two connected components U and V. Further let $u_{\max}$ be some point of U with maximal first coordinate, and $v_{\min}$ a point of V with minimal first coordinate. Then $u_{\max} + (1, 0, \ldots, 0) \in N(X)$. Thus by translating V such that $u_{\max} + (1, 0, \ldots, 0) = v_{\min}$ we get a set X' with $|X'| = |X|$ and $n(X') \leq n(X) - 1$. $\square$

In the following, all considered sets $X \subset \mathbb{Z}^d$ will be finite and connected.

Proposition 1.5. *The optimal neighborhood size is increasing with the cardinality of the point set: if X and Y are optimal sets with $|X| < |Y|$, then $n(X) \leq n(Y)$.*

Wang and Wang show this for standard spheres [WW77a], and thus it has to be true for all optimal sets.

Proposition 1.6. *Consider the ball B_r^d, with $|B_r^d| = s$. Then the neighborhood of any set $X \subset \mathbb{Z}^d$ with $|X| = s + 1$ has size $n(X) \geq n(B_r^d) + d - 1$.*

Proof. This follows again directly from the results in [WW77a]. The standard sphere with $s+1$ points is B_r^d plus a point $x \in N(B_r^d)$ in the positive orthant. The grid point x has coordinate sum r, and thus it has in each coordinate direction one neighbor with coordinate sum $r + 1$. All these neighbors must lie in $N(B_r^d \cup \{x\}) \backslash N(B_r^d)$. Thus the neighborhood of $B_r^d \cup \{x\}$ has at least size $n(B_r^d) + d - 1$.

As standard spheres are optimal, any set of cardinality $s + 1$ needs to have at least this neighborhood size. $\square$

Given a set $X \in \mathbb{Z}^d$, we want to consider slices of X with respect to some coordinate direction. To this end we denote by

$$L_{k,l}(X) = \{x \in X : x_k = l\}$$

the *l^{th} k-level* of X. Every k-level lies in a $(d-1)$-dimensional hyperplane, and we denote by $L_{k,l}^{d-1}(X) \in \mathbb{Z}^{d-1}$ the $(d-1)$-dimensional set that results from $L_{k,l}(X)$ by omitting the k-th coordinate:

$$L_{k,l}^{d-1}(X) = \left\{(x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_d) \in \mathbb{Z}^{d-1} : (x_1, \ldots, x_d) \in L_{k,l}(X)\right\}.$$

In the other direction, given some set $Y \in \mathbb{Z}^{d-1}$ then we define the *l^{th} k-extension* of Y as the set $Y_{k,l}^d \in Z^d$ that is obtained by adding a (new) k-th coordinate with value l:

$$Y_{k,l}^d = \left\{(x_1, \ldots, x_d) \in \mathbb{Z}^d : x_k = l, (x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_d) \in Y \subset \mathbb{Z}^{d-1}\right\}$$

Observation 1.7. *The size of the neighborhood of some k-level, $n(L_{k,l}(X))$, equals the size of the neighborhood of $L^{d-1}_{k,l}(X)$ (in $d-1$ dimensions) plus two times the size of X.*

Observation 1.8. *Let $X \subset \mathbb{Z}^d$ and let $L_{k,l}(X)$ be some level (in any coordinate-direction) with nonempty interior $I = \operatorname{int} L^{d-1}_{k,l}(X)$. Then adding (any subset of) the k-extensions $I^d_{k,l-1}$ and $I^d_{k,l+1}$ to X does not increase the size of the neighborhood $n(X)$.*

By adding we here mean taking the union of the point sets.

Observation 1.9. *If $X \subset \mathbb{Z}^d$ contains some level $L_{k,l}(X)$ such that the following two statements hold:*

1. $L^{d-1}_{k,j+1}(X) \subseteq \operatorname{int} L^{d-1}_{k,j}(X)$ *for all* $j \geq l$*;*
2. $L^{d-1}_{k,j-1}(X) \subseteq \operatorname{int} L^{d-1}_{k,j}(X)$ *for all* $j \leq l$*,*

then the size of the neighborhood of X is $n(X) = n(L_{k,l}(X))$.

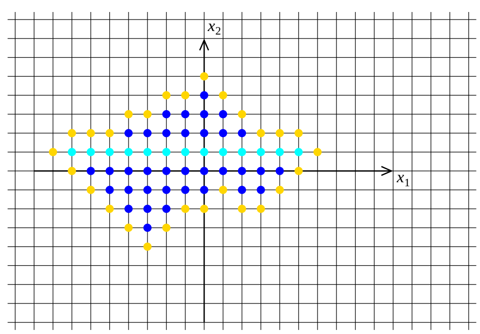

Figure 2: The highlighted level, $L_{2,1}(X)$, satisfies Observation 1.9.

Observation 1.10. *The cardinality of any ball $|B^d_r|$ is odd.*

Proof. The result is trivial for $d = 1$. For $d > 1$ a ball consists of an odd number of balls of dimension $d - 1$. $\square$

2 Uniqueness for Balls

2.1 Dimension 2

Given a set $X \subset \mathbb{Z}^2$, assume w.l.o.g. that the origin $(0, 0)$ is part of X, and consider the following four *diagonal tangents* of X:

$$\begin{array}{ll} t_{++}: & +x_1 + x_2 = c_{++} \geq 0 \\ t_{+-}: & +x_1 - x_2 = c_{+-} \geq 0 \\ t_{-+}: & -x_1 + x_2 = c_{-+} \geq 0 \\ t_{--}: & -x_1 - x_2 = c_{--} \geq 0 \end{array}$$

where $c_{\pm,\pm}$ are chosen so that for each equality there is some point in X that satisfies it and every point in X satisfies the inequalities obtained by replacing $=$ with $\leq$. The *diagonal hull* $DH(X)$ of X is the set of all integer points in the region bounded by these tangents. Further, X is called *diagonal convex* if $X = DH(X)$.

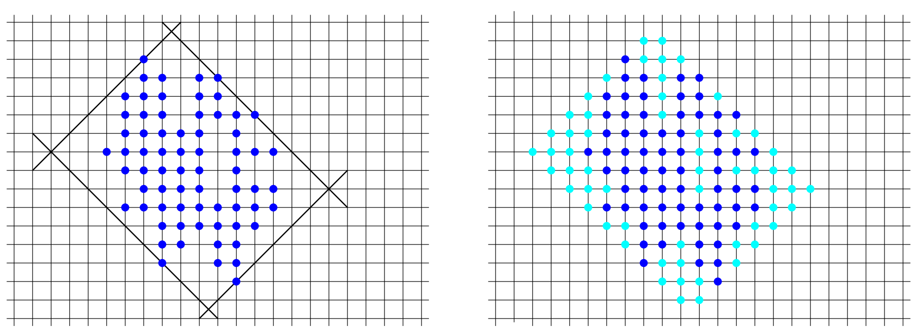

Figure 3: A set with its diagonals (left) and the diagonal hull of the set (right).

If a set $X \subset \mathbb{Z}^2$ is diagonal-convex, then each point on the boundary lies on at most two of the diagonal tangents, and X resembles a parallelogram. However, if two diagonal tangents intersect in a non-integer point, then there is a (connected) portion of ∂X of cardinality 2 that is parallel to some coordinate direction. We will refer to such parts as *axis-aligned* components (of ∂X). Accordingly, for each of the diagonal tangents t we call $\partial X \cap t$ a *diagonal* (of ∂X).

Note that ∂X might have none, two, or four axis-aligned components. For example, the diagonal hull of the set X in Figure 3 has two axis-aligned components, both of which are parallel to the horizontal axis.

Proposition 2.1. *Let $X \subset \mathbb{Z}^2$, then $n(DH(X)) \le n(X)$.*

Proof. $DH(X)$ is obtained from X by repeatedly applying the operation from Observation 1.8 until there are no points in any direction that can be added. □

Theorem 2.2. *Balls B_r^2 are unique optimal sets of their cardinality.*

Proof. Assume that $X \subset \mathbb{Z}^2$ is a set with $|X| = |B_r^2|$ and $n(X) = n(B_r^2)$ for some $r \in \mathbb{Z}_+$. Then X has to be diagonal convex. Otherwise $|DH(X)| > |B_r^2|$ and $n(DH(X)) \le n(X) = n(B_r^2)$, which contradicts Proposition 1.6.

There are four cases for the possible number and relative positions of the axis-aligned components: ∂X can have four, two parallel (opposite), none, or two orthogonal axis-aligned components of size 2 (see Figure 4).

Our main strategy will be to transfer parts of ∂X to some part of $N(X)$ without increasing the size of the neigborhood. For these new sets it is then easy to see that they cannot simultaneously be optimal and have the cardinality of a ball.

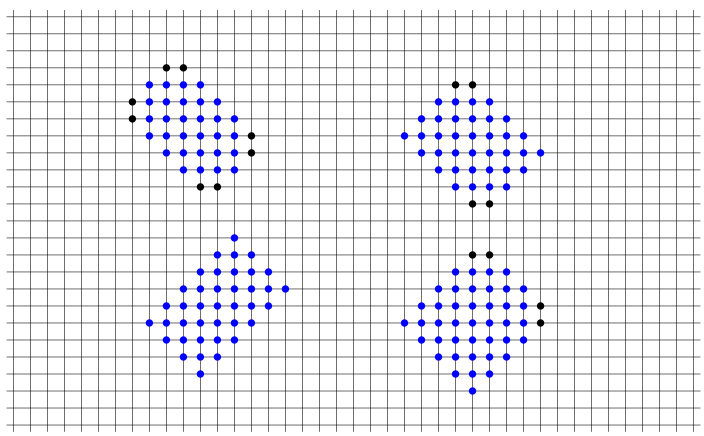

Figure 4: The four basic shapes of diagonal convex sets in $\mathbb{Z}^2$.

Case 1. ∂X has four axis-aligned components.

Consider two opposite diagonals. They both have the same length k, and are adjacent to $k+1$ points of the neighborhood. Remove all points from one of these diagonals, and add them along the other diagonal (from bottom to top, see Figure 5). This results in a set X' with $n(X') = n(X)$ that is not diagonal convex (as it contains an axis-aligned component of size 3), which is a contradiction to the assumption that X is optimal and has the cardinality of a ball.

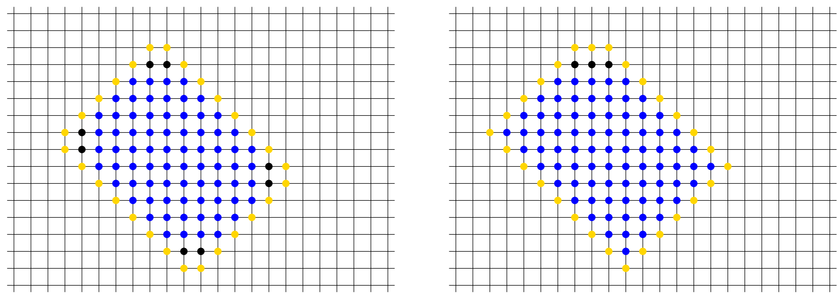

Figure 5: Case 1: Arranging all points from a diagonal along the opposite diagonal gives the same neighborhood-size and a non-diagonal of size 3.

Case 2. ∂X has two parallel non-diagonals.

Consider the levels in the coordinate direction k in which the axis-aligned components both constitute a level. Then every k-level has even cardinality, and thus $|X|$ is even. By Observation 1.10, this is a contradiction to X having the cardinality of a ball.

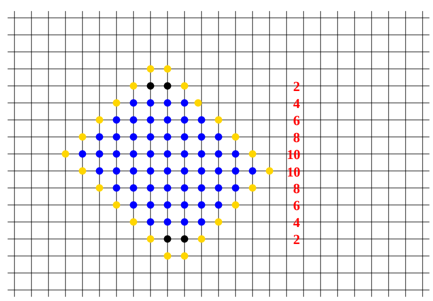

Figure 6: Case 2: Even parity.

Case 3. ∂X has no axis-aligned components.

Let k and l the lengths of the diagonals (each pair of opposite diagonals has the same length).

Case 3.1 If $k = l$, then X is a ball.

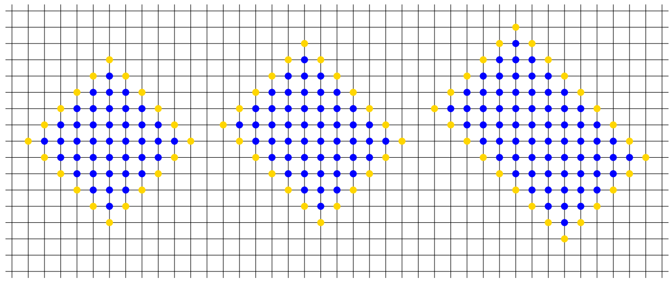

Figure 7: Case 3: $k = l$ (left), $k = l + 1$ (middle), and $k > l + 1$ (right).

Case 3.2 If $k > l$, then remove all points from one of the shorter diagonals and add them along one of the (at least before) longer diagonals.
The resulting set X' has $n(X') = n(X)$. Further, for $k = l + 1$ it is a diagonal convex set with two parallel axis-aligned components of size 2, while for $k > l + 1$, it is not diagonal convex. In either case, we get a contradiction to the assumption that X is optimal and has the cardinality of a ball.

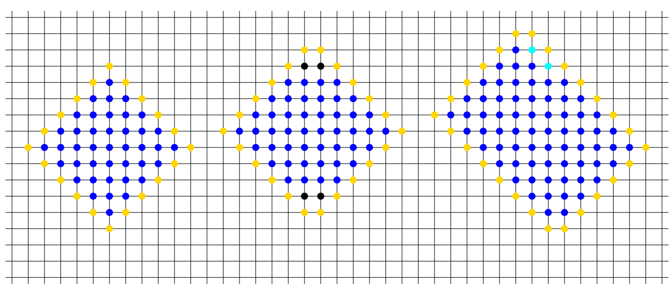

Figure 8: Case 3: changed point sets. Turquoise points are additional ones.

Case 4. ∂X has two orthogonal axis-aligned components.
Consider the diagonal that connects the two axis-aligned components, and say it has length k. Then the opposite diagonal has length $k + 1$ and the two other diagonals both have length l.

Case 4.1. If $k = l$, then this is exactly a ball minus one diagonal. These are optimal sets, but obviously cannot have the cardinality of a ball.

Case 4.2. If $k = l - 1$, then this is exactly a ball plus one diagonal. Again,

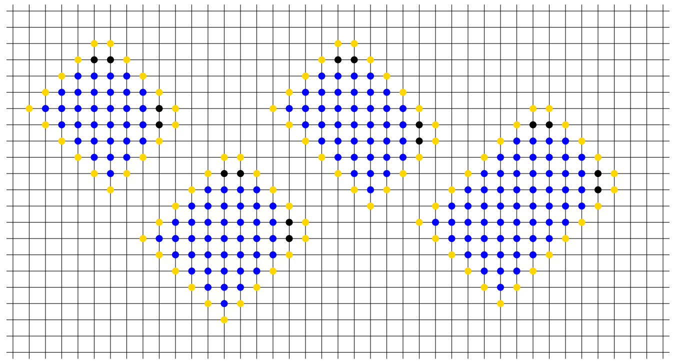

Figure 9: Case 4: left to right: $k = l$, $k = l - 1$, $k < l - 1$, and $k > l$.

these are optimal sets, but cannot have the cardinality of a ball.

Case 4.3. If $k < l - 1$, then we remove the diagonal of length k and add it again along one of the diagonals of length l.
In the resulting point set there is at least one point missing in this new diagonal, and adding it does not increase the size of the neighborhood. By Proposition 1.6 this contradicts our assumptions.

Case 4.4. If $k > l$, then we remove a diagonal of length l and add it along a diagonal of length k. Again, there is at least one point missing in the new diagonal and adding it does not increase the size of the neighborhood.

□

2.2 The General Case: d Dimensions

In [WW77a] the authors define a transformation of one set to another of the same cardinality: The *k-normalization* $N_k(X)$ of a set $X \subset \mathbb{Z}^d$ is obtained by:

1. replacing all (nonempty) k-levels of X by $(d-1)$-dimensional standard spheres of the same cardinality, and
2. changing the order of the levels such that the largest one is the 0^{th} k-level and the remaining ones are arranged half of them above and half of them below the 0^{th} k-level in a way that $|L_{k,i}| \geq |L_{k,i+1}|$ for $i \geq 0$, and $|L_{k,j}| \geq |L_{k,j-1}|$ for $j \leq 0$.

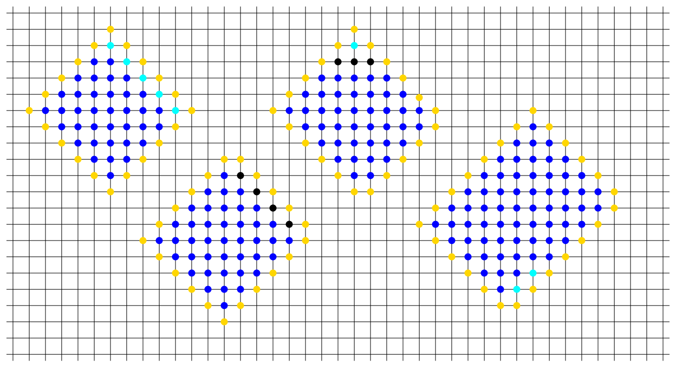

Figure 10: Case 4: changed point sets. Turquoise points are additional ones.

To show that standard spheres are optimal sets, they take an arbitrary set and repeatedly apply 1-normalization and d-normalization to it. They prove that this series of transformations terminates with a standard sphere after a finite number of steps, and that it does not increase the size of the neighborhood.

We are going to use their proof to show the uniqueness of balls as optimal sets.

Theorem 2.3. *Balls B_r^d are unique optimal sets of their cardinality.*

Proof. The proof is by induction on the dimension d, the induction base $d = 2$ being Theorem 2.2 from the last section.

For the induction step consider some optimal set $X \subset \mathbb{Z}^d$ with $|X| = |B_r^d|$ for some r. If we repeatedly apply 1-normalization and d-normalization to X as in [WW77a], then X is transformed to B_r^d. Consider the set Y that occurs in this transformation process exactly before the last normalization step. We assume the last step is in direction 1 to reduce the number of variables in the following.

Then there is a one-to-one correspondence between the 1-levels of Y and the 1-levels of B_r^d, such that all cardinalities match. Note that the levels of B_r^d are $(d-1)$-dimensional balls. Define

$$\begin{aligned} l_{\min} &:= \min\{l \in \mathbb{Z} : L_{1,l}(Y) \neq \emptyset\}, \text{ and} \\ l_{\max} &:= \max\{l \in \mathbb{Z} : L_{1,l}(Y) \neq \emptyset\}. \end{aligned}$$

Then as a lower bound for the size of the neighborhood of Y we have

$$n(Y) \geq |L_{1,l_{\max}}(Y)| + |L_{1,l_{\min}}(Y)| + \sum_{l=l_{\min}}^{l_{\max}} n(L_{1,l}^{d-1}(Y))$$

Now as every level $L_{1,l}(Y)$ has the cardinality of a ball $B_{r_l}^{d-1}$ for some r_l, it follows by the induction hypothesis that they all really have to be these balls (as otherwise $n(L_{1,l}^{d-1}(Y)) > n(B_{r_l}^{d-1})$ and thus $n(X) \geq n(Y) > n(B_r^d)$, which would be a contradiction to the assumption that X is optimal).

Finally consider the order of the 1-levels in Y. If they are not ordered in the same way as for the according ball B_r^d, then there exist two adjacent levels $L_{1,i}(Y) = B_{r_i}^{d-1}$ and $L_{1,j}(Y) = B_{r_j}^{d-1}$ such that $r_j < r_i - 1$. But then

$$I = \operatorname{int}\left(L_{1,i}^{d-1}(Y)\right) \setminus L_{1,j}^{d-1}(Y) \neq \emptyset$$

and thus by Observation 1.8 we can add points to Y without increasing the size of the neighborhood $n(Y)$. This is, once again, a contradiction to Proposition 1.6 since Y was assumed to be optimal and of the same cardinality as some B_r^d.

Thus $X = Y = B_r^d$ which completes the proof. □

3 Necessary Conditions for Optimal Sets

3.1 Back to Dimension 2

In Proposition 1.4 we showed that connectedness is a necessary condition for a set $X \subset \mathbb{Z}^2$ to be optimal. As a first step towards further conditions we consider the shapes of X and $Y = X \cup N(X)$ for diagonal convex sets, and the relation beween $n(X)$ and $n(Y)$.

Proposition 3.1. *If a set $X \subset \mathbb{Z}^2$ is diagonal convex, then $Y = X \cup N(X)$ is again diagonal convex and $n(Y) = n(X) + 4$.*

Proof. The lengths of the axis-aligned parts of $N(Y)$ and $N(X)$ are identical, while the lengths of the diagonal parts of $N(Y)$ are the lengths of the diagonal parts of $N(X)$ plus one. □

This proposition tells us that diagonal convex sets have a shape that is "stable" under the operation of adding the neighborhood, and that the size of the neighborhood behaves in a nice way. But there is a far larger class of sets that behaves in essentially the same way:

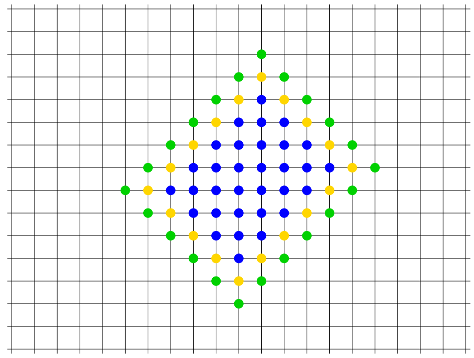

Figure 11: A diagonal convex set X, together with $N(X)$ and $N(X \cup N(X))$.

Consider some set $X \subset \mathbb{Z}^2$, such that the neighborhood $N(X)$ is a simple cycle. By this we mean that for any two points in $N(X)$ there are exactly two disjoint paths between them (where paths are defined in the same way as previously for the definition of connectedness).

Note that if we regard the cycle $N(X)$ as a (not necessarily convex) polygon, the grid points in the interior of the polygon are all points of X. Observe that the inside and outside angles at any $v \in N(X)$ are at least 90 degrees, and at least 135 degrees if one of the polygon edges adjacent to v is in a coordinate direction. An interior angle smaller than this would give $d(v, X) > 1$, and an exterior angle would lead to a subcycle of length 3 in $N(X)$.

We will proceed through the cycle $N(X)$ in the counter-clockwise direction and consider the occuring direction changes with respect to the oriented coordinate directions (e_i, σ) with $i \in \{1, 2\}$ and $\sigma \in \{+, -\}$:

- Choose as starting (and ending) point the topmost point of the tangent $t_{++} : x_1 + x_2 = c > 0$.
- Choose as starting (and ending) direction $(e_1, -)$.
- When proceeding through the cycle, remember the current coordinate direction (e_i, σ), and the number of occured direction changes. In every step x_k to x_{k+1}
  1. keep the direction (e_i, σ) if $\sigma = \mathrm{sign}(x_{k+1} - x_k)_i$,
  2. otherwise change the direction to (e_j, σ') with $j \neq i$ and $\sigma' = \mathrm{sign}(x_{k+1} - x_k)_j$.

- as the starting point is reached with a direction different from $(e_1, -)$, the turn to this direction (according to the turning rule above) counts an as occuring turn.

Every considered (and counted) direction change is a (left or right) turn by 90 degrees. As we require the starting direction to be identical to the ending direction, we turn 360 degrees in total. Thus we count an even number $k \geq 4$ of direction changes (each of the oriented directions at least once).

Also, by the above observations about the occuring angles, we notice that for every turn there is a diagonal part of $N(X)$ of size at least 2, that could be seen as being in either of the two directions before and after the turn. We will call such a diagonal part of $N(X)$ a *connecting diagonal*.

We call a set X *close-to-convex* if $N(X)$ is a cycle and if in the process described above there are four changes in direction (i.e. every oriented coordinate direction appears exactly once).

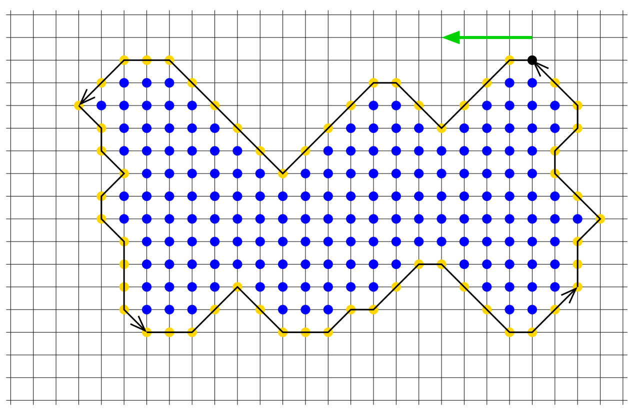

Figure 12: Proceeding through the neighborhood of a close-to-convex set. Direction changes occur at the black arrows.

Observation 3.2. *If a set $X \subset \mathbb{Z}^2$ is close-to-convex, then $Y = X \cup N(X)$ is again close-to-convex, and $n(Y) = n(X) + 4$. Moreover, Y has the same shape as X despite for the four connecting diagonals which each get longer by one. See Figure 13 for an example.*

Note that each connecting diagonal is identical to the intersection of $X \cup N(X)$ with one of its diagonal tangents.

Observation 3.3. *The standard spheres that are presented in [WW77a] are optimal sets that are close-to-convex, and for any standard sphere S, $S \cup N(S)$*

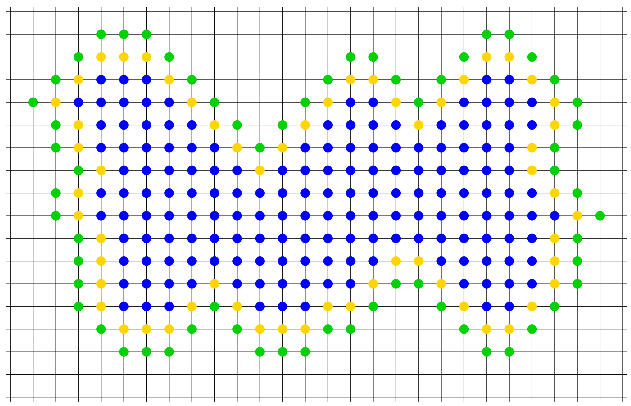

Figure 13: A close-to-convex set and two layers of neighborhood.

is again a standard sphere (and thus optimal). This implies that for any optimal set $X \subset \mathbb{Z}^2$ the inequality $n(Y) \geq n(X)+4$ holds for $Y = X \cup N(X)$.

Let us go back to general connected sets $X \subset \mathbb{Z}^2$. We denote the cycle $C(X) \subseteq N(X)$ for which X lies in the interior of the polygon defined by this cycle as the *cycle that surrounds* X. Further we call the (finite) set $\mathrm{cl}(X)$ of grid points enclosed by $C(X)$ the *closure* of X.

An ordered subset $\{x_1, x_2, \ldots, x_n\} \subset \mathbb{Z}^2$ is a *lattice path* if the elements are distinct and $d(x_i, x_{i+1}) = 1$ for all i. For emphasis, we note that every lattice path is a path, as defined in section 1, but not conversely.
A *hole* in X is a subset $H \subset \mathrm{cl}(X) \setminus X$ such that

1. H is connected, and
2. for every $h \in H$, every lattice path from h to $C(X)$ contains some element of X.

Proposition 3.4. *If $X \subset \mathbb{Z}^2$ is connected, then for $Y = X \cup N(X)$ we have $n(Y) \leq n(X) + 4$.*

Proof. For any close-to-convex set this is obviously true.

Now if X is not close-to-convex, then either its neighborhood $N(X)$ does not form a cycle or we will get more than four turns when proceeding through the cycle like described above.

In the latter case assume we have counted $2k + 4$ turns. Then we have counted exactly $k + 4$ left turns and k right turns. For every left turn the connecting diagonal gets longer by (at most) one, while for each right turn the connecting diagonal gets shorter by (at least) one (see Figure 14). All parts in between are just translated by 1 and are thus (at most) as long as they were.

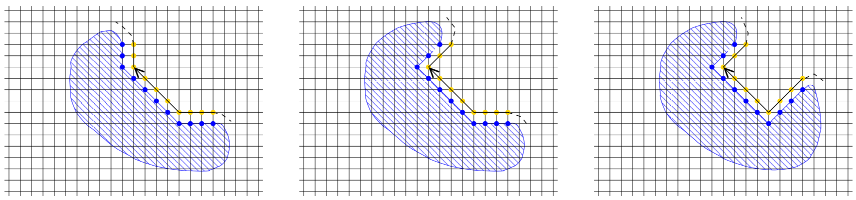

Figure 14: All possible right turns (up to symmetry).

Note that $N(Y)$ need not be a cycle, and that the *at most* and *at least* statements from above stem from the fact that points of $N(Y)$ might be created duplicately from more than one part of $N(X)$, see Figures 15 and 16 for examples.

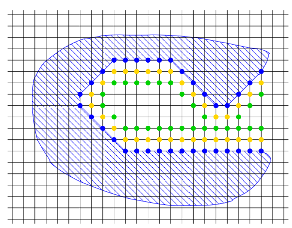

Figure 15: $N(Y)$ does not form a cycle: creating a hole.

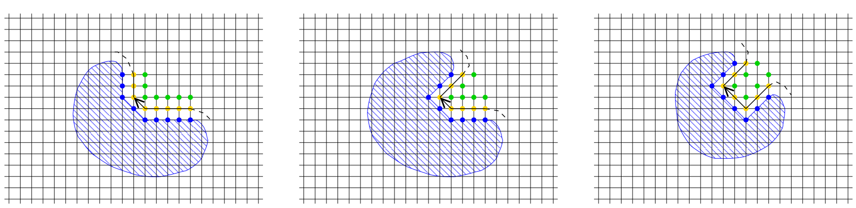

Figure 16: $N(Y)$ does not form a cycle: Right turns with short connecting diagonal.

Now what is left is the situation when $N(X)$ is not a cycle. Here, consider the cycle $C(X) \subset N(X)$ that surrounds X.

For the part of the neighborhood of $N(Y)$ that lies outside of $C(X)$ the same arguments as above can be applied.

For the inside part observe that every component of $N(Y)$ corresponds to a hole of $X \cup N(X)$, see Figure 17, and thus the size of the neighborhood inside $C(X)$ is decreasing.

□

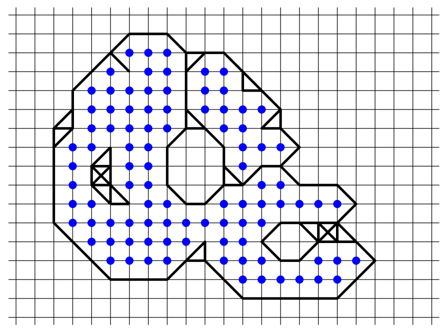

Figure 17: A connected set X visualizing some possible shapes. The lines indicate all paths in $N(X)$.

Proposition 3.5. *If $X \subset \mathbb{Z}^2$ is not close-to-convex, then X cannot be optimal.*

Proof. Assume that X is connected but not close-to-convex, and consider again the cycle $C(X) \subset N(X)$ that surrounds X.

If $C(X) \subsetneq N(X)$, then the set Y containing all points inside this cycle $C(X)$ has $|Y| > |X|$ and $n(Y) < n(X)$. Thus X cannot be optimal.

If $C(X) = N(X)$ then consider $Z = X \cup N(X)$. From the proof of Proposition 3.4 we know that if $N(Z)$ is a cycle again, then the length of every right turn connecting diagonal is shorter than the according one in $N(X)$. Thus, after finitely many steps of adding the neighborhood we obtain a set that is not optimal (as its neighborhood is not a cycle). But as Proposition 3.4 holds for every step of adding the neighborhood, X cannot be optimal either. □

Next we show that adding the neighborhood to a set does not carry us away from optimality. More precisely, we have:

Proposition 3.6. *Consider any connected set $X \subset \mathbb{Z}^2$ and its union with its neighborhood $X' = X \cup N(X)$. Let Y and Y' be the standard spheres of* *cardinality $|Y| = |X|$ and $|Y'| = |X'|$, respectively. If $n(X) = n(Y) + k$ for some $k \geq 0$, then $n(X') \leq n(Y') + k$.*

Proof. $|Y'| = |X'| = |X| + n(X) = |Y| + n(Y) + k \geq |Y \cup n(Y)|$. Thus from Observation 1.5, it follows that

$$n(Y') \geq n(Y \cup n(Y)) = n(Y) + 4 = n(X) - k + 4 = n(X') - k.$$

□

Now we can state the answer to Problem 1.1, i.e., we have shown that, at least in dimension 2, the set of optimal sets is closed under the operation of adding the neighborhood. We record this solution in the following corollary.

Corollary 3.7. *If a set $X \in \mathbb{Z}^2$ is optimal, then $n(X \cup N(X)) = n(X) + 4$ and $X \cup N(X)$ is also optimal.*

References

[Bez89] S L Bezrukov, *On the construction of solutions of a discrete isoperimetric problem in hamming space*, Mathematics of the USSR-Sbornik **63** (1989), no. 1, 81–96.

[Bez94] S.L. Bezrukov, *Isoperimetric problems in discrete spaces*, Extremal Problems for Finite Sets **3** (1994), 59–91.

[BS02] Sergei L. Bezrukov and Oriol Serra, *A local-global principle for vertex-isoperimetric problems*, Discrete Math. **257** (2002), no. 2-3, 285–309.

[CL69] G.F. Clements and B. Lindström, *A generalization of a combinatorial theorem of Macaulay*, J. Combinatorial Theory Ser. A **7** (1969), 230–238.

[Har66] L.H. Harper, *Optimal numberings and isoperimetric problems on graphs*, J. Combin. Theory **1** (1966), 385–393.

[Mac27] F. S. Macaulay, *Some Properties of Enumeration in the Theory of Modular Systems*, Proc. London Math. Soc. **s2-26** (1927), no. 1, 531–555.

[WW77a] Da-Lun Wang and Ping Wang, *Discrete isoperimetric problems*, SIAM Journal on Applied Mathematics **32** (1977), no. 4, 860–870.

[WW77b] ———, *Extremal configurations on a discrete torus and a generalization of the generalized macaulay theorem*, SIAM Journal on Applied Mathematics **33** (1977), no. 1, 55–59.