Shooting to find orbits

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Goals

In this talk we describe very easy, heuristic tools for the following.

- find (symmetric) periodic orbits with the computer
- compute the Conley-Zehnder index

Remark

On Friday I will describe a general scheme to make the numerical work rigorous: this includes taking care of rounding errors, truncation errors, and uniqueness in a neighborhood.
In other words, today's methods are just heuristics.

Symmetric periodic orbits

We consider a symplectic manifold (M, ω) with antisymplectic involution ι, so $\iota^* \omega = -\omega$.

Example

As a basic example, consider $(M = \mathbb{R}^4, \omega = \sum_i dp_i \wedge dq_i)$ with

$$\iota(q_1, q_2, p_1, p_2) = (q_1, -q_2, -p_1, p_2).$$

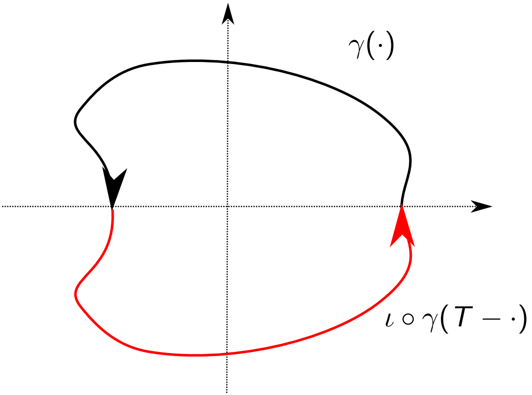
The fixed point set of ι is Lagrangian. In the example, $Fix(\iota)$ consists of points $(q_1, 0, 0, p_2)$.

Definition

A **symmetric, periodic orbit** of an ι-invariant H is a periodic orbit γ of X_H such that $\iota \circ \gamma(t) = \gamma(T - t)$.

Doubling of chords

Given a chord γ from $Fix(\iota)$ to itself, we obtain a symmetric, periodic orbit by doubling.



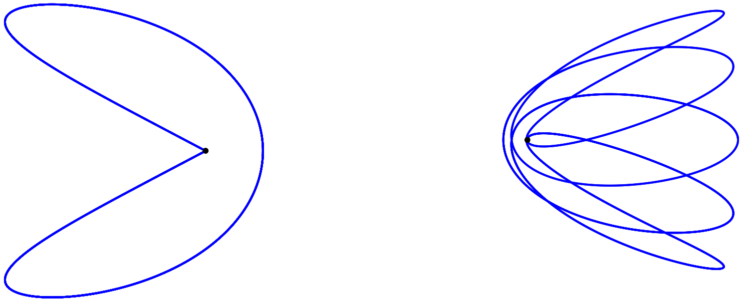
Symmetric Hamiltonians

Mechanical Hamiltonians with a potential that is symmetric in the x-axis are invariant. An interesting example is the diamagnetic Kepler problem.
In cylindrical coordinates, its Hamiltonian is given by

$$H = \frac{1}{2}\left(p_\rho^2 + p_z^2 + \frac{p_\vartheta^2}{\rho^2} - B p_\vartheta \right) + \frac{B^2}{8}\rho^2 - \frac{1}{\sqrt{\rho^2 + z^2}}$$

When the angular momentum $p_\vartheta = 0$, this gives an invariant Hamiltonian on $\mathbb{R}^4$.

Some symmetric periodic orbits in the diamagnetic Kepler problem



Review of PRTBP

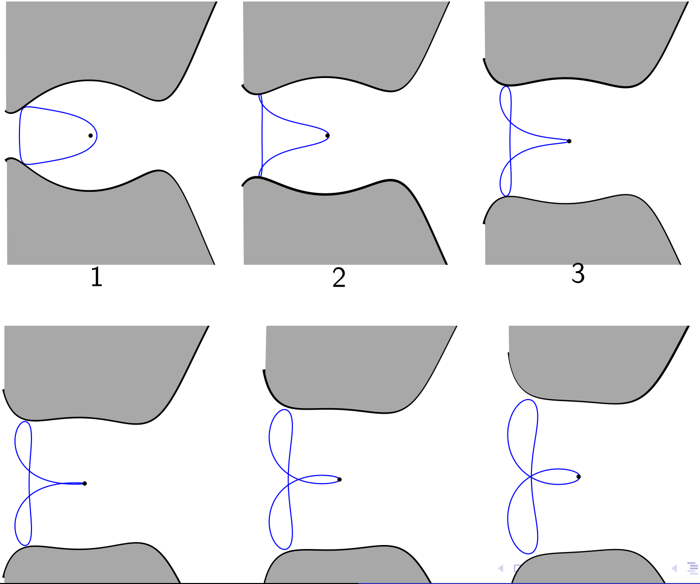
We consider a "massless" particle affected by by particles at $q_i(t)$ for $i = 1, 2$.

$$H(q, p) = \frac{1}{2}\|p\|^2 - \sum_{i=1}^{2} \frac{m_i}{\|q - q_i(t)\|}$$

Suppose now q_1 and q_2 move around each other in circular orbits. It was discovered by Jacobi that then the problem turns out to admit an integral after going to a rotating frame.
This is the **planar, restricted three-body problem** (PRTBP).
The **Jacobi Hamiltonian** is the autonomous Hamiltonian

$$H(q, p) = \frac{1}{2}\|p\|^2 + p^t J_0 q - \frac{1-\mu}{\|q+\mu\|} - \frac{\mu}{\|q-1+\mu\|}$$

A family of symmetric orbits in RTBP ($\mu = 0.99$) all with $\mu_{CZ} = 3$ (after Moser regularization).



Shooting algorithm for finding symmetric periodic orbits

Input: An invariant Hamiltonian H, a starting point q, a stepsize δ and an energy level c

Output: Some approximations of symmetric periodic orbits

1. $i \leftarrow 1$
2. find p such that $H(q, 0, 0, p) = c$, so $x = (q, 0, 0, p) \in L$
3. by numerical integration find approximate minimal t such that $q_2(Fl_t^{X_H}(x)) = 0$.
4. record the slope: $s_i \leftarrow p_1(Fl_t^{X_H}(x))$. $q \leftarrow q + \delta$. $i \leftarrow i + 1$
5. if $i = 2$ go to step 2.
6. (sign change) if $s_i \cdot s_{i-1} < 0$, find 0 of the slope function by Newton iterations. Record the new starting q' and half-period t'.
7. if $q < q_{max}$ go to step 2. Else return recorded data.

A good way to find t such that $q_2(Fl_t^{X_H}(x)) = 0$ is the following:

1. find t_0 such that $q_2(Fl_{t_0}^{X_H}(x)) < 0$ and $t_1 = t_0 + h$ such that $q_2(Fl_{t_1}^{X_H}(x)) > 0$.
2. from $x' = Fl_{t_0}^{X_H}(x)$ integrate the ODE from $q_2 = q_2(Fl_{t_0}^{X_H}(x)$ to 0

$$
\begin{aligned}
\dot{q}_1 &= f_{q_1}/f_{q_2} \\
\dot{q}_2 &= 1 \\
\dot{p}_1 &= f_{p_1}/f_{q_2} \\
\dot{p}_2 &= f_{p_2}/f_{q_2} \\
\dot{t} &= 1/f_{q_2}
\end{aligned}
$$

Remark

The choice of numerical integration scheme (Runge-Kutta, symplectic integrator, Taylor) affects the results and speed considerably.

The standard algorithm

The above is a really a cartoon (but very effective) of the standard algorithm. The standard way goes as follows

1. take a local surface of section L and $x \in L$
2. by numerical integration find the approximate minimal $t > 0$ such that $\tau(x) := Fl_t^X(x) \in L$; at the same time compute the variational equation
3. we get a numerical approximation of the first return map $\tau(x)$ and its derivative $d_x\tau$
4. if $\tau(x)$ is close to x, use Newton iterations to obtain a fixed point.

Remark

The earlier version is essentially the same using a 1-dimensional surface of section (the Lagrangian intersection with the energy surface).

Conley-Zehnder index

An important invariant of Hamiltonian periodic orbits is the so-called Conley-Zehnder index. Intuitively, it is a "winding number" of the linearized flow.
More precisely, let us consider a path of symplectic matrices,

$$\Psi : [0,1] \longrightarrow Sp(2n).$$

with $\Psi(0) = \mathrm{Id}$ and $\det(\Psi(1) - \mathrm{Id}) \neq 0$.

Definition

The **Maslov cycle** is the (singular variety)

$$V = \{A \in Sp(2n) \mid \det(A - \mathrm{Id}) = 0\}$$

Maslov cycle

The **Conley-Zehnder index** of a path of symplectic matrices is the intersection number of Ψ with the Maslov cycle. This can be computed with the crossing formula.

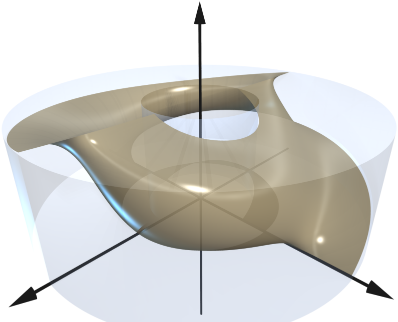


Figure: Maslov cycle in $Sp(2) \cong S^1 \times \mathbb{R}^2$

Quaternions to construct global frames

Now the **Conley-Zehnder index** of a periodic orbit on a star-shaped hypersurface in $\mathbb{R}^4$

1. Symplectically trivialize the symplectic complement of $span(X, X_H)$ (the contact structure) by the global frame

$$U = j\nabla H - \frac{\lambda(j\nabla H)}{\lambda(X_H)} X_H, \quad V = k\nabla H - \frac{\lambda(k\nabla H)}{\lambda(X_H)} X_H$$

   Rescale U such that $d\lambda(\tilde{U}, V) = 1$.

2. Use the variational equations to compute the linearized flow $dFl_t^{X_H}(U)$, $dFl_t^{X_H}(V)$. Decompose these vectors into the frame to get $\Psi(t)$. $\mu_{CZ}(\gamma) := \mu_{CZ}(\Psi)$.

Note $X_H = i\nabla H$.

Remark

Global trivializations can also explicitly be written down for $\mathbb{R}P^3$.

Proposition (Long, Hutchings)

Let (Y, λ) be contact manifold, and γ a non-degenerate periodic Reeb orbit with symplectic trivialization Φ. Let ϑ denote the rotation number of γ with respect to Φ. Then $\mu_{CZ}(\gamma^k) = \lfloor k\vartheta \rfloor + \lceil k\vartheta \rceil$.

Remark

- *If $|trace(\gamma)| < 2$, then γ is elliptic (and the rotation number is irrational since γ is assumed to be non-degenerate.*
- *If $trace(\gamma) > 2$, then γ is hyperbolic*
- *If $trace(\gamma) < -2$, then γ is negative hyperbolic (γ^2 is a bad orbit).*

Algorithm to get the CZ-index of a (approximate) periodic orbit

Input: Approximate starting point and period of a non-degenerate periodic orbit γ of a starshaped hypersurface in $\mathbb{R}^4$.

Output: Transverse Conley-Zehnder index of γ

1. Numerically integrate the variational equations to obtain an approximate, discretized path of symplectic matrices $\Psi(t_0)$, $\Psi(t_1)$,...
2. Compute $tr = Trace(\Psi(t_{final}))$ If $|tr| > 2$, the orbit is non-degenerate, hyperbolic. If $|tr| < 2$, the orbit is non-degenerate, elliptic.
3. Count axis crossings $\Psi(t_i)$ to obtain the rotation number
4. Apply the proposition.