Seminar | Mathematical Optimization and Quantum Information Theory

The quantum chromatic number

One can express finding the chromatic number $\chi(G)$ of a graph G as an integer optimization problem with $0/1$ variables. When one replaces the $0/1$ variables by positive semidefinite matrices (of arbitrary size), then one gets the quantum chromatic number $\chi_q(G)$ [1]. The quantum chromatic number is used to understand nonlocal games where the players are allowed to use entanglement.

The determination of both chromatic numbers is NP-hard [2]. So finding approximations through convex optimization hierarchies is of interest [3], [4]

Violation of Bell inequalities

Bell's theorem shows that classical ideas about locality and determinism impose constraints on how strong quantum correlations can be, known as a Bell inequalities [5]. The violation of these inequalities have profound consequences for the foundations of quantum mechanics, as well as practical applications (e.g. randomness amplification and quantum cryptography). Crucial for these applications is to know by how much the constraints can be violated. To answer this question one needs to solve a hierarchy of semidefinite programs, which give progressively tighter upper bounds to the violation [6].

Quantum steering

Steering is a non-classical phenomenon that formalizes what Einstein called "spooky action at a distance": If certain quantum systems are shared by two distant parties, a measurement performed by one party on their share of the system can instantly affect the quantum state of the other party. Steering can be rigorously characterized through semidefinite programming which provides efficient numerical methods to address a number of problems such as detection and quantification of this effect [7].

Entanglement detection

Entanglement is one of the most fascinating features of quantum mechanics and at the heart of interesting phenomena (e.g. Bell non-locality and steering), as well as technologies (e.g. quantum key distribution). The task of unambiguously detecting entanglement, or the absence thereof, in a quantum state is both important and surprisingly difficult. However, a hierarchy of semidefinite programs allows for answering this question [8]. This hierarchy is complete, in the sense that any entangled state is guaranteed to be detected at some finite point in the hierarchy.

Optimal strategies for distinguishing quantum states

The optimal probability of correctly distinguishing two classical probability distributions based on a single observation (e.g. distinguish a fair coin from a completely biased one based on a single coin toss) is proportional to the total variational distance of these distributions.

An analogous result is true for the task of distinguishing two quantum systems based on a single (arbitrary) measurement: the optimal success probability is proportional to the nuclear norm distance of the density operators that characterize these systems. The measurement achieving this optimal success probability is the solution of a semidefinite program and depends on the density operators [9]. This result can be generalized to the task of distinguishing more than two different quantum states [10].

Optimal strategies for distinguishing quantum channels

Quantum channel model the time evolution of quantum systems. The task of correctly distinguishing two channels based on a single evaluation is a natural generalization of the state discrimination task above. It turns out that the optimal success probability is again proportional to a norm distance between the two channels [11]. Unlike the total variation distance (ℓ_1-norm) and the nuclear norm, the norm in question – a norm of complete boundedness – in general does not admit a closed-form expression. However, it can be evaluated by means of a semidefinite program which renders its evaluation computationally tractable [12].

Quantum state estimation via compressed sensing

Compressed sensing is the art of solving ill-posed inverse problems via convex optimization. This novel approach has many applications including quantum state estimation: the task of recovering the state of a quantum system from measurement data. Quantum systems often exhibit additional structure: their states correspond to (approximately) low-rank matrices. Compressed sensing allows for exploiting this additional feature and considerably reduces the number of measurements required for quantum state estimation [13]. Mathematical proofs using convex optimization exist for different types of quantum measurements, see e.g. [14, Sec. 8] for a particularly clear argument that is applicable to a certain class of measurements.

Quantum speed-ups for semidefinite programming

Quantum computers harness the unique features of quantum mechanics and offer considerable speed-ups over classical computations. This also includes semidefinite programming: a recently proposed quantum algorithm [15] gives an unconditional square-root speed-up over any classical method for solving semidefinite programs.

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