Chapter III Eigenvalue optimization

Goal: Minimise largest eigenvalue of a symmetric matrix C which satisfies some LMIs.

Lemma 1 (Rayleigh-Ritz principle)
Let $C \in S^n$ be a symmmetric matrix and let $\lambda_{max}(C)$ [$\lambda_{min}(C)$] be the largest [the smallest] eigenvalue of C. Then,

$$\lambda_{max}(C) = \max_{x \in \mathbb{R}^n \setminus \{0\}} \frac{x^T C x}{x^T x} = \max_{x \in S^{n-1}} x^T C x,$$

$$\lambda_{min}(C) = \min_{x \in \mathbb{R}^n \setminus \{0\}} \frac{x^T C x}{x^T x} = \min_{x \in S^{n-1}} x^T C x,$$

where $S^{n-1} = \{x \in \mathbb{R}^n : x^T x = 1\}$ denotes the unit sphere.

Proof Use Lagrange multipliers:

$$f(x) = x^T C x, \quad g_1(x) = x^T x - 1.$$

Then: $\nabla f(x) = 2Cx$, $\nabla g_1(x) = 2x$.

If $x^* \in \mathbb{R}^n$ is a locally extreme point of f satisfying the constraint $g_1(x^*) = 0$, then there is a multiplier λ_1 with

$$2Cx^* = \lambda_1 2x^*.$$

Hence, x^* is an eigenvector of C with eigenvalue λ_1 and the statement of the lemma follows immediately. $\boxtimes$

<u>Theorem 2</u> The largest eigenvalue of $C \in S^n$ equals

$$\lambda_{max}(C) = \begin{array}{l} \max\ \langle C, X\rangle \\ \text{s.t. } X \in S^n_+ \\ \quad \langle I_n, X\rangle = 1 \end{array} = \begin{array}{l} \min\ y \\ \text{s.t. } yI_n - C \in S^n_+ \end{array}$$

For the smallest eigenvalue of C we have

$$\lambda_{min} = \begin{array}{l} \min\ \langle C, X\rangle \\ \text{s.t. } X \in S^n_+ \\ \quad \langle I_n, X\rangle = 1 \end{array} = \begin{array}{l} \max\ y \\ \text{s.t. } C - yI_n \in S^n_+ . \end{array}$$

Proof Consider the dual

$$d^* = \inf \{ y : y I_n - C \succeq 0 \}.$$

Then, $y^* = \lambda_{max}(C)$ is a feasible solution because for $x \in \mathbb{R}^n \setminus \{0\}$ we have

$$x^T (y^* I_n - C) x = y^* x^T x - x^T C x \geq 0$$

$$\iff y^* \geq \frac{x^T C x}{x^T x},$$

which is true by Rayleigh-Ritz. Hence,

$$d^* \leq y^* = \lambda_{max}(C).$$

Consider the primal

$$p^* = \sup \{ \langle C, X \rangle : X \in S_+^n, \langle I_n, X \rangle = 1 \}.$$

Define $X^* = x^*(x^*)^T$, where $x^* \in \mathbb{R}^n$ is eigenvector of C for the eigenvalue $\lambda_{max}(C)$ with $\|x^*\| = 1$. X^* is feasible and

$$\langle C, X^* \rangle = \langle C, x^*(x^*)^T \rangle = (x^*)^T C x^* = \lambda_{max}(C).$$

Hence, $p^* \geq \lambda_{max}(C)$.

Now apply weak duality

$$\lambda_{max}(C) \le p^* \le d^* \le \lambda_{max}(C).$$

$\boxtimes$

<u>Remark</u>. Can prove a similar statement for the sum of the k largest [smallest] eigenvalues of $C \in S^n$. $\to$ Fan's theorem.

- More on eigenvalue optimization: $\to$ Section "Convex spectral functions".