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Convex Optimization

Winter Term 2018/19

— Exercise Sheet 4 —

Exercise 4.1. Let $K \subseteq \mathbb{R}^n$ be a proper convex cone, and let $A \in \mathbb{R}^{m\times n}$ and $c \in \mathbb{R}^n$. Show: Exactly one of the following two alternatives holds:

(i) There is an $x \in K$ such that $Ax = 0$ and $c^{\mathsf{T}}x > 0$.

(ii) The system $A^{\mathsf{T}}y - c \in K^*$ is weakly feasible.

Exercise 4.2. Consider the following semidefinite program, which involves inequalities:

$$p^* = \sup\{\langle C, X\rangle : \langle A_j, X\rangle \leq b_j \text{ for } j = 1, \dots, m,\ X \succeq 0\}.$$

(a) Bring this program in standard primal form and write its dual conic program.

(b) Read the proof of Theorem 4.4 and discuss why the following is true: Suppose $p^* < \infty$ and there is a matrix $X \in \mathcal{S}^n_{++}$ with $\langle A_j, X\rangle \leq b_j$ for $j = 1, \dots, m$. Then the dual program attains the infimum and the duality gap is zero.

Exercise 4.3. (Hand-in) Let $K \subseteq \mathbb{R}^n$ be a proper convex cone, and let $A \in \mathbb{R}^{m\times n}$ and $c \in \mathbb{R}^n$. Show: Exactly one of the following two alternatives holds:

(i) There is an $x \in K \setminus \{0\}$ such that $Ax = 0$ and $c^{\mathsf{T}}x \geq 0$.

(ii) There is a $y \in \mathbb{R}^m$ such that $A^{\mathsf{T}}y - c \in \operatorname{int} K^*$.

Exercise 4.4. (Hand-in) What is the exact value of the minimal maximal eigenvalue of the following matrix $X \in \mathcal{S}^5$

$$\begin{pmatrix} 1 & X_{12} & 1 & X_{14} & X_{15} \\ X_{12} & 1 & 1 & X_{24} & X_{25} \\ 1 & 1 & 1 & X_{34} & 1 \\ X_{14} & X_{24} & X_{34} & 1 & 1 \\ X_{15} & X_{25} & 1 & 1 & 1 \end{pmatrix} ?$$

Hint: You can use a numerical SDP solver to “guess” the solution.

Hand-in: Until Wednesday November 7, 12:00 (noon).
Exercises 4.3 and 4.4 to be submitted to the “Convex optimization” mailbox in room 3.01 (Studierendenarbeitsraum) of the Mathematical Institute.

Online SDP Solver

NEOS solvers (search for: neos sdp csdp)

Example for an input in *sparse SDPA* format:

Input:	Explanation:
`2`	$m = 2$ (number of conditions)
`2`	The cone is $\mathcal{S}_+^{k_1} \times \mathcal{S}_+^{k_2}$ (two psd-cones)
`2 2`	$k_1 = 2$ and $k_2 = 2$
`10.0 20.0`	$b = (10, 20)^\mathsf{T}$

`0 1 1 1 1.0`
`0 1 2 2 2.0`
`0 2 1 1 3.0`
`0 2 2 2 4.0`
	$C = \begin{bmatrix} 1 & & & \\ & 2 & & \\ & & 3 & \\ & & & 4 \end{bmatrix}$

`1 1 1 1 1.0` `1 1 2 2 1.0`	$A_1 = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & & \\ & & & \end{bmatrix}$

`2 1 2 2 1.0`
`2 2 1 1 5.0`
`2 2 1 2 2.0`
`2 2 2 2 6.0`
	$A_2 = \begin{bmatrix} & & & \\ & 1 & & \\ & & 5 & 2 \\ & & 2 & 6 \end{bmatrix}$

(The empty lines in the input are for readability only)