Chapter IV Sum of squares

Notation Polynomials in n variables

- monomials $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} = x^\alpha$ $\quad \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$
- degree of x^α: $\deg x^\alpha = |\alpha| = \sum_{i=1}^n \alpha_i$
- polynomial $p = \sum_\alpha p_\alpha x^\alpha$ finite linear combination of monomials x^α with coefficients $p_\alpha \in \mathbb{R}$.
- polynomial ring: $\underbrace{\mathbb{R}[x_1, \dots, x_n]}_{\mathbb{R}[x]} = \{p : p = \sum p_\alpha x^\alpha \text{ poly.}\}$
- $\deg p = \sup\{|\alpha| : p_\alpha \neq 0\}$ degree of poly. p.
- $\mathbb{R}[x_1, \dots, x_n]_d = \mathbb{R}[x]_d = \{p \in \mathbb{R}[x] : \deg p \le d\}$
- fact: $\dim \mathbb{R}[x]_d = \binom{n+d}{d}$ and the monomials x^α, $|\alpha| \le d$, form a basis of $\mathbb{R}[x]_d$.

Def. 1 (a) $p \in \mathbb{R}[x]$ is a sum of squares (SOS) if $\exists\, q_1, \dots, q_m \in \mathbb{R}[x]$: $p = q_1^2 + \cdots + q_m^2$.

(b) $\Sigma_{n,d} = \{ p \in \mathbb{R}[x]_d : p \text{ is SOS} \}$

is called the <u>SOS cone</u>.

(c) $\mathcal{P}_{n,d} = \{ p \in \mathbb{R}[x]_d : p(x) \geq 0 \text{ for all } x \in \mathbb{R}^n \}$

cone of <u>nonnegative polynomials</u>. ($\to$ Ex 1.3)

<u>Obviously</u>: $\Sigma_{n,2d+1} = \Sigma_{n,2d}$; $\mathcal{P}_{n,2d+1} = \mathcal{P}_{n,2d}$;

$\Sigma_{n,d} \subseteq \mathcal{P}_{n,d}$.

<u>Theorem 2</u> "The Gram matrix method"

$$\Sigma_{n,2d} = \{ p \in \mathbb{R}[x]_{2d} : \exists Q \in S_+^{\binom{n+d}{d}} : p = [x]_d^T \, Q \, [x]_d \},$$

where $[x]_d \in \mathbb{R}[x]^{\binom{n+d}{d}}$ is the vector containing a basis (for example monomials) of $\mathbb{R}[x]_d$.

<u>Example</u> $n = 2$, $d = 2$

$$[x]_2 = (1, x_1, x_2, x_1 x_2, x_1^2, x_2^2)^T$$

Proof (of Thm. 2)

"$\subseteq$": For $p = \sum_{i=1}^{m} q_i^2$ write $q_i = \sum_\alpha q_{i\alpha} x^\alpha$

$$= \underline{q_i}^T [x]_d,$$

with $\underline{q_i} = (q_{i\alpha}) \in \mathbb{R}[x]^{\binom{n+d}{d}}$.

Then, $q_i^2 = \left(\underline{q_i}^T [x]_d\right)^2 = [x]_d^T \underline{q_i}\, \underline{q_i}^T [x]_d$,

and
$$p = \sum_{i=1}^{m} [x]_d^T \underline{q_i}\, \underline{q_i}^T [x]_d$$
$$= [x]_d^T \underbrace{\left(\sum_{i=1}^{m} \underline{q_i}\, \underline{q_i}^T\right)}_{\in S_+^{\binom{n+d}{d}}} [x]_d .$$

"$\supseteq$": Suppose $\exists\, Q \in S_+^{\binom{n+d}{d}}$: $p = [x]_d^T Q [x]_d$.

Apply spectral decomposition $Q = \sum_{j=1}^{\binom{n+d}{d}} \lambda_j u_j u_j^T$.

Then,
$$p = \sum_{j=1}^{\binom{n+d}{d}} \Big(\underbrace{(\sqrt{\lambda_j}\, u_j)^T [x]_d}_{=q_j}\Big)^2 .$$

$\boxtimes$

Remarks • Theorem 2: Decision problem "$p \in \Sigma_{n,2d}$?" can be reduced to SDP feasibility.

- rank Q determines the number of summands in an SOS representation of p.
- Q is not uniquely determined.
- Decision problem "$p \in \mathcal{P}_{n,4}$" is NP-hard.
- Blekherman (2006): $\mathcal{P}_{n,2d}$ is much bigger than $\Sigma_{n,2d}$. ($\to$ seminar)

But:

Theorem 3 (Hilbert, 1888)

$$\Sigma_{n,2d} = \mathcal{P}_{n,2d} \iff \begin{array}{l} n=1 \quad \text{or} \\ d=1 \quad \text{or} \\ n=2,\ d=2. \end{array}$$

Proof • $n=1$ ($\to$ Ex.)

- $d=1$: Cholesky factorization
- $n=2, d=2$: not easy, not here ($\to$ seminar)

• all other cases: use variant of Motzkin poly.

$$M(x,y) = x^2y^2(x^2+y^2-3)+1 \in \mathcal{P}_{2,6} \setminus \Sigma_{2,6}.$$

Proof $M \in \mathcal{P}_{2,6}$:

If $x^2+y^2-3 \geq 0$, then $M(x,y) \geq 0$.

If $x^2+y^2-3 < 0$, define $z^2 = 3-x^2-y^2$ (>0).

By the AM-GM-inequality:
(arithmetic mean - geometric mean)

$$\underset{\left(-M(x,y)+1\right)^{1/3}}{\overset{\shortparallel}{\left(x^2y^2z^2\right)^{1/3}}} \leq \frac{x^2+y^2+z^2}{3} = 1 \implies M(x,y) \geq 0.$$

$M \notin \Sigma_{2,6}$: Suppose $M = \sum_{k=1}^{m} q_k^2$. Then $\deg q_k \leq 3$.

Coefficients of q_k:

x^3 : 0
y^3 : 0
x^2 : 0
y^2 : 0
x : 0
y : 0

So $q_k = a_k xy^2 + b_k x^2y + c_k xy + d_k$

Coefficient of x^2y^2 in M is -3.

So: $\sum_{k=1}^{m} c_k^2 = -3$ $\lightning$. $\boxtimes$

History

- Hilbert (1888): $\Sigma_{n,2d} \neq \mathcal{P}_{n,2d}$ in general
- Hilbert (1900): 17^{th} problem

$$\mathcal{P}_{n,2d} = \left\{ p : p = \sum_{i=1}^{m} \left(\frac{f_i}{g_i}\right)^2, \ f_i, g_i \in \mathbb{R}[x] \right\} ?$$

- Artin (1927): YES!

For Motzkin polynomial

$$M(x,y) = \frac{x^2y^2(x^2+y^2+1)(x^2+y^2-2)^2 + (x^2-y^2)^2}{(x^2+y^2)^2}$$

sum of squares of four rational functions.

Literature: K. Schmüdgen – Around Hilbert's 17^{th} problem, ISMP, 2012.