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Convex Optimization

Winter Term 2018/19

— Exercise Sheet 8 —

Exercise 8.1. Determine the Shannon capacity of all graphs having four vertices.

Exercise 8.2. Show that $\vartheta(C_5) \leq \sqrt{5}$, using the formulation of Exercise 8.4 for the theta number.
Hint: Consider the following vectors in $\mathbb{R}^3$: $c = (0,0,1)$, $u_k = (s\cos(2k\pi/5), s\sin(2k\pi/5), t)$ for $k = 1,2,3,4,5$, where the scalars $s, t \in \mathbb{R}$ are chosen in such a way that $u_1, \ldots, u_5$ form an orthonormal representation of C_5. Recall $\cos(2\pi/5) = \frac{\sqrt{5}-1}{4}$.
This is the original proof of Lovász, known as the *umbrella construction*.

Exercise 8.3. (Hand-in) Let $G = (V, E)$ be a graph.

(a) Show: $\vartheta(G) = \min\{\lambda_{\max}(Z) \ : \ Z \in \mathcal{S}^V, \ Z = J + T, \ T_{i,j} = 0 \ \text{ if } \{i,j\} \notin E\}$.

(b) Assume that G is *regular*, i.e. e is an eigenvector of the adjacency matrix A_G of G. Show:

$$\vartheta(G) \leq |V| \frac{-\lambda_{\min}}{\lambda_{\max} - \lambda_{\min}},$$

where $\lambda_{\min}$ is the smallest and $\lambda_{\max}$ is the largest eigenvalue of A_G.

Exercise 8.4. (Hand-in) Let $G = (V = [n], E)$ be a graph. Consider the graph parameter

$$\vartheta_1(G) = \min_{c, u_i} \max_{i \in V} \frac{1}{(c^\top u_i)^2},$$

where the minimum is taken over all unit vectors c and all orthonormal representations $u_1, \cdots, u_n$ of G (i.e., $u_1, \ldots, u_n$ are unit vectors satisfying $u_i^\top u_j = 0$ for all pairs $\{i,j\} \in \overline{E}$).
Show: $\vartheta(G) = \vartheta_1(G)$.
Hint: Use the dual formulation of $\vartheta(G)$. For the inequality $\vartheta(G) \leq \vartheta_1(G)$, consider the vectors $v_i = c - \frac{u_i}{c^\top u_i}$ for $i \in [n]$. For the inequality $\vartheta_1(G) \leq \vartheta(G)$, show that there exists a nonzero vector c which is orthogonal to suitable vectors $x_1, \ldots, x_n$, and consider the vectors $u_i = \frac{c + x_i}{\sqrt{t}}$.

Hand-in: Until Wednesday December 5, 12:00 (noon).
Exercises 8.3 and 8.4 to be submitted to the “Convex optimization” mailbox in room 3.01 (Studierendenarbeitsraum) of the Mathematical Institute.