Notes on action selectors

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1 Notations, conventions and known results

Let (M,ω) be a closed symplectic manifold such that $\omega|_{\pi_2(M)}=0$. Let J be a smooth ω-compatible almost complex structure on M, meaning that

$$g_J(\xi,\eta):=\omega(J\xi,\eta), \qquad \forall \xi,\eta\in T_xM,\ \forall x\in M,$$

is a Riemannian metric on M. The associated norm is denoted by $|\cdot|_J$. We denote by X_H the Hamiltonian vector field associated to a Hamiltonian $H\in C^\infty(M)$, that is

$$\omega(X_H,\cdot)=dH.$$

The Hamiltonian action functional on the space of contractible loops $C^\infty_{\mathrm{contr}}(\mathbb{T},M)$ associated to a Hamiltonian $H\in C^\infty(\mathbb{T}\times M)$ has the form

$$\mathbb{A}_H(x):=\int_{\mathbb{D}}\bar{x}^*(\omega)+\int_{\mathbb{T}}H(t,x(t))\,dt,$$

where $\bar{x}\in C^\infty(\mathbb{D},M)$ is such that $\bar{x}|_{\partial\mathbb{D}}=x$. The first integral does not depend on the choice of the extension $\bar{x}$ of x because ω vanishes on $\pi_2(M)$. The critical points of $\mathbb{A}_H$ are precisely the elements of $\mathscr{P}(H)$, the set of contractible 1-periodic orbits of X_H. By Ascoli-Arzelà theorem, $\mathscr{P}(H)$ is a compact subset of $C^\infty_{\mathrm{contr}}(\mathbb{T},M)$.

The space $C^\infty(\mathbb{R}\times\mathbb{T},M)$ is endowed with the C^∞_{loc} topology, which is metrizable and complete. We shall identify $C^\infty(\mathbb{R}\times\mathbb{T},M)$ with $C^\infty(\mathbb{R},C^\infty(\mathbb{T},M))$ and we use the notation

$$u(s)=u(s,\cdot)\in C^\infty(\mathbb{T},M), \qquad \forall s\in\mathbb{R}.$$

The additive group $\mathbb{R}$ acts on $\mathbb{C}^\infty(\mathbb{R}\times\mathbb{T},M)$ by translations

$$(\sigma,u)\mapsto\tau_\sigma u, \qquad \text{where } (\tau_\sigma u)(s):=u(\sigma+s).$$

The L^2-negative gradient equation for the functional $\mathbb{A}_H$ is the Floer equation

$$\partial_s u+J(u)\big(\partial_t u-X_H(t,u)\big)=0. \tag{1}$$

If u is a solution of (1), then the function $s\mapsto\mathbb{A}_H(u(s,\cdot))$ is decreasing and

$$\lim_{s\to-\infty}\mathbb{A}_H(u(s,\cdot))-\lim_{s\to+\infty}\mathbb{A}_H(u(s,\cdot))=E(u):=\int_{\mathbb{R}\times\mathbb{T}}\big|\partial_s u\big|_J^2\,ds\,dt.$$

It might also be useful to recall that u is a solution of (1) and $\frac{d}{ds}\mathbb{A}_H(u(s,\cdot))\big|_{s=s_0}=0$ for some $s_0\in\mathbb{R}$, then u is a trivial cylinder, meaning that $u(s,t)=x(t)$ with $x\in\mathscr{P}(H)$ (see [HZ94, Chapter 6, Lemma 2]).

Now let $H\in C^\infty(\mathbb{R}\times\mathbb{T}\times M)$ be such that $\partial_s H$ has compact support and set

$$H_-(t,x):=H(-s,t,x) \quad\text{and}\quad H_+(t,x):=H(s,t,x) \quad \text{for } s \text{ large}.$$

If u solves the s-dependent Floer equation

$$\partial_s u + J(u)\big(\partial_t u - X_H(s,t,u)\big) = 0, \tag{2}$$

then $s \mapsto \mathbb{A}_{H(s,\cdot,\cdot)}(u(s,\cdot))$ is decreasing for $|s|$ large and, for every $s_0 < s_1$,

$$\mathbb{A}_{H(s_0,\cdot,\cdot)}(u(s_0)) - \mathbb{A}_{H(s_1,\cdot,\cdot)}(u(s_1)) = \int_{[s_0,s_1]\times\mathbb{T}} \left|\partial_s u\right|_J^2 \, ds\, dt - \int_{[s_0,s_1]\times\mathbb{T}} \partial_s H(s,t,u(s,t))\, ds\, dt. \tag{3}$$

It follows that

$$\lim_{s\to-\infty} \mathbb{A}_{H_-}(u(s,\cdot)) - \lim_{s\to+\infty} \mathbb{A}_{H_+}(u(s,\cdot)) = E(u) - \int_{\mathbb{R}\times\mathbb{T}} \partial_s H(s,t,u(s,t))\, ds\, dt.$$

Set

$$\mathscr{U}(H) := \{u \in C^\infty(\mathbb{R}\times\mathbb{T}, M) \mid u \text{ is a solution of (2) with } E(u) < \infty\}.$$

We recall that a subset $\mathscr{U}$ of $C^\infty(\mathbb{R}\times\mathbb{T}, M)$ is said to be bounded if for every multi-index $\alpha \in \mathbb{N}^2$, $|\alpha| \geq 1$, there holds

$$\sup_{u\in\mathscr{U}} |\partial_s^{\alpha_1}\partial_t^{\alpha_2} u|_J < \infty.$$

Bounded subsets are relatively compact in the C^∞_{loc} topology. The following result is a simple generalization of [HZ94, Chapter 6, Corollary 1 and Proposition 11]:

Proposition 1.1. *Let $H \in C^\infty(\mathbb{R}\times\mathbb{T}\times M)$ be such that $\partial_s H$ has compact support. Then $\mathscr{U}(H)$ is a bounded closed subset of $C^\infty(\mathbb{R}\times\mathbb{T}, M)$. Moreover, if $u \in \mathscr{U}(H)$ then the set*

$$\alpha\text{-}\lim(u) := \left\{ \lim_{n\to\infty} \tau_{s_n} u \mid s_n \to -\infty \textit{ is such that } \tau_{s_n} u \textit{ converges} \right\}$$

is a non-empty, compact and connected subset of $\mathscr{U}(H_-)$ which consists of trivial cylinders of the form $v(s,t) = x(t)$ for $x \in \mathscr{P}(H_-)$.

Given $z \in \mathbb{R}\times\mathbb{T}$, we denote by

$$\mathrm{ev}_z : C^\infty(\mathbb{R}\times\mathbb{T}, M) \to M, \qquad \mathrm{ev}_z(u) := u(z)$$

the evaluation map at z. Denote by $\check{H}^*$ the Alexander-Spanier cohomology functor. The following result is proved by a simple modification of the arguments of [HZ94, Chapter 6, pages 245-249]:

Proposition 1.2. *For every $z \in \mathbb{R}\times\mathbb{T}$ the homorphism*

$$\left(\mathrm{ev_z}|_{\mathscr{U}(H)}\right)^* : H^*(M) \cong \check{H}^*(M) \to \check{H}^*(\mathscr{U}(H))$$

which is induced by the map

$$\mathrm{ev_z}|_{\mathscr{U}(H)} : \mathscr{U}(H) \to M$$

is injective. In particular, the above map is surjective.

Propositions 1.1 and 1.2 can be applied in particular to Hamiltonians $H \in C^\infty(\mathbb{T}\times M)$ which do not depend on s. It is in this generality that they are proved in [HZ94, Chapter 6]. In this case, $\mathscr{U}(H)$ is translation-invariant. Translations define a continuous flow on the metric space $\mathscr{U}(H)$

$$\tau : \mathbb{R}\times\mathscr{U}(H) \to \mathscr{U}(H), \qquad \tau(\sigma, u) := \tau_\sigma u,$$

whose equilibrium points are precisely the trivial cylinders $u(s,t) = x(t)$ with $x \in \mathscr{P}(H)$. The flow τ is gradient-like with respect to the continuous function

$$a_H : \mathscr{U}(H) \to \mathbb{R}, \qquad a(u) := \mathbb{A}_H(u(0,\cdot)),$$

meaning that

$$a_H\big(\tau_{\sigma'} u\big) \leq a_H\big(\tau_\sigma u\big), \qquad \text{if } \sigma' > \sigma,$$

and the inequality is strict whenever u is not an equilibrium point of the flow.

We recall the following general fact about gradient-like flows:

Theorem 1.3. *Let $\mathscr{U}$ be a compact metric space and let τ be a continuous flow on $\mathscr{U}$ which is gradient-like with respect to the continuous function $a : \mathscr{U} \to \mathbb{R}$. Then for every $\alpha \in \check{H}^*(\mathscr{U})$, $\alpha \neq 0$, the level*

$$c(\alpha, a, \mathscr{U}) := \inf \left\{ \lambda \in \mathbb{R} \mid \text{the image of } \alpha \text{ by the homorphism } \check{H}^*(\mathscr{U}) \to \check{H}^*(\{a < \lambda\}) \text{ is non-zero} \right\}.$$

is such that there exists an equilibrium point $x \in \mathscr{U}$ for τ for which $a(x) = c(\alpha)$. Furthermore, τ has at least

$$\text{cup-length}\,(\mathscr{U}) + 1$$

equilibrium points.

needs clarification

Remark 1.4. *In particular,*

$$c(1, a, \mathscr{U}) = \min_{\mathscr{U}} a.$$

There are also other possible definitions for critical levels which are associated to cohomology classes (as sup of something). But maybe they can be reduced to these ones by considering $-a$.

2 The minimal selector

Let $H \in C^\infty(\mathbb{T} \times M)$ be a Hamiltonian. We would like to define an action selector for the Hamiltonian H.

The definition. Let $\mathscr{K}(H)$ be the set of s-dependent Hamiltonians $K \in C^\infty(\mathbb{R} \times \mathbb{T} \times M)$ such that $\partial_s K$ has compact support and $K_- = H$.

Let $K \in \mathscr{K}(H)$ and assume that $\partial_s K$ is supported in $[-s_0, s_0] \times \mathbb{T} \times M$. If $u \in \mathscr{U}(K)$, then the function $s \mapsto \mathbb{A}_H(u(s))$ is decreasing and bounded on $(-\infty, -s_0]$. Therefore, the function

$$a_H^- : \mathscr{U}(K) \to \mathbb{R}, \qquad a_H^-(u) := \lim_{s \to -\infty} \mathbb{A}_H(u(s)) = \sup_{s \leq -s_0} \mathbb{A}_H(u(s)),$$

is well-defined. Being the supremum of a family of continuous functions, the function a_H^- is lower semi-continuous. As such, it has a minimum on the compact space $\mathscr{U}(K)$.

Definition 2.1. *Let $H \in C^\infty(\mathbb{T} \times M)$ and $K \in \mathscr{K}(H)$. We set*

$$A^-(K) := \min_{u \in \mathscr{U}(K)} a_H^-(u), \qquad A(H) := \sup_{K \in \mathscr{K}(H)} A^-(K).$$

The number $A^-(K)$ is a critical value of $\mathbb{A}_H$. Indeed, if $a_H^-(u) = A^-(K)$ then every v in the set α-lim(u) (see Proposition 1.1) is of the form $v(s,t) = x(t)$ with $x \in \mathscr{P}(H)$ and $a_H^-(u) = \mathbb{A}_H(x)$. Since the set of critical values of $\mathbb{A}_H$ is closed, also $A(H)$ is a critical value of the same functional. We refer to the function

$$A : C^\infty(\mathbb{T} \times M) \to \mathbb{R}$$

as to the minimal action selector.

First properties. The first easy properties of the action selector A are:

$$A(H) = 0, \qquad \text{if } H \equiv 0, \tag{4}$$

$$A(H + c) = A(H) + c, \qquad \forall c \in \mathbb{R},\ \forall H \in C^\infty(\mathbb{T} \times M). \tag{5}$$

Indeed, the first property follows from the fact that for the Hamiltonian $H \equiv 0$, $\mathscr{P}(H)$ consists of all the constant loops, which have action zero. The second property follows from the identities $a_{H+c}^- = a_H^- + c$ and $\mathscr{K}(H + c) = \mathscr{K}(H) + c$. Less trivial is the following:

Proposition 2.2 (Monotonicity)**.** *If $H_0, H_1 \in C^\infty(\mathbb{T} \times M)$ are such that $H_0 \geq H_1$, then $A(H_0) \geq A(H_1)$.*

Proof. Fix $\epsilon > 0$. We shall prove that

$$\sup_{K_0 \in \mathscr{K}(H_0)} \min_{\mathscr{U}(K_0)} a_{H_0}^- \geq \sup_{K_1 \in \mathscr{K}(H_1)} \min_{\mathscr{U}(K_1)} a_{H_1}^- - \epsilon, \tag{6}$$

and the thesis will follow from the arbitrariness of ϵ. Proving (6) is equivalent to showing that for every K_1 in $\mathscr{K}(H_1)$ there exists K_0 in $\mathscr{K}(H_0)$ such that

$$\min_{\mathscr{U}(K_0)} a_{H_0}^- \geq \min_{\mathscr{U}(K_1)} a_{H_1}^- - \epsilon. \tag{7}$$

Up to a translation, we may assume that

$$K_1(s,t,x) = H_1(t,x), \qquad \forall s \leq 0.$$

Let $\varphi \in C^\infty(\mathbb{R})$ be a real function such that $\varphi' \geq 0$, $\varphi(s) = 0$ for $s \leq 0$ and $\varphi(s) = 1$ for $s \geq 1$. For $\lambda \in \mathbb{R}$ we define $K_0^\lambda \in \mathscr{K}(H_0)$ by

$$K_0^\lambda(s,t,x) := \varphi(s-\lambda)K_1(s,t,x) + \big(1 - \varphi(s)\big)H_0(t,x).$$

We claim that there exists $\lambda \leq -1$ such that (7) holds with $K_0 = K_0^\lambda$. Arguing by contradiction, we assume that for every $\lambda \leq -1$ there is a u_λ in $\mathscr{U}(K_0^\lambda)$ such that

$$a_{H_0}^-(u_\lambda) < \min_{\mathscr{U}(K_1)} a_{H_1}^- - \epsilon. \tag{8}$$

Let $(\lambda_n) \subset (-\infty, -1]$ be such that $\lambda_n \to -\infty$. Up to the replacement of (λ_n) with a subsequence, we may assume that (u_{λ_n}) converges to some u in $\mathscr{U}(K_1)$.

We fix a number $s \leq 0$. If $\lambda_n \leq s - 1$, then by the action-energy identity (3)

$$\begin{aligned}
&a_{H_0}^-(u_{\lambda_n}) \geq \mathbb{A}_{H_0}\big(u_{\lambda_n}(\lambda_n)\big) \\
&= \mathbb{A}_{H_1}\big(u_{\lambda_n}(s)\big) + \int_{[\lambda_n,s]\times\mathbb{T}} \big|\partial_s u_{\lambda_n}\big|_J^2\,ds\,dt - \int_{[\lambda_n,s]\times\mathbb{T}} \varphi'(s-\lambda_n)(H_1 - H_0)(t,u_{\lambda_n})\,ds\,dt \\
&\geq \mathbb{A}_{H_1}\big(u_{\lambda_n}(s)\big),
\end{aligned}$$

where we have used the hypothesis $H_1 - H_0 \leq 0$. By taking the limit for $n \to \infty$, we deduce that

$$\liminf_{n\to\infty} a_{H_0}^-(u_{\lambda_n}) \geq \mathbb{A}_{H_1}\big(u(s)\big),$$

and by taking the supremum over all $s \leq 0$,

$$\liminf_{n\to\infty} a_{H_0}^-(u_{\lambda_n}) \geq a_{H_1}^-(u).$$

Together with (8), this implies the chain of inequalities

$$a_{H_1}^-(u) \leq \liminf_{n\to\infty} a_{H_0}^-(u_{\lambda_n}) \leq \min_{\mathscr{U}(K_1)} a_{H_1}^- - \epsilon,$$

which is the desired contradiction because $u \in \mathscr{U}(K_1)$. $\square$

Remark 2.3. *The hypothesis of Proposition 2.2 can be somehow weakened: it is enough to assume that*

$$\int_{\mathbb{T}} \max_{x\in M}\big(H_1(t,x) - H_0(t,x)\big)\,dt \leq 0.$$

Can this be useful? For instance, this implies that the averaging process

$$\overline{H}(x) := \int_{\mathbb{T}} H(t,x)\,dt, \qquad \forall x \in M,$$

does not change the value of the action selector: $A(\overline{H}) = A(H)$.

Monotonicy and property (5) have the following standard consequence:

Proposition 2.4. *The action selector A is 1-Lipschitz with respect to the sup-norm on $C^\infty(\mathbb{T}\times M)$:*

$$\big|A(H_1)-A(H_0)\big| \leq \|H_1-H_0\|_\infty, \qquad \forall H_0,H_1\in C^\infty(\mathbb{T}\times M).$$

Proof. If we apply Proposition 2.2 to the inequalities

$$H_0-\|H_1-H_0\|_\infty \leq H_1 \leq H_0+\|H_1-H_0\|_\infty,$$

we find

$$A\big(H_0-\|H_1-H_0\|_\infty\big) \leq A(H_1) \leq A\big(H_0+\|H_1-H_0\|_\infty\big).$$

Together with (5), these inequalities take the form

$$A(H_0)-\|H_1-H_0\|_\infty \leq A(H_1) \leq A(H_0)+\|H_1-H_0\|_\infty,$$

which is equivalent to the thesis. □

Remark 2.5. *The strongest version of Proposition 2.2 mentioned in Remark 2.3 implies the finer estimates*

$$\big|A(H_1)-A(H_0)\big| \leq \sup_{x\in M}\left|\int_{\mathbb{T}}\big(H_1(t,x)-H_0(t,x)\big)\,dt\right| \leq \int_{\mathbb{T}}\sup_{x\in M}\big|H_1(t,x)-H_0(t,x)\big|\,dt.$$

An equivalent definition. Given $K\in\mathscr{K}(H)$, consider the set

$$\mathscr{U}_{\mathrm{ess}}(K) := \Big\{u\in C^\infty(\mathbb{R}\times\mathbb{T},M) \mid u=\lim_{n\to\infty}\tau_{s_n}u_n \text{ where } s_n\to-\infty \text{ and } (u_n)\subset\mathscr{U}(K)\Big\}.$$

As we show below, $\mathscr{U}_{\mathrm{ess}}(K)$ is a compact τ-invariant subspace of $\mathscr{U}(H)$. It is the space of cylinders in $\mathscr{U}(H)$ which are essential with respect to K, in the sense that they survive through the homotopy K. We shall prove that the minimal action selector A can be expressed as:

$$A(H) = \sup_{K\in\mathscr{K}(H)}\ \min_{\mathscr{U}_{\mathrm{ess}}(K)} a_H. \tag{9}$$

We begin with the following:

Proposition 2.6. *The set $\mathscr{U}_{\mathrm{ess}}(K)$ is a compact τ-invariant subspace of $\mathscr{U}(H)$.*

Proof. Let $u=\lim\tau_{s_n}u_n$ be an element of $\mathscr{U}_{\mathrm{ess}}(K)$. Since $v=\tau_{s_n}u_n$ solves the equation

$$\partial_s v + J(v)\big(\partial_t v - X_{\tau_{s_n}K}(s,t,v)\big)=0,$$

and $\tau_{s_n}K$ converges to $K_-=H$, u is a solution of the s-independent Floer equation defined by H. Moreover,

$$E(u)\leq\liminf_{n\to\infty}E(\tau_{s_n}u_n)=\liminf_{n\to\infty}E(u_n)\leq\sup_{v\in\mathscr{U}(K)}E(v)<+\infty.$$

Therefore, $\mathscr{U}_{\mathrm{ess}}(K)$ is contained in $\mathscr{U}(H)$. If $\sigma\in\mathbb{R}$, then

$$\tau_\sigma u=\lim_{n\to\infty}\tau_{s_n+\sigma}u_n$$

is in $\mathscr{U}_{\mathrm{ess}}(K)$, which is then τ-invariant. If

$$v^h=\lim_{n\to\infty}\tau_{s_n^h}u_n^h, \qquad \text{where} \quad \lim_{n\to\infty}s_n^h=-\infty, \quad \forall h\in\mathbb{N},$$

and (v^h) converges to $v\in C^\infty(\mathbb{R}\times\mathbb{T},M)$, a standard diagonal argument implies the existence of a diverging sequence $(n_h)\subset\mathbb{N}$ such that

$$\lim_{h\to\infty}\operatorname{dist}\big(\tau_{s^h_{n_h}}u^h_{n_h},v^h\big)=0, \qquad \lim_{h\to\infty}s^h_{n_h}=-\infty,$$

where dist is a distance on $C^\infty(\mathbb{R}\times\mathbb{T},M)$. Therefore $\tau_{s^h_{n_h}}u^h_{n_h}$ converges to v, which hence belongs to $\mathscr{U}_{\mathrm{ess}}(K)$. This shows that $\mathscr{U}_{\mathrm{ess}}(K)$ is a closed subspace of $\mathscr{U}(H)$. Since the latter space is compact, so is $\mathscr{U}_{\mathrm{ess}}(K)$. □

Formula (9) is an immediate consequence of the following:

Proposition 2.7. $A^-(K) = \min_{\mathscr{U}(K)} a_H^- = \min_{\mathscr{U}_{\mathrm{ess}}(K)} a_H$.

Proof. Let $u \in \mathscr{U}(K)$ be a minimizer of a_H^- and let $v(s,t) = x(t)$, $x \in \mathscr{P}(H)$, be an element of α-$\lim(u)$. Then

$$a_H(v) = \mathbb{A}_H(x) = a_H^-(u),$$

so

$$\min_{\mathscr{U}_{\mathrm{ess}}(K)} a_H \leq a_H(v) = a_H^-(u) = \min_{\mathscr{U}(K)} a_H^-.$$

Conversly, let $v \in \mathscr{U}_{\mathrm{ess}}(K)$ be a minimizer of a_H. Then

$$v = \lim_{n\to\infty} \tau_{s_n} u_n, \qquad \text{where } s_n \to -\infty \text{ and } (u_n) \subset \mathscr{U}(K).$$

Up to a subsequence, we may assume that (u_n) converges to some $u \in \mathscr{U}(K)$. If s is small enough, then

$$\mathbb{A}_H(u(s)) = \lim_{n\to\infty} \mathbb{A}_H(u_n(s)) \leq \lim_{n\to\infty} \mathbb{A}_H(u_n(s_n)) = \lim_{n\to\infty} a_H(\tau_{s_n} u_n) = a_H(v).$$

By taking the limit for $s \to -\infty$, we find

$$a_H^-(u) \leq a_H(v),$$

which implies that

$$\min_{\mathscr{U}(K)} a_H^- \leq a_H^-(u) \leq a_H(v) = \min_{\mathscr{U}_{\mathrm{ess}}(K)} a_H.$$

□

The space $\mathscr{U}_{\mathrm{ess}}(K)$ satisfies the analogue of Proposition 1.2:

Proposition 2.8. *For every $z \in \mathbb{R} \times \mathbb{T}$ the homorphism*

$$\left(\mathrm{ev_z}|_{\mathscr{U}_{\mathrm{ess}}(K)}\right)^* : H^*(M) \cong \check{H}^*(M) \to \check{H}^*(\mathscr{U}_{\mathrm{ess}}(K))$$

which is induced by the map

$$\mathrm{ev_z}|_{\mathscr{U}_{\mathrm{ess}}(K)} : \mathscr{U}_{\mathrm{ess}}(K) \to M$$

is injective. In particular, the above map is surjective.

Proof. Let $\mathscr{W} \subset C^\infty(\mathbb{R} \times \mathbb{T}, M)$ be a neighborhood of $\mathscr{U}_{\mathrm{ess}}(K)$. We claim that if $s_0 \in \mathbb{R}$ is small enough then $\tau_{s_0}\mathscr{U}(K) \subset \mathscr{W}$: if not, we could find sequences $s_n \to -\infty$ and $(u_n) \subset \mathscr{U}(K)$ such that $\tau_{s_n} u_n$ is not in $\mathscr{W}$. But by compactness, $(\tau_{s_n} u_n)$ has a converging subsequence, whose limit is by definition an element of $\mathscr{U}_{\mathrm{ess}}$. Therefore this subsequence must eventually belong to $\mathscr{W}$, which is a contradiction.

If s_0 is as above, we denote by

$$i : \mathscr{U}_{\mathrm{ess}} \hookrightarrow \mathscr{W} \quad \text{and} \quad j : \tau_{s_0}\mathscr{U}(K) \hookrightarrow \mathscr{W}$$

the inclusion mappings. Then, if $z = (s,t) \in \mathbb{R} \times \mathbb{T}$ and $z' = (s + s_0, t)$, we get the commutative diagram

$$\begin{tikzcd}
\check{H}^*(\mathscr{W}) \arrow[r, "j^*"] \arrow[d, "i^*"'] & \check{H}^*\big(\tau_{s_0}\mathscr{U}(K)\big) \arrow[r, "\tau_{s_0}^*"] & \check{H}^*\big(\mathscr{U}(K)\big) \\
\check{H}^*\big(\mathscr{U}_{\mathrm{ess}}(K)\big) & \check{H}^*(M) \arrow[l, "\mathrm{ev}^*"'] \arrow[ul, "\mathrm{ev}_z^*"'] \arrow[u, "\mathrm{ev}_z^*"'] \arrow[ur, "\mathrm{ev}_{z'}^*"']
\end{tikzcd}$$

Proposition 1.2 and the fact that $\tau_{s_0}^*$ is an isomorphism imply that $(\mathrm{ev}_z|_{\tau_{s_0}\mathscr{U}(K)})^*$ is injective. Then so is $(\mathrm{ev}_z|_{\mathscr{W}})^*$. Since this is true for every neighborhood $\mathscr{W}$ of $\mathscr{U}_{\mathrm{ess}}(K)$, the continuity of Alexander-Spanier cohomology implies that the homomorphism $(\mathrm{ev}_z|_{\mathscr{U}_{\mathrm{ess}}(K)})^*$ is injective. □

Autonomous Hamiltonians. Let $H \in C^\infty(H)$ be an autonomous Hamiltonian. In this case, the critical points of H are the constants orbits of X_H, and in particular they are elements of $\mathscr{P}(H)$. Moreover, the solutions of the Floer equation (1) which do not not depend on t, that is $u(s,t) = u(s)$, are the solutions of the ODE

$$u'(s) + \nabla H(u(s)) = 0,$$

so they are the negative-gradient flow lines of H. In general, the vector field X_H could have other non-constant contractible orbits and the Floer equation could have other t-dependent bounded energy solutions, but when this does not happen it is easy to calculate the value of the minimal action selector:

Proposition 2.9. *Let $H \in C^\infty(H)$ be an autonomous Hamiltonian. Assume that $\mathscr{P}(H)$ consists only of constant orbits and that every solution of the Floer equation (1) with bounded energy does not depend on t. Then*

$$A(H) = \min_M H.$$

Proof. Since $\min_M H$ is the minimal critical value of $\mathbb{A}_H$, we must show that for every $K \in \mathscr{K}(H)$ there holds $A^-(K) \leq \min_M H$. Assume by contradiction that there is a $K \in \mathscr{K}(H)$ with

$$A^-(K) > \min_M H.$$

By the hypothesis that $\mathscr{U}(H)$ is the set of gradient flow lines of H and by the characterization of $A^-(K)$ of Proposition 2.7, we deduce that $\mathscr{U}_{\text{ess}}(K)$ is a set of gradient flow lines of H which is contained in

$$\big\{x \in M \mid H(x) \geq A^-(K)\big\} \subsetneq M.$$

But this violates the surjectivity of the map $\mathrm{ev}_z|_{\mathscr{U}_{\text{ess}}(K)}$ of Proposition 2.8. □

Remark 2.10. *The hypothesis of the above proposition are satisfied when the autonomous Hamiltonian H is a Morse function, with a Morse-Smale gradient, and with dH which is C^1-small. Is it true also if we just ask that dH is C^1-small?* needs clarification

The same conclusion holds under the following assumptions, which are neither stronger nor weaker than those considered above:

Proposition 2.11. *Let $H \in C^\infty(H)$ be an autonomous Hamiltonian with exactly two critical values. Assume also that X_H has no non-constant contractible 1-periodic orbits. Then*

$$A(H) = \min_M H.$$

Proof. In this case $\mathbb{A}_H$ has exactly two critical values, $\min_M H$ and $\max_M H$. We must exclude that for some $K \in \mathscr{K}(H)$ the minimum of a_H on $\mathscr{U}_{\text{ess}}(K)$ is $\max_M H$. But since

$$\max_{\mathscr{U}_{\text{ess}}(K)} a_H \leq \max_{\mathscr{U}(H)} a_H = \max_{\mathscr{P}(H)} \mathbb{A}_H = \max_M H,$$

the latter fact forces $\mathscr{U}_{\text{ess}}(K)$ to consist only of the constant cylinders defined by the maximum points of H. This violates the surjectivity of the evaluation map $\mathrm{ev}_z|_{\mathscr{U}_{\text{ess}}(K)}$ of Proposition 2.8. □

Remark 2.12. *Is the following fact true? If the C^1 norm of the smooth vector field X on the compact manifold M is small enough, then X has no non-constant closed orbits of period 1 or less.* needs clarification

Remark 2.13. *One could also restrict the class $\mathscr{K}(H)$ by requiring that $K(s,t) = 0$ for s large enough. In this case, removal of singularities shows that we are actually dealing with open discs (or equivalently, punctured spheres), which are J-holomorphic near the origin and satisfy the Floer equation on a collar of the boundary, which is equipped with cylindrical coordinates. These are exactly the objects which are used in the PSS isomorphism. Notice also that the proof of Proposition 1.2 requires seeing the full cylinders as limits of longer and longer sausages: in the latter approach, capping at the positive end becomes unnecessary.* what else should we include here?

3 More general action selectors

Let $H \in C^\infty(\mathbb{T} \times M)$ and $K \in \mathscr{K}(H)$. Let ξ be a non-zero cohomology class in $H^*(M) \cong \check{H}^*(M)$. By Proposition 1.2, the cohomology mclass

$$\xi_K := \big(\mathrm{ev}|_{\mathscr{U}(K)}\big)^* \xi \in \check{H}^*\big(\mathscr{U}(K)\big),$$

is non-zero.

Definition 3.1. *Set*

$$A^-(\xi, K) := \inf \Big\{ a \in \mathbb{R} \ \Big| \ \textit{the image of } \xi_K \textit{ by the homomorphism} \\ \check{H}^*(\mathscr{U}(K)) \to \check{H}^*\big(\{u \in \mathscr{U}(K)) \mid a_H^-(u) < a\}\big) \textit{ is non-zero} \Big\},$$

and

$$A(\xi, H) := \sup_{K \in \mathscr{K}(H)} A^-(\xi, K).$$

By the functoriality of $\check{H}^*$, the set which appears in the definition of $A^-(\xi, K)$ is an interval which is unbounded from above.

Let us check that $A^-(\xi, K)$ is a critical value of $\mathbb{A}_H$. Since $a_H^-(\mathscr{U}(K))$ is contained in the set of critical values of $\mathbb{A}_H$ and the latter set is closed, it is enough to prove that

$$A^-(\xi, K) \in \overline{a_H^-(\mathscr{U}(K))}.$$

Arguing by contradiction, we can find $\epsilon > 0$ such that

$$\big[A^-(\xi, K) - \epsilon, A^-(\xi, K) + \epsilon\big] \cap a_H^-(\mathscr{U}(K)) = \emptyset.$$

Therefore,

$$\big\{u \in \mathscr{U}(K) \mid a_H^-(u) < A^-(\xi, K) + \epsilon\big\} = \big\{u \in \mathscr{U}(K) \mid a_H^-(u) < A^-(\xi, K) - \epsilon\big\}.$$

The above identity contradicts the definition of $A^-(\xi, K)$, which requires the image of ξ_K by the homomorphism

$$\check{H}^*\big(\mathscr{U}(K)\big) \to \check{H}^*\Big(\big\{u \in \mathscr{U}(K) \mid a_H^-(u) < A^-(\xi, K) + \epsilon\big\}\Big)$$

to be non-zero and its image by the homomorphism

$$\check{H}^*\big(\mathscr{U}(K)\big) \to \check{H}^*\Big(\big\{u \in \mathscr{U}(K) \mid a_H^-(u) < A^-(\xi, K) - \epsilon\big\}\Big)$$

to be zero.

In the particular case $\xi = 1$, we have

$$A^-(1, K) = \min_{u \in \mathscr{U}(K)} a_H^-(u),$$

and we find the minimal action selector of the previous section: $A(1, H) = A(H)$. to be completed

References

[HZ94] H. Hofer and E. Zehnder, *Symplectic invariants and Hamiltonian dynamics*, Birkhäuser, Basel, 1994.