Asymptotic Optimality When Dimension Exceeds Sample Size

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The Convolution Theorem does not apply to the James-Stein estimator or its numerous practical descendants because they are not "regular". The Local Asymptotic Minimax (LAM) Theorem, in which limiting sample size greatly exceeds parameter dimension, does not reveal the domain or magnitude of the risk improvement achievable by Stein shrinkage when dimension is much larger than sample size.

Stein (1956, *Inadmissibility of the usual estimator for the mean of a multivariate normal distribution*, Third Berkeley Symposium) sketched a dimensional asymptotics approach to assessing estimation of the mean. Filling in details of his remarks leads by natural implications to an asymptotic minimax bound, as dimension of the mean tends to infinity, that is achieved by the James-Stein estimator but not by the sample mean. This implied asymptotic minimax bound turns out to be a special case of the later Pinsker (1980, *Optimal filtration of square-integrable signals in Gaussian white noise*, Problems Inform. Transmission) asymptotic minimax bound, albeit obtained by an orthogonal invariance argument rather than by Pinsker's more general Bayes method.

For specifics of Stein's (1956) approach and its initial implications, see Beran (1996, *Stein estimation in high dimensions: a retrospective*, Madan Puri Festschrift, E. Brunner and M. Denker, eds.). Practical descendants of James-Stein include multiple shrinkage, submodel selection, and penalized least squares estimators whose tuning parameters are selected to minimize estimated risk. For an exact treatment of multiple shrinkage, see Stein (1966, *An approach to the recovery of inter-block information in balanced incomplete block designs*, Festschrift for Jerzy Neyman, F. N. David, ed.). For further useful shrinkage estimators and their dimensional asymptotics, see Beran and Dümbgen (1998, *Modulation estimators and confidence sets*, Ann. Statist.) and Beran (2014, *Hypercube estimators: penalized least squares, submodel selection and numerical stability*, Computational Statistics and Data Analysis).