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Convex Optimization

Winter Term 2018/19

— Exercise Sheet 2 —

Exercise 2.1. Let $K \subseteq \mathbb{R}^n$ be a proper convex cone. Show that x lies in the topological interior of K if and only if $y^{\mathsf{T}} x > 0$ holds for all $y \in K^* \setminus \{0\}$.

Exercise 2.2. Given $x_1, \ldots, x_n \in \mathbb{R}$, consider the following matrix

$$X = \begin{pmatrix} 1 & x_1 & \ldots & x_n \\ x_1 & x_1 & 0 & 0 \\ \vdots & 0 & \ddots & 0 \\ x_n & 0 & 0 & x_n \end{pmatrix}.$$

That is, $X \in \mathcal{S}^{n+1}$ is the matrix indexed by $\{0, 1, \ldots, n\}$, with entries $X_{00} = 1$, $X_{0i} = X_{i0} = X_{ii} = x_i$ for $i \in \{1, \ldots, n\}$, and all other entries are equal to 0.

Show that X is positive semidefinite if and only if $x_i \geq 0$ for all $i \in \{1, \ldots, n\}$ and $\sum_{i=1}^n x_i \leq 1$.

Exercise 2.3. (Hand-in) Let $v_1, \ldots, v_k \in \mathbb{R}^n$ be unit vectors. The lines $\{\lambda v_i : \lambda \in \mathbb{R}\}$ are called *equiangular* if $v_i^{\mathsf{T}} v_j = \pm\alpha$ for all $i, j \in [k]$ with $i \neq j$ and some fixed $\alpha \in \mathbb{R}$.
Show that the matrix

$$\frac{1}{r - \frac{1}{2}} \left(r I_{rt} - J_r \otimes I_t + \frac{1}{2} J_{rt} \right)$$

determines a set of rt equiangular lines in dimension $(r-1)t + 1$. Here I denotes the identity matrix and J the all-ones-matrix where each entry equals 1.

Exercise 2.4. (Hand-in)

(a) Let $A = (A_{ij}) \in \mathcal{S}^n$ with $A_{ij} = \frac{1}{i+j}$ for all $1 \leq i, j \leq n$. Show: A ist positive semidefinite.

Hint: For $x \in \mathbb{R}^n$ consider the integral

$$\int_0^\infty \left(\sum_{k=1}^n x_k e^{-kt} \right)^2 dt.$$

(b) Show: If $X, Y \in \mathcal{S}^n_+$, then $X \circ Y \in \mathcal{S}^n_+$. What happens when X, Y are both positive definite?

(c) Let $X \in \mathcal{S}^n$ with $X_{ij} \in [-1, 1]$ for $i, j \in [n]$. Show: If X is positive semidefinite, then $\arcsin(X)$ is also positive semidefinite, where $\arcsin(X) \in \mathcal{S}^n$ is componentwise defined by

$$(\arcsin(X))_{ij} = \arcsin(X_{ij}) = \sum_{k=0}^{\infty} \frac{(2k-1)!!}{(2k)!!} \frac{X_{ij}^{2k+1}}{2k+1}.$$

The double factorial $m!!$ is equal to $m \cdot (m-2) \cdot (m-4) \cdots 4 \cdot 2$ if m is even and equal to $m \cdot (m-2) \cdot (m-4) \cdots 3 \cdot 1$ if m is odd.

Hand-in: Until Wednesday October 24, 12:00 (noon).
Exercises 2.3 and 2.4 to be submitted to the "Convex optimization" mailbox in room 3.01 (Studierendenarbeitsraum) of the Mathematical Institute.