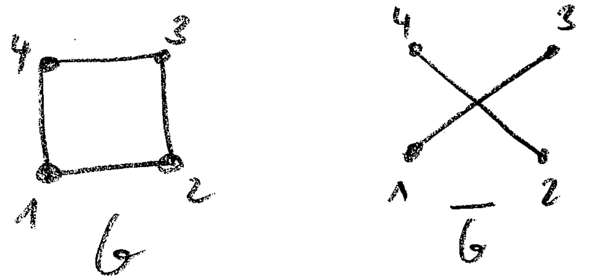
Chapter VI SDP relaxations for α and χ.

§1 Basic definitions

Let $G = (V, E)$ be an undirected graph. Its <u>complementary graph</u> $\overline{G} = (V, \overline{E})$ is defined by $\overline{E} = \{ \{i,j\} : i,j \in V, i \neq j, \{i,j\} \notin E \}$.



A set $I \subseteq V$ is called <u>independent (stable)</u> if for all $i,j \in I$: $\{i,j\} \notin E$. The <u>independence (stability) number</u> of G is

$$\alpha(G) = \max \{ |I| : I \subseteq V \text{ independent} \}.$$

A set $C \subseteq V$ is called a <u>clique</u> in G if C is independent in $\overline{G}$. The clique number of G is

$$\omega(G) = \alpha(\overline{G}).$$

A <u>k-coloring</u> of G, $k \in \mathbb{N}$, is a map $\varphi : \{1, \dots, k\} \to V$ so that $\varphi(i) \neq \varphi(j)$ for all $\{i,j\} \in E$. In other words, a k-coloring of G is a partition $V = I_1 \,\dot\cup\, \dots \,\dot\cup\, I_k$ into k independent sets. The <u>chromatic number</u> of G is

$$\chi(G) = \min\{k : \exists\ k\text{-coloring of } G\}$$

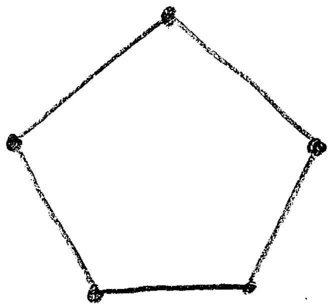
<u>Facts</u> The decision problems

- $\alpha(G) \geq k$?
- $\omega(G) \geq k$?
- $\chi(G) \leq k$?

are all NP-complete.

Even the restricted problem: Is "$\chi(G) \leq 3$?" for planar graphs G is NP-complete. On the other hand $\chi(G) \leq 4$ for every planar graph G by the 4-color theorem of Appel and Haken (1976). It is also easy to check "$\chi(G) \leq 2$?", meaning is G bipartite ?

Clearly, $\omega(G) \le \chi(G)$, but in general $\omega(G) < \chi(G)$. For example, cycle graph C_{2n+1} of odd length:



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$$\omega(C_{2n+1}) = 2$$
$$\chi(C_{2n+1}) = 3$$

and $\omega(\overline{C_{2n+1}}) = n = \alpha(C_{2n+1})$

$$\chi(\overline{C_{2n+1}}) = n+1.$$

§2 Lovász' Sandwich Theorem

Def. 1 For $G = (V, E)$ define the Lovász ϑ-number by

$$\vartheta(G) = \sup \langle J, X \rangle \qquad (*)$$
$$\text{s.t. } X \in \mathcal{S}_+^V$$
$$\langle I, X \rangle = 1$$
$$X_{ij} = 0 \quad \text{for } \{i,j\} \in E,$$

where $J = ee^T$ and $e^T = (1, \dots, 1) \in \mathbb{R}^V$.

<u>Remark</u>. $\vartheta(G)$ is a semidefinite relaxation of a quadratic program with 0/1-constraint which computes $\alpha(G)$ ($\rightarrow$ V.1).

- The dual of $(*)$ is

$$\begin{aligned} \inf\ & t \\ \text{s.t. } & t,\ y_{ij} \in \mathbb{R} \quad \{i,j\} \in E \\ & \underbrace{tI + \sum y_{ij} E_{ij} - J}_{=K} \in S_+^V \end{aligned}$$

$$= \begin{aligned} \inf\ & t \\ \text{s.t. } & t \in \mathbb{R},\ K \in S_+^V \\ & K_{ii} = t-1 \quad \text{for } i \in V \\ & K_{ij} = -1 \quad \text{for } \{i,j\} \notin E \text{ and } i \neq j. \end{aligned} \qquad (**)$$

Now $X = \frac{1}{|V|} I$ is strictly feasible for $(*)$ and $K = (|V|+1)I - J$ is strictly feasible for $(**)$, so by strong duality, the sup in $(*)$ is attained, and the inf in $(**)$ is attained.

Theorem 2 (Lovász' Sandwich Theorem)

$$\alpha(G) \leq \vartheta(G) \leq \chi(\overline{G}).$$

Proof $\alpha(G) \leq \vartheta(G)$: Let $I \subseteq V$ be an independent set. Define its characteristic vector $\chi^I \in \mathbb{R}^V$ by

$$(\chi^I)_i = \begin{cases} 1, & \text{if } i \in I, \\ 0, & \text{otherwise.} \end{cases}$$

Then, the matrix $X = \frac{1}{|I|} \chi^I (\chi^I)^T \in S_+^V$ is a feasible solution for $\vartheta(G)$ with $\langle J, X \rangle = |I|$.

Hence, $\alpha(G) \leq \vartheta(G)$.

$\vartheta(G) \leq \chi(\overline{G})$: Let X be a feasible solution for $\vartheta(G)$ and let $V = C_1 \dot\cup \ldots \dot\cup C_k$ be a partition of V into k cliques of G.

Claim: $\langle J, X \rangle \leq k$ (and so $\vartheta(G) \leq \chi(\overline{G})$).

Proof We have $\sum_{i=1}^{k} \chi^{C_i} = e = (1, \ldots, 1)^T$ since we have a partition.

Furthermore,

$$0 \leq Y = \sum_{i=1}^{k} (kX^{C_i} - e)(kX^{C_i} - e)^T$$

$$= k^2 \sum_{i=1}^{k} X^{C_i}(X^{C_i})^T - e \underbrace{\sum_{i=1}^{k} k(X^{C_i})^T}_{ke^T}$$

$$- \underbrace{\left(\sum_{i=1}^{k} kX^{C_i}\right)}_{ke} e^T + \sum_{i=1}^{k} ee^T$$

$$= k^2 \sum_{i=1}^{k} X^{C_i}(X^{C_i})^T - kJ, \qquad J = ee^T,$$

and

$$0 \leq \langle X, Y \rangle = k^2 \langle X, \sum_{i=1}^{k} X^{C_i}(X^{C_i})^T \rangle - k\langle J, X\rangle$$

$$= k^2 \underbrace{\mathrm{Tr}(X)}_{=1} - k\langle J, X\rangle$$

C_i's are cliques,
$X_{vw} = 0$ if $\{v,w\} \in E$

$$= k^2 - k\langle J, X\rangle$$

$$\Longrightarrow \langle J, X\rangle \leq k.$$

$\boxtimes$

<u>Remark</u> • If one replaces in the definition of $\vartheta(G)$ the cone S_+^V by CP^V, then one gets equality between the conic program and $\alpha(G)$.
(→ Ex. 3.4).

§ 3 The Shannon capacity of a graph

Claude E. Shannon (1916–2001) : founder of (mathematical) information theory

basic model in information theory

$$\boxed{\text{transmitter}} \xrightarrow[\text{message}]{X} \boxed{\text{(noisy) channel}} \xrightarrow{Y} \boxed{\text{receiver}}$$

<u>problem</u>: usually $Y \neq X$

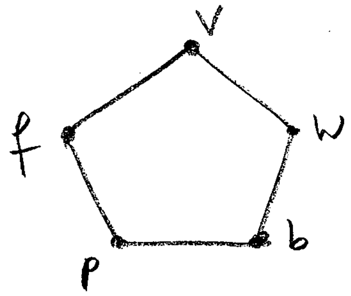
<u>goal</u>: recover X from Y

<u>message</u> X is a sequence of symbols from a (finite) alphabet V.

<u>noisy channel</u> can confuse symbols which are similar.

model this by <u>similarity</u> / <u>confusion graph</u> $G = (V, E)$, where two symbols $i, j \in V$ can be confused iff $\{i,j\} \in E$.

<u>example</u>



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information rate for zero error communication:

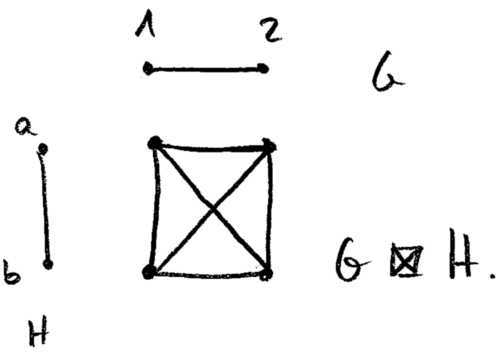
- send only symbols which cannot be confused (zero error)
- related to independence number $\alpha(G)$ $\quad (\alpha(C_5) = 2)$
- trick/approach to improve information rate: send more symbols at the same time: use V^k instead of V.

Def. 1 Let $G = (V, E)$, $H = (W, F)$ be graphs. Define the strong graph product of G and H by

$$G \boxtimes H = \Big(V \times W,\ \big\{ \{(v_1, w_1), (v_2, w_2)\} : (v_1 = v_2 \wedge \{w_1, w_2\} \in F) \vee (w_1 = w_2 \wedge \{v_1, v_2\} \in E) \vee (\{v_1, v_2\} \in E \wedge \{w_1, w_2\} \in F) \big\} \Big).$$

Notation: $G^{\boxtimes k} = \underbrace{G \boxtimes G \boxtimes \cdots \boxtimes G}_{k\text{-times}}$

example



Lemma 2 $\alpha(G \boxtimes H) \geq \alpha(G)\,\alpha(H)$, in particular

$$\sqrt[k]{\alpha(G^{\boxtimes k})} \geq \alpha(G).$$

Proof Let $I \subseteq V$ independent in G, and $J \subseteq W$ indep. in H. Then, $I \times J$ independent in $G \boxtimes H$. $\boxtimes$

Def. 3 The Shannon capacity of G is

$$\Theta(G) = \sup_{k \in \mathbb{N}} \sqrt[k]{\alpha(G^{\boxtimes k})} \qquad (\geq \alpha(G) \text{ by Lemma 2}).$$

Example $\Theta(C_5) \geq \sqrt{5} > 2$

Because $\alpha(C_5^{\boxtimes 2}) \geq 5$. $I = \{(1,1), (2,3), (3,5), (4,2), (5,4)\}$ is an independent set in $C_5^{\boxtimes 2}$.

Shannon (1956): 1) Computed $\Theta(G)$ for all graphs G up to 5 vertices, except $G = C_5$.
→ exercise.

2) Is $\Theta(C_5) = \sqrt{5}$?

3) How to compute $\Theta(G)$?

Lovász (1979): Answered 2) → this lecture.

3) No algorithm is known. Major open problem.

Even the value $\Theta(C_7)$ is not known. Currently,

$$\Theta(C_7) \geq (367)^{1/5} \approx 3{,}2578\ldots$$

Polak, Schrijver (2018), $\alpha(C_7^{\boxtimes 5}) \geq 367$.

$$\Theta(C_7) \leq 3{,}3177\ldots$$

Lovász (1979), also this lecture.

Theorem 4 $\vartheta(G \boxtimes H) = \vartheta(G) \cdot \vartheta(H)$

Proof Recall the Kronecker product $A \otimes B \in \mathbb{R}^{mp \times nq}$ of two matrices $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{p \times q}$ defined by

$$(A \otimes B)_{((i,h),(j,k))} = A_{i,j} B_{h,k}.$$

See Chapter II. 2.13, 14.

$\vartheta(G \boxtimes H) \geq \vartheta(G) \cdot \vartheta(H)$:

Consider primal

$$\vartheta(G) = \max\{\langle J, X\rangle : X \in S_+^V,\ \langle I, X\rangle = 1,$$
$$X_{ij} = 0 \text{ if } \{i,j\} \in E\}.$$

Let X be feasible for $\vartheta(G)$, Y feasible for $\vartheta(H)$.
Then $X \otimes Y$ feasible for $\vartheta(G \boxtimes H)$ (check it!), and

$$\langle J, X \otimes Y \rangle = \langle J, X \rangle \cdot \langle J \cdot Y \rangle.$$

$\vartheta(G \boxtimes H) \leq \vartheta(G)\,\vartheta(H)$

Consider dual

$$\vartheta(G) = \min \{ t : t \in \mathbb{R},\ Z \in S^V,$$
$$Z - J \in S^V_+,$$
$$Z_{ii} = t \quad \text{for } i \in V,$$
$$Z_{ij} = 0 \quad \text{if } i \neq j \text{ and } \{i,j\} \notin E \}.$$

Let (r, X) be feasible for $\vartheta(G)$, let (s, Y) be feasible for $\vartheta(H)$. Then $(rs, X \otimes Y)$ feasible for $\vartheta(G \boxtimes H)$ because:

- $(X \otimes Y)_{((i,h),(i,h))} = X_{ii} Y_{hh} = rs.$

- $\{(i,h),(j,k)\} \notin E(G \boxtimes H)$, then

$$(X \otimes Y)_{((i,h),(j,k))} = X_{ij} Y_{hk} = 0$$

- $X \otimes Y - J \otimes J \succeq 0$:

By assumption $X - J \succeq 0$ and $Y - J \succeq 0$.

Hence, $(X - J) \otimes (Y - J) \succeq 0$

$$\overset{\shortparallel}{X \otimes Y} - X \otimes J - J \otimes Y + J \otimes J \succeq 0 \qquad (*)$$

We also know

$$(X - J) \otimes J \succeq 0 \quad \text{and} \quad J \otimes (Y - J) \succeq 0.$$

Therefore,

$$X \otimes J - J \otimes J + J \otimes Y - J \otimes J \succeq 0 \qquad (**)$$

$(*) + (**)$ gives

$$X \otimes Y - J \otimes J \succeq 0. \qquad \boxtimes$$

<u>Corollary 5</u> $\Theta(G) \le \vartheta(G)$.

<u>Proof</u> $\sqrt[k]{\alpha(G^{\boxtimes k})} \le \sqrt[k]{\vartheta(G^{\boxtimes k})} \underset{\text{Thm. 4}}{=} \sqrt[k]{\vartheta(G)^k} = \vartheta(G)$. $\boxtimes$

<u>Theorem 6</u> $\vartheta(C_5) = \sqrt{5}$.

<u>Proofs:</u>
1. By computer → exercise
2. Will show $\vartheta(C_n)\,\vartheta(\overline{C_n}) = n$, hence $\vartheta(C_5) = \sqrt{5}$ because $C_5 \cong \overline{C_5}$.
3. Will compute $\vartheta(C_n) = \dfrac{n \cos(\pi/n)}{1 + \cos(\pi/n)}$ when n is odd.
4. Lovász' umbrella (→ Proofs from the Book) $\boxtimes$

§ 4 More properties of ϑ

Clearly: $\alpha(G) \cdot \chi(G) \geq |V|$.

Similar relation for ϑ:

Theorem 1 $\vartheta(G) \cdot \vartheta(\overline{G}) \geq |V|$.

Proof Consider dual SDP:

$$\vartheta(G) = \min \{ t : t \in \mathbb{R}, Z \in S^V, \\ Z - J \in S^V_+, \\ Z_{ii} = t \text{ for } i \in V, \\ Z_{ij} = 0 \text{ for } i \neq j \text{ and } \{i,j\} \notin E \}.$$

Let (t, X) be optimal for $\vartheta(G)$ and $(\overline{t}, \overline{X})$ opt. for $\vartheta(\overline{G})$. Then,

i) $\langle X, J \rangle \geq |V|^2$ and $\langle \overline{X}, J \rangle \geq |V|^2$

because

$$0 \leq \langle X - J, J \rangle = \langle X, J \rangle - \underbrace{\langle J, J \rangle}_{= |V|^2}.$$

ii) $\langle X, \overline{X} \rangle = \sum_{i,j \in V} X_{ij} \overline{X_{ij}} = \sum_{i \in V} X_{ii} \overline{X_{ii}} = |V| \cdot t \cdot \overline{t}$

$= |V| \, \vartheta(G) \, \vartheta(\overline{G})$.

iii) $\langle X, \bar{X} \rangle \geq |V|^2$

$$0 \leq \langle X - J, \bar{X} \rangle = \langle X, \bar{X} \rangle - \langle X, J \rangle \overset{i)}{\leq} \langle X, \bar{X} \rangle - |V|^2$$

Now the theorem follows from ii) and iii). $\boxtimes$

<u>Goal:</u> If G is <u>vertex transitive</u>, then also $\vartheta(G) \cdot \vartheta(\bar{G}) \leq |V|$ holds.

(similarly: $\alpha(G)\, \chi^*(G) = |V|$ for vertex transitive G, where $\chi^*(G)$ is the fractional chromatic number of G).

<u>Def. 2</u> Let $G = (V, E)$ be a graph.

a) The <u>automorphism group</u> of G is

$$\mathrm{Aut}(G) = \{ \sigma : V \to V : \sigma \text{ permutation and } \{i,j\} \in E \Leftrightarrow \{\sigma(i), \sigma(j)\} \in E \}.$$

b) G is called <u>vertex transitive</u> (<u>homogeneous</u>) if for all $i, j \in V$ there is $\sigma \in \mathrm{Aut}(G)$: $\sigma(i) = j$.

<u>Fact</u> Decision problem "Given G, is $\mathrm{Aut}(G) = \{id\}$?" is very interesting and strongly related to the graph isomorphism problem. $\rightsquigarrow$ VL Theoretische Informatik.

Def. 3 Let $\sigma : V \to V$ be a permutation. Define the permutation matrix $P^\sigma \in \mathbb{R}^{V \times V}$ by

$$P^\sigma_{ij} = \begin{cases} 1, & \text{if } j = \sigma(i), \\ 0, & \text{otherwise.} \end{cases}$$

Let $X \in S^V$ be a symmetric matrix. Define

$$\sigma(X) = P^\sigma X (P^\sigma)^T = \left(X_{\sigma(i),\sigma(j)} \right)_{i,j \in V}.$$

Lemma 4 Let $G = (V, E)$ be a graph and $\sigma \in \mathrm{Aut}(G)$.
Let X be feasible for

$$\vartheta(G) = \max \{ \langle J, X \rangle : X \in S^V_+,\ \langle I, X \rangle = 1,\ X_{ij} = 0 \text{ if } \{i,j\} \in E \}.$$

Then $\sigma(X)$ is feasible as well, with $\langle J, \sigma(X) \rangle = \langle J, X \rangle$.

Proof Clearly: $\langle J, \sigma(X) \rangle = \langle J, P^\sigma X (P^\sigma)^T \rangle$

$$= \langle (P^\sigma)^T J P^\sigma, X \rangle = \langle J, X \rangle.$$

and $\langle I, \sigma(X) \rangle = \langle I, X \rangle$.

Furthermore, $\sigma(X)_{ij} = X_{\sigma(i)\sigma(j)} = 0$ if $\{i,j\} \in E$ since $\sigma \in \mathrm{Aut}(G)$. $\boxtimes$

Lemma 5 There exists an optimal solution X^* which is invariant under $\mathrm{Aut}(G)$, i.e. $\forall \sigma \in \mathrm{Aut}(G)$: $\sigma(X^*) = X^*$.

Proof Let X be optimal for $\vartheta(G)$, which exists by strong duality. Then

$$X^* = \frac{1}{|\mathrm{Aut}(G)|} \sum_{\sigma \in \mathrm{Aut}(G)} \sigma(X)$$

is optimal for $\vartheta(G)$ by Lemma 4. Moreover, X^* is invariant under $\mathrm{Aut}(G)$. $\boxtimes$

Lemma 6 Let $G=(V,E)$ be a vertex transitive graph. Then there is an optimal solution X^* of $\vartheta(G)$ with

$$X^*_{ii} = \frac{1}{|V|} \quad \text{for all } i \in V,$$

$$X^* e = \frac{\vartheta(G)}{|V|} e, \quad \text{where } e^T = (1, \dots, 1).$$

Proof Let X^* be an optimal solution invariant under $\mathrm{Aut}(G)$ like in Lemma 5. By Ex. 7.3 we have $X^* e = \vartheta(G) \operatorname{diag} X^*$.

To show: $\operatorname{diag} X^* = \frac{e}{|V|}$.

Since G is vertex transitive we have for all $i,j \in V$

$$X^*_{jj} = \sigma(X^*)_{ii} = X^*_{ii}$$

for $\sigma \in \mathrm{Aut}(G)$ with $\sigma(i) = j$. So all diagonal elements of X^* coincide and since $\mathrm{Tr}(X^*) = 1$ we have $X^*_{ii} = \frac{1}{|V|}$. $\boxtimes$

<u>Theorem 7</u> Let G be a vertex transitive graph. Then $\vartheta(G)\,\vartheta(\overline{G}) = |V|$.

<u>Proof</u> Need to show: $\vartheta(G)\,\vartheta(\overline{G}) \leq |V|$.

Let X^* be an optimal solution of the primal SDP as in Lemma 5. Define a feasible solution (t,Y) of the dual SDP $\vartheta(\overline{G})$ as follows: $Y = \frac{|V|^2}{\vartheta(G)} X^*$, $t = \frac{|V|}{\vartheta(G)}$. Then $\vartheta(G)\,\vartheta(\overline{G}) \leq \vartheta(G) \frac{|V|}{\vartheta(G)} = |V|$.

Now check that Y is indeed feasible:

- $Y_{ij} = 0$ for $i \neq j$, $\{i,j\} \in E(G)$: ✓
- $Y - J \in S^V_+$: $\frac{|V|^2}{\vartheta(G)} X^* - J \succeq 0$ because

$$\frac{|V|^2}{\vartheta(G)} X^* e - Je = \frac{|V|^2}{\vartheta(G)} \frac{\vartheta(G)}{|V|} e - |V| e = 0.$$

(eigenspaces of J are $\mathbb{R}e$ and $(\mathbb{R}e)^\perp$). $\boxtimes$

<u>Corollary 8</u> $\vartheta(C_n)\,\vartheta(\overline{C_n}) = n$

$\longrightarrow \vartheta(C_5) = \sqrt{5}.$

§ 5 Computing ϑ for Cayley graphs of finite Abelian groups

Let $(G,+)$ be a finite Abelian group. Then

$$G \cong \mathbb{Z}/_{n_1\mathbb{Z}} \times \dots \times \mathbb{Z}/_{n_r\mathbb{Z}}$$

with $n_1 \mid n_2 \mid \dots \mid n_r$, and $|G| = n_1 \cdots n_r$.
These elementary divisors determine G uniquely.
($\rightarrow$ modules over principal domains / Smith normal form)

Def 1 For $\Sigma \subseteq G$ with $\Sigma = -\Sigma$ define the Cayley graph of G and Σ by

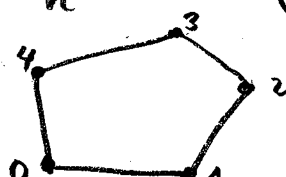
$$\mathrm{Cayley}(G, \Sigma) = (V, E)$$

with

$$V = G$$

$$E = \{\{x,y\} : x, y \in G,\ x - y \in \Sigma\}$$

Example $C_n = \mathrm{Cayley}(\mathbb{Z}/_{n\mathbb{Z}}, \{\pm 1\})$



<u>Goal</u>: Compute $\vartheta(\text{Cayley}(G, \Sigma))$, in particular $\vartheta(C_n)$ for n odd.

<u>Observation</u> G is a subgroup of $\text{Aut}(\text{Cayley}(G,\Sigma))$.
Because for $a \in G$ we have

$$x - y \in \Sigma \iff (a+x) - (a+y) \in \Sigma.$$

We will apply the discrete Fourier transform to simplify the SDP $\vartheta(\text{Cayley}(G,\Sigma))$ to an LP. Sometimes this LP can be solved by hand, as in the case of C_n.

<u>Def. 2</u> The unit circle

$$\mathbb{T} = \{ z \in \mathbb{C} : z\overline{z} = 1 \}$$

is called the <u>torus</u> in $\mathbb{C}$. A group homomorphism

$$\chi : (G,+) \to (\mathbb{T}, \cdot), \quad \chi(x+y) = \chi(x)\chi(y) \quad \forall x,y \in G$$

is called a <u>character</u> of G.

Example $G = \mathbb{Z}/n\mathbb{Z}$, $a \in G$ defines the character

$$\chi_a(x) = e^{\frac{2\pi i a x}{n}},$$

since

$$\chi_a(x+y) = e^{\frac{2\pi i a(x+y)}{n}} = e^{\frac{2\pi i a x}{n}} \cdot e^{\frac{2\pi i a y}{n}}$$

$$= \chi_a(x)\,\chi_a(y).$$

The dual group of G

Characters form a group themselves with pointwise multiplication: $(\chi\psi)(x) = \chi(x)\psi(x)$. This is $\hat{G}$, the dual group of G.

- unit element: trivial character $e(x) = 1 \quad \forall x \in G$
- inverse: $\chi^{-1} = \overline{\chi}$
- product of two characters is character again:

$$(\chi\psi)(x+y) = \chi(x+y)\psi(x+y) = \chi(x)\chi(y)\psi(x)\psi(y)$$

$$= (\chi\psi)(x)\,(\chi\psi)(y).$$

In the following view character χ of G as vector in complex vector space $\mathbb{C}^G$.

Lemma 3 (orthogonality relation)

Let G be a finite Abelian group. Let χ, ψ be characters of G. Then,

$$\chi^* \psi = \begin{cases} |G| , & \text{if } \chi = \psi, \\ 0 , & \text{otherwise.} \end{cases}$$

Proof If $\chi = \psi$, then

$$\chi^* \chi = \sum_{x \in G} \overline{\chi}(x) \chi(x) = \sum_{x \in G} 1 = |G|.$$

If $\chi \neq \psi$, then $\exists\, y \in G$: $\chi(y) \neq \psi(y)$, that is $(\overline{\chi} \psi)(y) \neq 1$. Furthermore,

$$(\overline{\chi} \psi)(y)\ \chi^* \psi = (\overline{\chi} \psi)(y) \sum_{x \in G} \overline{\chi}(x) \psi(y)$$

$$= \sum_{x \in G} \overline{\chi}(x+y)\, \psi(x+y)$$

$$= \sum_{x \in G} \overline{\chi}(x)\, \psi(x)$$

$$= \chi^* \psi . \qquad \Rightarrow \quad \chi^* \psi = 0.$$

$\boxtimes$

Corollary 4 For $G = \mathbb{Z}/_{n_1\mathbb{Z}} \times \dots \times \mathbb{Z}/_{n_r\mathbb{Z}}$ every element $(a_1, \dots, a_r) \in G$ defines a character

$$\chi_{(a_1,\dots,a_r)}(x_1,\dots,x_r) = \prod_{j=1}^{r} e^{\frac{2\pi i a_j x_r}{n_j}}.$$

The map $(a_1,\dots,a_r) \mapsto \chi_{(a_1,\dots,a_r)}$ is a group isomorphism between G and $\hat{G}$. In particular, $\chi_{(a_1,\dots,a_r)}$ forms an ONB of $\mathbb{C}^G$.

Proof • Clear: $\chi_{(a_1,\dots,a_r)}$ is a character.

• Also clear: $(a_1,\dots,a_r) \mapsto \chi_{(a_1,\dots,a_r)}$ is a group homomorphism, which is injective, so $|G| \leq |\hat{G}|$.

• By Lemma 3: $|\hat{G}| \leq \dim \mathbb{C}^G = |G|$.

$\Rightarrow |G| = |\hat{G}| \Rightarrow$ Claim. $\boxtimes$

So we can represent every vector $f \in \mathbb{C}^G$ in the form

$$f(x) = \sum_{\chi \in \hat{G}} f_\chi \, \chi(x) \quad \text{for } f_\chi \in \mathbb{C}.$$

This is called the <u>discrete Fourier transform</u> (DFT), the coefficients f_χ are the <u>Fourier coefficients</u> of f.

<u>Theorem 4</u> A vector $f \in \mathbb{C}^G$ with DFT

$$f(x) = \sum_{\chi \in \hat{G}} f_\chi \, \chi(x) \quad , \quad f_\chi \in \mathbb{C},$$

define a matrix $F \in \mathbb{C}^{G \times G}$ entrywise by

$$F_{x,y} = f(x-y) \quad , \quad x, y \in G.$$

Then:

(a) $F^* = F \iff f_\chi \in \mathbb{R} \;\; \forall \chi \in \hat{G}$

(b) F is Hermitian positive semi-definite:
$\forall z \in \mathbb{C}^G : \;\; z^* F z \geq 0 \iff f_\chi \geq 0 \;\; \forall \chi \in \hat{G}.$

(c) F has only real entries $\iff f_\chi = \overline{f_{\bar{\chi}}} \;\; \forall \chi \in \hat{G}.$

<u>Proof</u>

(a) $F^* = F \iff F_{x,y} = \overline{F_{y,x}} \;\; \forall x, y \in G$

$$\iff \sum_{\chi \in \hat{G}} f_\chi \, \chi(x-y) = \underbrace{\overline{\sum_{\chi \in \hat{G}} f_\chi \, \chi(y-x)}}_{= \sum_{\chi \in \hat{G}} \overline{f_\chi} \, \chi(x-y)} \quad \forall x, y \in G$$

$\iff f_\chi = \overline{f_\chi} \;\; \forall \chi \in \hat{G}.$

$\iff f_\chi \in \mathbb{R} \;\; \forall \chi \in \hat{G}.$

(b) Sufficiency: Suppose $f_\chi \geq 0 \ \forall \chi \in \hat{G}$.

Then, for $z \in \mathbb{C}^G$:

$$z^* F z = \sum_{x,y \in G} \overline{z(x)}\, F_{xy}\, z(y)$$

$$= \sum_{x,y \in G} \overline{z(x)} \sum_{\chi \in \hat{G}} f_\chi\, \chi(x-y)\, z(y)$$

$$= \sum_{\chi \in \hat{G}} f_\chi \sum_{x,y \in G} \bar{z}(x)\, \chi(x)\, \overline{\chi(y)}\, z(y)$$

$$= \sum_{\chi \in \hat{G}} f_\chi\, |z^* \chi|^2 \geq 0.$$

Necessity: Suppose F is Hermitian positive semidefinite. Then for $\psi \in \hat{G}$, viewed as $\psi \in \mathbb{C}^G$ we have

$$0 \leq \psi^* F \psi = \sum_{\chi \in \hat{G}} f_\chi\, |\psi^* \chi|^2 = f_\psi\, |G|^2$$

by the orthogonality relation. Hence, $f_\psi \geq 0$.

(c) Similar to (a)

$$\sum_\chi f_\chi\, \chi(x-y) = F_{x,y} = \overline{F_{x,y}} = \overline{\sum f_\chi\, \chi(x-y)}$$

$$= \sum \overline{f_\chi}\, \overline{\chi(x-y)}.$$

$$\Rightarrow f_\chi = \overline{f_{\bar{\chi}}}.$$

More context

Every matrix $F \in \mathbb{C}^{G\times G}$ which is defined by $F_{x,y} = f(x-y)$ for some $f \in \mathbb{C}^G$ is called a <u>G-circulant matrix</u>.

For example $G = \mathbb{Z}/5\mathbb{Z}$ and

$$F = \begin{array}{c c} & \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} f(0) & f(4) & f(3) & f(2) & f(1) \\ f(1) & f(0) & f(4) & f(3) & f(2) \\ f(2) & f(1) & f(0) & f(4) & f(3) \\ f(3) & f(2) & f(1) & f(0) & f(4) \\ f(4) & f(3) & f(2) & f(1) & f(0) \end{bmatrix} \end{array}$$

We have for $\chi \in \hat{G}$

$$\begin{aligned} [F\chi]_x &= \sum_{y\in G} F_{x,y}\, \chi(y) \\ &= \sum_{y\in G} f(x-y)\, \chi(y) \\ &= \sum_{y\in G} \sum_{\psi\in\hat{G}} f_\psi\, \psi(x-y)\, \chi(y) \\ &= \sum_{\psi\in\hat{G}} f_\psi\, \psi(x)\, \psi^*\chi \overset{\text{orthonormality relation}}{=} |G|\, f_\chi\, \chi(x) \end{aligned}$$

Thus χ is an eigenvector of F with eigenvalue $|G|\, f_\chi$.

Theorem 5

$$\vartheta(\mathrm{Cayley}(G,\Sigma)) = \max |G|^2 f_e$$
$$\text{s.t. } f_\chi \geq 0 \quad \forall \chi \in \hat{G}$$
$$f_\chi = f_{\bar{\chi}} \quad \forall \chi \in \hat{G}$$
$$\sum_{\chi \in \hat{G}} f_\chi = \frac{1}{|G|}$$
$$\sum_{\chi \in \hat{G}} f_\chi \chi(z) = 0 \text{ for } z \in \Sigma.$$

The SDP collapses to an LP.

Proof Let X^* be optimal for

$$\vartheta(\mathrm{Cayley}(G,\Sigma)) = \max\{\langle J, X\rangle : X \succeq 0, \mathrm{Tr}(X) = 1, X_{x,y} = 0 \text{ if } x-y \in \Sigma\}.$$

We may assume that $X^*_{x,y} = X^*_{x',y'}$ whenever $x-y = x'-y'$. This follows from a symmetrization argument: If X^* is optimal, then also

$$Y^*_{x,y} = \frac{1}{|G|} \sum_{z \in G} X^*_{x-z,y-z}$$

is optimal. So we can represent X^* as $X^*_{x,y} = f(x-y)$ for some $f \in \mathbb{C}^G$ as in Thm. 4.

Let $f(x) = \sum_{\chi \in \hat{G}} f_\chi \, \chi(x)$ the DFT of f. Now we express the maximization problem in the variables f_χ, $\chi \in \hat{G}$.

- $f_\chi \geq 0$, $f_\chi = f_{\bar{\chi}}$ follows from Thm. 4 since $X^* \in S_+^G$ is real positive semidefinite.

- $\mathrm{Tr}(X^*) = 1$

$$1 = \mathrm{Tr}(X^*) = \sum_{x \in G} X^*_{x,x} = \sum_{x \in G} \sum_{\chi \in \hat{G}} f_\chi \overbrace{\chi(x-x)}^{=1}$$

$$= \sum_{x \in G} \sum_{\chi \in \hat{G}} f_\chi = |G| \sum_{\chi \in \hat{G}} f_\chi .$$

- $\langle J, X^* \rangle = e^T X^* e = \sum_\chi f_\chi |e^T \chi|^2 = |G|^2 f_e$, orthogonality relation.

where $e \in \hat{G}$ is the trivial character.

- $X^*_{x,y} = 0$ if $x - y$ translates to

$$0 = X^*_{x,y} = \sum_\chi f_\chi \, \chi(z) \quad \text{for} \quad z = x - y .$$

$\boxtimes$

Theorem 6

$$\vartheta(C_n) = \frac{n\cos\left(\frac{\pi}{n}\right)}{1+\cos\left(\frac{\pi}{n}\right)} \quad \text{for } n \text{ odd.}$$

Proof We have $C_n = \mathrm{Cayley}(\mathbb{Z}/n\mathbb{Z}, \{\pm 1\})$.

Now apply Thm. 5

$$\begin{aligned}\vartheta(C_n) = \ & \max\ n^2 f_0 \\ & \text{s.t. } f_0, f_1, \ldots, f_{(n-1)/2} \geq 0 \\ & f_0 + 2\sum_{j=1}^{(n-1)/2} f_j = \frac{1}{n} \\ & f_0 + 2\sum_{j=1}^{(n-1)/2} f_j \cos\left(\frac{2\pi j}{n}\right) = 0\end{aligned}$$

To maximise f_0 one has to minimise $\sum_{j=1}^{(n-1)/2} f_j$.

For this, set all $f_j = 0$, $j \geq 1$, except for $j = \frac{n-1}{2}$ because the function $j \mapsto \cos\left(\frac{2\pi j}{n}\right)$ attains its minimum value which is

$$\cos\left(\frac{2\pi\left(\frac{n-1}{2}\right)}{n}\right) = \cos\left(\frac{\pi(n-1)}{n}\right) = -\cos\left(\frac{\pi}{n}\right).$$

Now solve the linear system

$$\begin{aligned} f_0 + 2 f_{\frac{n-1}{2}} &= \frac{1}{n} \\ f_0 - 2 f_{\frac{n-1}{2}} \cos\left(\frac{\pi}{n}\right) \end{aligned} \quad \text{yields } f_0 = \frac{\cos\left(\frac{\pi}{n}\right)}{n\left(1+\cos\left(\frac{\pi}{n}\right)\right)}. \quad \boxtimes$$