Persistence modules in symplectic topology

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based on joint works with Egor Shelukhin, Vukašin Stojisavljević

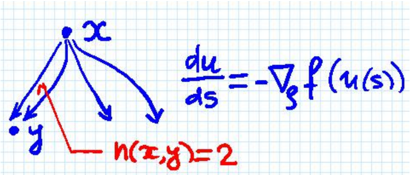
and a survey (in progress) with Jun Zhang

Morse homology

f-Morse function, ρ-generic metric, $\mathcal{F}$-field.

Filtered complex: $C_t = \mathcal{F} \cdot \mathrm{Crit}_t(f)$ - span of critical points x of f with value $f(x) < t$, $t \in \mathbb{R}$.

Differential: $d : C_t \to C_t$, $dx = \sum n(x, y)y$, where $n(x, y)$-number of gradient lines of f connecting x and y.



Problem: Find homology of (C_t, d) with computer.

Difficulty: Count orbits connecting approximate crit. pts.?

New approach needed!

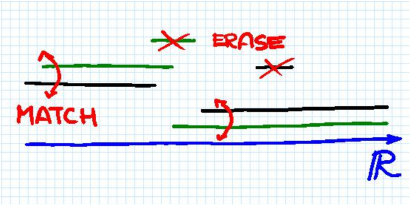
Edelsbrunner, Harer, Carlsson,... Last decade in the context of topological data analysis.

Barcode $\mathcal{B} = \{I_j, m_j\}$-finite collection of intervals I_j with multiplicities m_j, $I_j = (a_j, b_j]$, $a_j < b_j \leq +\infty$.

Bottleneck distance between barcodes: $\mathcal{B}, \mathcal{C}$ are δ-matched, $\delta > 0$ if after erasing some intervals in $\mathcal{B}$ and $\mathcal{C}$ of length $< 2\delta$ we can match the rest in 1-to-1 manner with error at most δ at each end-point.

$$d_{bot}(\mathcal{B}, \mathcal{C}) = \inf \delta \ .$$

Figure: Matching

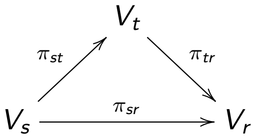


Persistence modules

$\mathcal{F}$ – a field.

Persistence module: a pair (V, π), where V_t, $t \in \mathbb{R}$ are $\mathcal{F}$-vector spaces, $\dim V_t < \infty$, $V_s = 0$ for all $s \ll 0$.

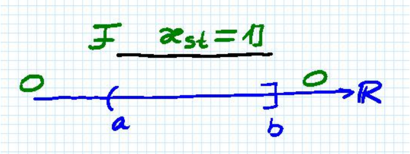
$\pi_{st} : V_s \to V_t$, $s < t$ linear maps: $\forall s < t < r$



Regularity: For all but finite number of jump points $t \in \mathbb{R}$, there exists a neighborhood U of t such that π_{sr} is an isomorphism for all $s, r \in U$. Extra assumption ("semicontinuity") at jump points.

Interval module $(\mathcal{F}(a,b],\kappa)$, $a \in \mathbb{R}$, $b \in \mathbb{R} \cup +\infty$:
$\mathcal{F}(a,b]_t = \mathcal{F}$ for $t \in (a,b]$ and $\mathcal{F}(a,b]_t = 0$ otherwise;
$\kappa_{st} = \mathbb{1}$ for $s,t \in (a,b]$ and $\kappa_{st} = 0$ otherwise.

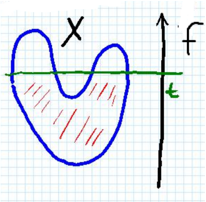
Figure: Interval module



Structure theorem: For every persistence module (V,π) there exists unique barcode $\mathcal{B}(V) = \{(I_j, m_j)\}$ such that $V = \bigoplus \mathcal{F}(I_j)^{m_j}$.

X-closed manifold, $f : X \to \mathbb{R}$-Morse function.
Persistence module $V_t(f) := H_*(\{f < t\}, \mathcal{F})$. The persistence morphisms are induced by the inclusions
$\{f < s\} \hookrightarrow \{f < t\}, \;\; s < t$.

Figure: Sublevels



Robustness: $||f|| := \max |f|$-uniform norm.
$(C^\infty(X), ||\cdot||) \to (\text{Barcodes}, d_{bot}), \;\; f \mapsto \mathcal{B}(V(f))$ is Lipshitz.

Symplectic preliminaries

(M^{2n}, ω)–symplectic manifold
ω– **symplectic form**. Locally $\omega = \sum_{i=1}^{n} dp_i \wedge dq_i$.
Examples of closed symplectic manifolds:

- Surfaces with area forms;
- Products.

M-phase space of mechanical system. **Energy determines evolution:** $F : M \times [0,1] \to \mathbb{R}$ – Hamiltonian function (energy).
Hamiltonian system:

$$\begin{cases} \dot{q} = \frac{\partial F}{\partial p} \\ \dot{p} = -\frac{\partial F}{\partial q} \end{cases}$$

Family of **Hamiltonian diffeomorphisms**

$$f_t : M \to M, \quad (p(0), q(0)) \mapsto (p(t), q(t))$$

Key feature: $\phi_t^* \omega = \omega$.

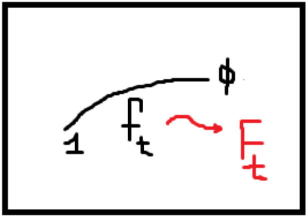
Hamiltonian diffeomorphisms

(M, ω)-closed symplectic manifold. $Ham(M, \omega)$ - group of Hamiltonian diffeomorphisms.

$Ham \subset Symp_0$. $Ham = Symp_0$ if $H^1(M, \mathbb{R}) = 0$.

Hofer's length: For a Hamiltonian path $\alpha = \{f_t\}$, $f_0 = \mathbb{1}$, $f_1 = \phi$ length$(\alpha) = \int_0^1 ||F_t||\, dt$, where F_t - normalized (zero mean) Hamiltonian of α.

Figure: Path α



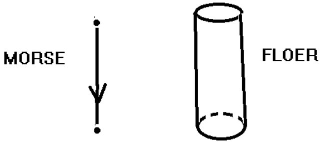
Hofer's metric

Put $d_H(\mathbb{1}, \phi) = \inf_\alpha \text{length}(\alpha)$, where α-path between $\mathbb{1}$ and ϕ.
$d_H(\phi, \psi) := d_H(\mathbb{1}, \phi\psi^{-1})$ - Hofer's metric, 1990

- non-degenerate Hofer, P., Viterbo, Lalonde-McDuff
- biinvariant
- essentially unique non-degenerate Finsler metric on *Ham* associated to a *Ham*-invariant norm on the Lie algebra $C^\infty(M)_{normalized}$ Buhovsky-Ostrover, 2011

LM- space of contractible loops $z : S^1 \to M$
$F(x,t)$- 1-periodic Hamiltonian, $\phi_F \in Ham$ - time one map
Action functional: $\mathcal{A}_F(z) : LM \to \mathbb{R}$, $z \mapsto \int_0^1 F(z(t),t)dt - \int_D \omega$
D-disc spanning z. Well defined if $\pi_2(M) = 0$
Critical points: 1-periodic orbits of Hamiltonian flow
Gradient equation: Cauchy-Riemann (Gromov's theory, 1985)
Gradient lines connecting critical points – **Fredholm problem**

Figure: Gradient lines:



Count of connecting lines: Floer homology HF

Floer persistence module

For $s \in \mathbb{R}$ get family of vector spaces $HF(\{\mathcal{A}_F < s\})$ with natural morphisms (as in Morse theory).
Under certain assumptions on (M, ω) (apsherical, atoroidal,...)

- the module depends only on the time one map $\phi \in Ham(M, \omega)$ of the Hamiltonian flow of F.
- There exists a version of Floer persistence module $HF(\phi)_\alpha$ built on non-contractible closed orbits in the free homotopy class α.

Robustness

Theorem (P.-Shelukhin, 2016)

For closed (M, ω) with $\pi_2(M) = 0$, the map

$$(Ham\,, d_{Hofer}) \to (Barcodes, d_{bot})\ ,$$

$\phi \mapsto \mathcal{B}$ *(persist. module associated to Floer theory of* ϕ*) is Lipschitz.*

Lipschitz functions on barcodes include some known numerical invariants of Hamiltonian diffeomorphisms: spectral invariants (Viterbo, Schwarz, Oh); boundary depth (Usher)

Generalizations

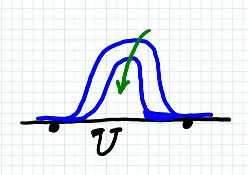
Generalization to arbitrary symplectic manifolds, including theory of persistence modules over Novikov rings (Usher-Zhang)
Applications to geometry of *Ham*.

Symplectic homology of domains

(Floer-Hofer)

$U \subset \mathbb{R}^{2n}$ - domain, $\mathcal{C}(U)$- compactly supported non-negative Hamiltonians.

$H, F \in \mathcal{C}(U), H \leq F \Rightarrow$ natural morphism $HF^t(F) \to HF^t(H)$, $t > 0$.



$$P_i^t(U) := SH_i^{e^t}(U) := \lim_{\leftarrow} HF_i^{[e^t,+\infty)}(H),\ i \geq 2n$$

point-wise fin. dim. persistence module - the same theory

Symplectic Banach-Mazur (pseudo)-distance

(Ostrover,P.,Usher,Gutt,Zhang)
$\mathcal{S}$-set of starshaped domains in $\mathbb{R}^{2n}$.
For $U, V \in \mathcal{S}$ set

$$\rho(U,V) = \inf \left\{ \ln C \in [0,\infty) \;\middle|\; \begin{array}{c} \exists \;\; \frac{1}{C}U \xrightarrow{\phi} V \xrightarrow{\psi} CU \\ \text{s.t. } \psi \circ \phi \text{ is ``unknotted''} \end{array} \right\}$$

and $d_{SBM}(U,V) = \max\{\rho(U,V), \rho(V,U)\}$.

Map: $U \in \mathcal{S} \rightsquigarrow$ barcode $\mathcal{B}_i(U)$ of symplectic homology $P_i^t(U)$.

Theorem (Robustness)

$U \to \mathcal{B}_i(U)$ is Lipschitz for d_{SBM} on $\mathcal{S}$ and d_{bot} on barcodes.

In progress: Applications to geometry of $(\mathcal{S}, d_{SBM})$.

Further directions

Persistence of higher algebraic structures: Floer persistence module carries a natural operation $HF_s(\phi) \otimes HF_t(\psi) \to HF_{s+t}(\phi\psi)$ (pair-of-pants product). How to use it?
First steps (when $\phi = \mathbb{1}$) - P-Shelukhin-Stojisavljević.
Explore structures on SH (co-product?).

"Learning" symplectic mfds and their morphisms: Reconstruct (with a controlled error) "hard" invariants of a symplectic manifold or a symplectic diffeomorphism, given its (discrete) approximation.

Persistence and (de)quantization: Reconstruct (with a controlled error) "hard" invariants of a Hamiltonian function/diffeomorphism or subset from its (Berezin-Toeplitz) quantization. Analogy with the previous problem due to remainders (error terms) of the (de)quantization. In progress.

Conclusion

GEOMETRY $\rightsquigarrow$ **ALGEBRA (pers. mod.)** $\rightsquigarrow$ **BARCODE**

GEOMETRY	ALGEBRA
$C^\infty(M)$, uniform norm	Morse homology
$Ham(M)$, Hofer's metric	Floer homology
starsh. domains, Banach-Mazur dist.	sympl. homology

Application: distance to powers

(M, ω)-closed symplectic manifold, $k \geq 2$ - integer.
$\mathrm{Powers}_k = \{\phi = \psi^k | \ \psi \in Ham\}$- Hamiltonian diffeomorphisms admitting a root of order k.

Theorem (P.-Shelukhin)

Let Σ be a closed oriented surface of genus ≥ 4 equipped with an area form σ, and $k \geq 2$ an integer. Then

$$\sup_{\phi \in Ham} d(\phi, \mathrm{Powers}_k) = +\infty \ .$$

Comments

Extensions: Proved for various split manifolds of the form $\Sigma \times M$ (P.-Shelukhin, Zhang, P-S-Stojisavljević)

Idea: Diffeomorphism ϕ induces $\mathbb{Z}_p$-action by conjugation on the persistence module of $HF^t(\phi^p)$. Look at persistence eigenmodule corresponding to the primitive p-th root of unity. Involves Floer homology for non-contractible loops.

Application: coarse geometry of *Ham*

Theorem

For closed surface of genus ≥ 4, any asymptotic cone of Ham contains a free group with two generators.

D. Alvarez-Gavela, V. Kaminker, A. Kislev, K. Kliakhandler, A. Pavlichenko, L. Rigolli, D. Rosen, O. Shabtai, B. Stevenson, J. Zhang, 2015.