Convex Optimization (WS 2018/19)

Chapter I Introduction

Mathematical program ($\hat{=}$ mathematical optimization problem)

given: functions $f_0, f_1, \dots, f_m : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$

constants $b_1, \dots, b_m \in \mathbb{R}$

find:
$$\begin{aligned} \inf\ & f_0(x) \\ \text{s.t.}\ & x \in \mathbb{R}^n \\ & f_j(x) \le b_j, \quad j \in [m] = \{1, \dots, m\}. \end{aligned}$$

Vocabulary objective function f_0; constraint fun $f_1, \dots, f_m$, optimization variable x; feasible solution x w/ $f_j(x) \le b_j$ $\forall j \in [m]$; optimal solution x w/ x feasible and $f_0(x) \le f(x')$ $\forall x'$ feasible

Important examples

(i) Linear programs (LPs)

- all functions $f_0, f_1, \dots, f_m$ are linear

- LPs are quite well-understood in theory and practice. ($\rightsquigarrow$ OR)

(ii) <u>Nonlinear programs</u> (NLPs)

- functions $f_0, f_1, \dots, f_m$ can be nonlinear.

For example (From the SIAM 100-Digit Challenge):

Global minimum of

$$f_0(x,y) = e^{\sin(50x)} + \sin(60e^y) + \sin(70\sin x) + \sin(\sin(80y)) - \sin(10(x+y)) + (x^2+y^2)/4 \; ?$$

2720 critical points in $[-1,+1]^2$, optimal sol. $(-0{,}0244\ldots, 0{,}2106\ldots)$.

<u>But</u>: NLPs too general for developing useful mathematical theory.

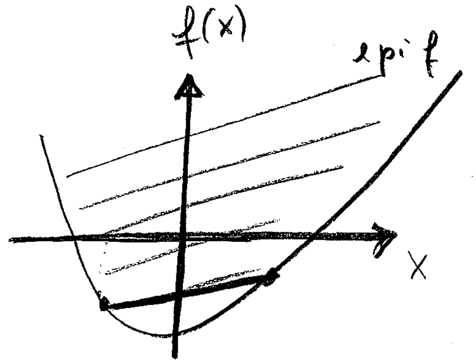
(iii) <u>Convex programs</u>

- all functions $f_0, \dots, f_m$ are convex.

Recall:

Def 1 Function $f: \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ is convex if its epigraph $\operatorname{epi} f = \{(x,\alpha) \in \mathbb{R}^{n+1} : f(x) \le \alpha\}$ is a convex set, or equivalently, if f satisfies Jensen's inequality

$$\forall t \in [0,1],\ x,y \in \mathbb{R}^n: \quad f(tx + (1-t)y) \le t f(x) + (1-t) f(y).$$



Fundamental properties of convex programs

(a) Set of feasible solutions is convex.

(b) Local minima are global minima

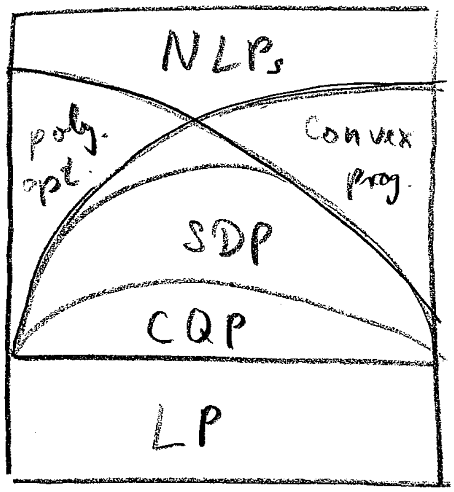
(c) Frequently, but not always, convex programs can be solved efficiently

(d) Convex programs have a nice duality theory.

(iv) <u>polynomial optimisation</u>

- $f_0, \dots, f_m$ are polynomials
- (almost) as difficult as NLPs
- can be transformed into (very high-dim.) convex programs (⇝ Seminar).

<u>Landscape of NLPs</u>



<u>central subject.</u>

$\boxed{\text{SDP}}$ = semidefinite programs

CQP = conic quadratic programs