On the infinity Laplace operator

Petri Juutinen

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$$-\frac{d}{dx}\big(H(x, u(x), Du(x)) \cdot H_p(x, u(x), Du(x)\big) = 0 \tag{2}$$

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- If $H(x,r,p) = \frac{1}{2}|p|^2$, then (2) reduces to

$$\Delta_\infty u := \left(D^2 u\, Du\right) \cdot Du = \sum_{i,j=1}^{n} u_{x_i} u_{x_j} u_{x_i x_j} = 0. \tag{3}$$

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If $Du \neq 0$, this implies

$$\Delta_\infty u = -\frac{|Du|^2}{p-2} \Delta u,$$

and thus letting $p \to \infty$ we recover the infinity Laplace equation $\Delta_\infty u = 0$.

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- applications: image processing, shape metamorphism, differential games etc.
- stochastic version: the random turn Tug-of-War of Peres, Schramm, Sheffield and Wilson (J. Amer. Math. Soc. 2008)

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However, the non-homogeneous equations such as

$$\Delta_\infty u(x) = f(x)$$

and

$$\Delta_\infty^S u(x) = f(x)$$

are, of course, not equivalent.

Viscosity solutions

Definition

Let $\Omega \subset \mathbb{R}^n$ be an open set. An upper semicontinuous function $u : \Omega \to \mathbb{R}$ is a viscosity subsolution of (3) in Ω if, whenever $\hat{x} \in \Omega$ and $\varphi \in C^2(\Omega)$ are such that

(i) $u(\hat{x}) = \varphi(\hat{x})$,

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A lower semicontinuous function $v : \Omega \to \mathbb{R}$ is a viscosity supersolution of (3) in Ω if $-v$ is a viscosity subsolution.
Finally, a continuous function $h : \Omega \to \mathbb{R}$ is a viscosity solution of (3) in Ω if it is both a viscosity subsolution and a viscosity supersolution.

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- *A function* $u \in C^2$ *is is a viscosity solution of (3) in* Ω *if and only if* $\Delta_\infty u(x) = 0$ *for all* $x \in \Omega$.

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- $u\colon \mathbb{R}^2 \to \mathbb{R}$, $u(x, y) = |x|^{4/3} - |y|^{4/3}$ *is a viscosity solution, and* $u \notin C^2$.

Absolute minimizers

Functionals of the form

$$I(v, \Omega) = \int_\Omega f(x, u(x), Du(x))\, dx$$

are set-additive. Thus if u minimizes $I(\cdot, \Omega)$ (with given boundary data), then it automatically also minimizes $I(\cdot, V)$, subject to its own boundary values, for every open $V \subset \Omega$.

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Definition

A locally Lipschitz continuous function $u:\Omega \to \mathbb{R}^m$, $m \geq 1$, is called an absolute minimizer of $S(\cdot,\Omega)$, if

$$S(u,V) \leq S(v,V)$$

for every $V \subset\subset \Omega$ and $v \in W^{1,\infty}(V) \cap C(\overline{V}))$ such that $v|_{\partial V} = u|_{\partial V}$.

Comparison principle

Theorem (Jensen 1993)

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, and suppose that u and v are a subsolution and a supersolution of (3) in Ω, respectively, such that $u \leq v$ on $\partial\Omega$. Then $u \leq v$ in Ω.

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- The main point of the proof is to approximate a given subsolution by subsolutions with non-vanishing gradient.
- The case of unbounded domains has been considered by Crandall, Gunnarsson and Wang (2007).

Existence

Theorem

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, and suppose that $g\colon \partial\Omega \to \mathbb{R}$ is continuous. Then there is a unique u such that

$$\begin{cases} \Delta_\infty u = 0 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases}$$

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The existence can be obtained using

1. Perron's method.
2. the approximation involving p-Laplace equation.
3. Tug-of-War.

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- u is locally Lipschitz continuous.
- if $n = 2$, $u \in C^{1,\alpha}$ (Savin 2005, Evans and Savin 2008).
- $C^{1,1/3}$ is the best one can hope for (in any dimension).

Comparison with cones

The cone functions

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Definition (Crandall, Evans, Gariepy 2001)

A function $u \in C(\Omega)$ enjoys comparison with cones from above if, whenever $V \subset\subset \Omega$ is open, $x_0 \notin V$, $a, b \in \mathbb{R}$, $a > 0$, are such that

$$u(x) \leq C(x) := a|x - x_0| + b \qquad \text{on } \partial V,$$

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A function $v \in C(\Omega)$ enjoys comparison with cones from below if $-v$ enjoys comparison with cones from above.
Finally, $u \in C(\Omega)$ enjoys comparison with cones if it enjoys comparison with cones both from above and below.

Equivalence

Theorem (Jensen 1993, Crandall et al 2001)

Let $u:\Omega\to\mathbb{R}$ *be a locally Lipschitz continuous function. Then the following are equivalent:*

1. u *is an absolute minimizer of the functional* $S(v)=\operatorname{ess\,sup}|Du|$.
2. u *is an AMLE (=absolutely minimizing Lipschitz extension): for every open* $V\subset\subset\Omega$ *we have* $Lip(u,V)=Lip(u,\partial V)$.
3. u *is a viscosity solution of the infinity Laplacian.*
4. u *enjoys comparison with cones.*

(4) $\implies$ (2): Since

$$u(z) - Lip(u, \partial V)|x - z| \leq u(x) \leq u(z) + Lip(u, \partial V)|x - z|$$

holds for $x, z \in \partial V$ and $V \subset\subset \Omega$

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$$Lip(u, \partial(V \setminus \{x\})) = Lip(u, \partial V)$$

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Hence $|u(x) - u(y)| \leq Lip(u, \partial V)|x - y|$, which shows that $Lip(u, V) = Lip(u, \partial V)$.

(3) $\implies$ (1): Let $u \in C^2(\Omega)$ satisfy $\Delta_\infty u(x) = 0$ in Ω, and suppose there is $V \subset\subset \Omega$, $x_0 \in V$ and $v \in W^{1,\infty}(\Omega) \cap C(\overline{\Omega})$ such that $u = v$ on ∂V and

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Combining these two inequalities with (4) yields $|u(y) - u(z)| > |v(y) - v(z)|$, which contradicts $u = v$ on ∂V.

Tug-of-war

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The value function $u_\varepsilon(x)$ of the above game is continuous for each $\varepsilon > 0$ and $u_\varepsilon \to u$ uniformly, where u is the unique viscosity solution of the infinity Laplacian so that $u = g$ on $\partial\Omega$.

The dynamic programming principle (DPP) reads

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$$(DPP) \implies \varphi(x_\varepsilon) \le \frac{1}{2}\Big(\max_{B_\varepsilon(x_\varepsilon)} \varphi + \min_{B_\varepsilon(x_\varepsilon)} \varphi \Big). \tag{5}$$

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For ε small, we have

$$\max_{B_\varepsilon(x_\varepsilon)} \varphi = \varphi(x_\varepsilon) + \varepsilon|D\varphi(x_\varepsilon)| + \frac{\varepsilon^2}{2} D^2\varphi(x_\varepsilon)\frac{D\varphi(x_\varepsilon)}{|D\varphi(x_\varepsilon)|} \cdot \frac{D\varphi(x_\varepsilon)}{|D\varphi(x_\varepsilon)|} + o(\varepsilon^2),$$

$$\min_{B_\varepsilon(x_\varepsilon)} \varphi = \varphi(x_\varepsilon) - \varepsilon|D\varphi(x_\varepsilon)| + \frac{\varepsilon^2}{2} D^2\varphi(x_\varepsilon)\frac{D\varphi(x_\varepsilon)}{|D\varphi(x_\varepsilon)|} \cdot \frac{D\varphi(x_\varepsilon)}{|D\varphi(x_\varepsilon)|} + o(\varepsilon^2).$$

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Substituting these to (5), dividing by ε^2 and letting $\varepsilon \to 0$ yields

$$0 \leq D^2\varphi(x)\frac{D\varphi(x)}{|D\varphi(x)|} \cdot \frac{D\varphi(x)}{|D\varphi(x)|} = \Delta_\infty^S \varphi(x).$$

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