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Convex Optimization

Winter Term 2018/19

— Exercise Sheet 3 —

Exercise 3.1. Consider the semidefinite optimization problem

$$\begin{aligned} &\text{maximize } X_{11} + X_{12} \\ &\text{subject to } 2X_{11} + X_{22} = 1, \\ &\qquad \begin{pmatrix} X_{11} & X_{12} \\ X_{12} & X_{22} \end{pmatrix} \succeq 0. \end{aligned}$$

(a) Draw the set of feasible solutions.

(b) Determine the optimal solutions.

(c) Determine the dual program and its optimal solutions.

Exercise 3.2. Markowitz Portfolio Optimization (Nobel-prize in Economics 1990):
We consider a portfolio problem with n assets or stocks $a_1, \dots, a_n$, which can be held over a fixed period of time, and a fixed budget B.
Consider the relative price change p_i of each asset, i.e., its change in price over the period divided by its price at the beginning, as a random variable. Suppose we know each mean $c_i = \mathbb{E}[p_i]$ and the covariance $C \in \mathcal{S}^n$ with $C_{ij} = \mathbb{E}[(p_i - c_i)(p_j - c_j)]$.
A vector $x \in \mathbb{R}^n_+$ with $\sum_{i=1}^n x_i = B$ is called a portfolio, where we invest x_i in a_i, for every $i \in [n]$.
The goal is to find a portfolio that maximizes the expected return $c^\mathsf{T} x$ with control over the risk, which is associated with the variance $\operatorname{Var}(p^\mathsf{T} x) = x^\mathsf{T} C x$.

(a) Formulate this problem as a conic program, e.g., by maximizing $c^\mathsf{T} x - \gamma\sqrt{\operatorname{Var}(p^\mathsf{T} x)}$, with a fixed risk-aversion parameter $\gamma \geq 0$, and using a cone of the form $\mathcal{L}^{k+1} \times \mathbb{R}^m_+$.

(b) Determine the dual program.

Exercise 3.3. (Hand-in) Show:

(a) $\mathcal{CP}_n = \operatorname{cone}\{xx^\mathsf{T} : x \in \mathbb{R}^n_+\}$ is a proper convex cone.

(b) $(\mathcal{CP}_n)^* = \mathcal{COP}_n$, where $\mathcal{COP}_n = \{X \in \mathcal{S}^n : x^\mathsf{T} X x \geq 0 \text{ for all } x \in \mathbb{R}^n_+\}$ is the cone of copositive matrices.

Exercise 3.4. (Hand-in) Let $G = (V, E)$ be a graph. The *independence number* $\alpha(G)$ of the graph is the maximal cardinality of a set $S \subseteq V$ such that $\{i, j\} \notin E$ for any $i, j \in S$. Show that $\alpha(G)$ equals the optimal value of the following conic program:

$$\begin{aligned} &\text{maximize } \langle J, A \rangle \\ &\text{subject to } A \in \mathcal{CP}_n, \\ &\qquad \langle I, A \rangle = 1, \\ &\qquad A_{ij} = 0 \text{ if } \{i, j\} \in E. \end{aligned}$$

Hand-in: Until Wednesday October 31, 12:00 (noon).
Exercises 3.3 and 3.4 to be submitted to the "Convex optimization" mailbox in room 3.01 (Studierendenarbeitsraum) of the Mathematical Institute.