Universität zu Köln
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Convex Optimization

Winter Term 2018/19

— Exercise Sheet 1 —

Exercise 1.1. Let $X \subseteq \mathbb{R}^n$. Prove: For every $c \in \operatorname{cone}(X)$ there are linearly independent vectors $x_1, \ldots, x_k \in X$ such that $c \in \operatorname{cone}\{x_1, \ldots, x_k\}$.

Exercise 1.2. Let $\mathcal{L}^{n+1}$ be the Lorentz cone

$$\mathcal{L}^{n+1} = \{(x,t) \in \mathbb{R}^{n+1} : \|x\| \leq t\}.$$

Show that $(\mathcal{L}^{n+1})^* = \mathcal{L}^{n+1}$.

Exercise 1.3. (Hand-in) Show: The set of non-negative polynomials of degree at most $2d$

$$\left\{(a_0, a_1, \ldots, a_{2d}) \in \mathbb{R}^{2d+1} : a_0 + a_1 x + \cdots + a_{2d} x^{2d} \geq 0 \text{ for all } x \in \mathbb{R}\right\}$$

is a proper convex cone for any $d \geq 0$.

Exercise 1.4. (Hand-in) Recall that a complex square matrix $A \in \mathbb{C}^{n \times n}$ is *Hermitian* (or self-adjoint) if $A = A^*$, i.e., $A_{ij} = \overline{A}_{ji}$ for all entries of A. The Hermitian matrices form a real vector space (of dimension n^2), with the Frobenius inner product $\langle A, B\rangle = \sum_{ij} \overline{A}_{ij} B_{ij} = Tr(A^* B)$.

A Hermitian matrix $M \in \mathbb{C}^{n \times n}$ is *positive semidefinite* if $z^* M z \geq 0$ for all $z \in \mathbb{C}^n$, or equivalently if all eigenvalues of M are non-negative.

Consider the set $\mathcal{H}^n_+$ of positive semidefinite complex $n \times n$-matrices as a subset of the Hermitian matrices . Show:

(a) $\mathcal{H}^n_+$ is a self-dual proper convex cone for any $n \geq 1$.

(b) $\mathcal{H}^2_+$ is isometric to $\mathcal{L}^{3+1}$.

Hand-in: Until Wednesday October 17, 12:00 (noon).
Exercises 1.3 and 1.4 to be submitted to the “Convex optimization” mailbox in room 3.01 (Studierendenarbeitsraum) of the Mathematical Institute.