Monte Carlo Methods in Financial Engineering

Paul Glassermann

1.1 Principles of Monte Carlo

Outline:

- 1.1.1 Introduction
- 1.1.2 First Examples
- 1.1.3 Efficiency of Simulation Estimators

Standard Call (European Call)

- Def.: The Right to buy the underlying asset at a fixed price at a fixed time

S(t) price of the stock

K strike

T expiration time

- Payoff $= \begin{cases} 0 & if\ S(T) \leq K \\ S(T) - K & if\ S(T) > K \end{cases} \quad =: \ (S(T) - K)^+$

Standard Call

- Expected present value: $E[e^{-rT}(S(T)-K)^+]$

- What is the value of S(T)?
  - Black-Scholes-modell $\frac{dS(t)}{S(t)} = r\,dt + \sigma\,dW(t)$ (GBM) (1.1)

$$S(T) = S(0)\exp(\left(r - \frac{1}{2}\sigma^2\right)T + \sigma W(T))$$

W(T)$\sim N(0,T)$, $Z \sim N(0,1)$ $$S(T) = S(0)\exp(\left(r - \frac{1}{2}\sigma^2\right)T + \sigma\sqrt{T}Z)$$

- Analytical solution

$$E[e^{-rT}(S-K)^+] = S\Phi\left(\frac{\log\left(\frac{S}{K}\right)+\left(r+\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right) - e^{-rT}K\Phi\left(\frac{\log\left(\frac{S}{K}\right)+\left(r-\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right)$$

Standard Call via Monte-Carlo

- $E[e^{-rT}(S(T) - K)^{+}]$
- Assume C_i $i.i.d.$ like $e^{-rT}(S(T) - K)^{+}$
  - Strong law of large numbers ⇒ $P\left(\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} C_i = E[C_i]\right) = 1$

  ⟶ Simulating n independent draws of C_i approximates E[C_i]

For i = 1, …, n

Generate Z_i

Set $S_i(T) = S(0)\exp(\left(r - \frac{1}{2}\sigma^2\right)T + \sigma\sqrt{T}Z_i)$

Set $C_i = e^{-rT}(S_i(T) - K)^{+}$

Set $\hat{C}_n = \frac{1}{n}\sum_{i=1}^{n} C_i$

Path Dependent Example

- Two modifications to the Standard Call, which makes simulations at intermediate timesteps necessary

1) Let the dynamics of the underlying asset S(t) be given by

$$dS(t) = rS(t)dt + \sigma\big(S(t)\big)S(t)dW(t)$$

- In most cases exists no analytical solution

  → use discrete approximation for samples of S(T)

$$\Delta t = \frac{T}{m},\ \ m \in \mathbb{N}$$

$$S(t+\Delta t) = S(t) + rS(t)\Delta t + \sigma\big(S(t)\big)S(t)\sqrt{\Delta t}Z$$

Euler-Approximation

For i = 1, …, n

For j = 1,…,m

Generate Z_{ij}

Set $S_i(t_j) = S_i(t_{j-1}) + rS_i(t_{j-1})(t_j - t_{j-1}) + \sigma(S_i(t_{j-1}))S_i(t_{j-1})\sqrt{(t_j - t_{j-1})}Z_{ij})$

Set $C_i = e^{-rT}(S_i(t_m) - K)^+$

Set $\hat{C}_n = \frac{1}{n}\sum_{i=1}^{n} C_i$

Asian Option

2) Options with payoffs that depend on the average level of the underlying asset

- $(\bar{S} - K)^+$ the assets payoff
  $0 = t_0 < t_1 < \cdots < t_m = T$
  T expiration time, K strike
  $\bar{S} = \frac{1}{m}\sum_{j=1}^{m} S(t_j)$

- Each replication $\bar{S}_i$ i=1,…,n requires m simulations of the stock
  $S(t_{j+1}) = S(t_j)\exp(\left(r - \frac{1}{2}\sigma^2\right)(t_{j+1} - t_j) + \sigma\sqrt{(t_{j+1} - t_j)}Z_{j+1})$

Asian Option

For i = 1, …, n

For j = 1,…,m

Generate Z_{ij}

Set $S_i(t_j) = S_i(t_{j-1})\exp(\left(r - \frac{1}{2}\sigma^2\right)(t_j - t_{j-1}) + \sigma\sqrt{(t_j - t_{j-1})}Z_{ij})$

Set $\bar{S} = \frac{1}{m}\sum_{j=1}^{m} S_i(t_j)$

Set $C_i = e^{-rT}(\bar{S} - K)^+$

Set $\hat{C}_n = \frac{1}{n}\sum_{i=1}^{n} C_i$

Efficiency of Simulation Estimators

- Criteria for comparing simulators
  - Computing time
  - Bias
  - Variance

Unbiased

- Unbiased $\Leftrightarrow E[\hat{C}_n] = C$
- $\hat{C}_n = \frac{1}{n}\sum_{i=1}^{n} C_i$ with C_i i.i.d. , E$[C_i]$=C , VarC_i=$\sigma_C^2 < \infty$
- Central limit theorem $\Rightarrow \sqrt{n}(\hat{C}_n - C) \approx N(0, \sigma_C^2)$

- s computational budget

  τ computing time per replication C_i

  $\left\lfloor \frac{s}{\tau} \right\rfloor$ number of replications
- CLT $\Rightarrow \sqrt{\left\lfloor \frac{s}{\tau} \right\rfloor}\,(\hat{C}_{\left\lfloor \frac{s}{\tau} \right\rfloor} - C) \approx N(0, \sigma_C^2)$

Unbiased

- $\left\lfloor \frac{s}{\tau} \right\rfloor \leq \frac{s}{\tau} \leq \left\lfloor \frac{s}{\tau} \right\rfloor + 1 \quad \Rightarrow \quad \frac{1}{\tau} - \frac{1}{s} \leq \frac{\left\lfloor \frac{s}{\tau} \right\rfloor}{s} \leq \frac{1}{\tau} \quad \Rightarrow \quad \lim_{s \to \infty} \frac{\left\lfloor \frac{s}{\tau} \right\rfloor}{s} = \frac{1}{\tau}$

  $\Rightarrow \hat{C}_{\left\lfloor \frac{s}{\tau} \right\rfloor} - C \approx N\left(0, \frac{\sigma_C^2 \tau}{s}\right)$

- In comparing estimators choose the one with lower value $\sigma_C^2 \tau$
  - σ_c usally unknown

    $\rightarrow$ sample standard deviation $\zeta_c = \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} (C_i - \hat{C}_n)^2}$

Random computing time

- (C_i, τ_i) $i.i.d.$ with τ_i computing time, C_i as before
- $N(s) = \sup\{n \geq 0 : \sum_{i=1}^{n} \tau_i \leq s\}$ number of replications

- CLT $\Rightarrow$ $\sqrt{N(s)}\left(\hat{C}_{N(s)} - C\right) \approx N(0, \sigma_C^2)$
- SLLN $\Rightarrow$ $\mathrm{P}(\lim_{s\to\infty} \frac{N(s)}{s} = \frac{1}{E[\tau]}) = 1$

$\Rightarrow$ $\hat{C}_{N(s)} - C \approx N(0, \frac{\sigma_C^2 E[\tau]}{s})$

$\rightarrow$ Compare estimators by the product $\sigma_C^2 E[\tau]$

Random computing time

- $T_{N(s)} \coloneqq \sum_{i=1}^{N(s)} \tau_i$
- $\frac{T_{N(s)}}{N(s)} \leq \frac{s}{N(s)} \leq \frac{T_{N(s)+1}}{N(s)}$, $\tau_1 \leq s$

- SLLN $\Rightarrow P(\lim_{s\to\infty} \frac{T_{N(s)}}{N(s)} = E[\tau_1]) = 1$ and $P(\lim_{s\to\infty} \frac{T_{N(s)+1}}{N(s)} = E[\tau_1]) = 1$

$\Rightarrow \quad P(\lim_{s\to\infty} \frac{s}{N(s)} = E[\tau_1]) = 1$

$\Leftrightarrow \quad P(\lim_{s\to\infty} \frac{N(s)}{s} = \frac{1}{E[\tau_1]}) = 1$

Random computing time

- $C_{N(s)}$ is biased

$$E[C_{N(s)}] = \begin{cases} C \ , & N(s) > 0 \\ 0 \ , & N(s) = 0 \end{cases}$$
$$= E[C I_{\{\tau_1 \le s\}}]$$
$$= C(1 - P(\tau_1 > s))$$
$$\neq C$$

- $P(\tau_1 > s) \to 0$ as $s \to \infty$
  - Bias vanishes as number of replications increase
  - $C_{N(s)}$ asymptotical unbiased

Biased estimator

- $Bias[\hat{C}_n] = E[\hat{C}_n] - C$

- Sources of bias
  - Discretization error
  - Nonlinear function of means

- Consider only estimators for which any bias can be eliminated through increasing computational effort

Discretization error Asian Option

- In example of the Asian option instead of (1.9)
  - $0 = t_0 < t_1 < \cdots < t_m = T$
  - $h = \frac{t_{i+1} - t_i}{k}$, j = 0,…,k-1

$$S(t_i + (j+1)h) = S(t_i + jh) + rS(t_i + jh)h + \sigma S(t_i + jh)\sqrt{h}Z_{j+1}$$

$\rightarrow$ model discretization error

- Approx. model draws closer to the exact one as $k \rightarrow \infty$
  - Bias vanishes under increasing computational effort

Discretization error Lookback Option

- Payoff $(\max_{0 \le t \le T} S(t) - S(T))$
- Expected present value $\alpha = E[e^{-rT}(\max_{0 \le t \le T} S(t) - S(T))]$
- Estimator $\hat{\alpha} = e^{-rT}(\max_{0 \le j \le m} S(t_j) - S(T))$

- $\max_{0 \le j \le m} S(t_j) \le \max_{0 \le t \le T} S(t)$ And $E[\hat{\alpha}] < \alpha \ almost\ sure$

  →Payoff discretization error

- Bias vanishes under additional computational effort as $m \to \infty$

Nonlinear function/Compound option

- Standard Call expiring at T_1 on a Standard Call expiring at T_2
- Expected present value of Call1:
  $C^{(1)} = E[e^{-rT_1}(\tilde{S}(T_1) - K_1)^+]$

- Underlying Asset $\tilde{S}(T_1)$ is an option on the stock S
  $C^{(2)}(x) = E[e^{-r(T_2-T_1)}(S(T_2) - K_2)^+ \mid S(T_1) = x]$
  $\tilde{S}(T_1) = C^{(2)}(S(T_1))$

- Simulate $S_i(T_1)$ for i=1,…,n and $S_{ij}(T_2)$ dependent on $S_i(T_1)$ for j=1,…,m
  $\hat{C}_m^{(2)}\big(S_i(T_1)\big) \quad = \frac{1}{m}\sum_{j=1}^{m} e^{-r(T_2-T_1)}(S_{ij}(T_2) - K_2)^+$
  $\hat{C}_n^{(1)} \quad = \frac{1}{n}\sum_{i=1}^{n} e^{-rT_1}(\,\hat{C}_m^{(2)}\big(S_i(T_1)\big) - K_1)^+$

Compound option

- $\hat{C}_m^{(2)}\big(S_i(T_1)\big)$ is unbiased $\quad E[\hat{C}_m^{(2)}\big(S_i(T_1)\big)] = C_m^{(2)}\big(S_i(T_1)\big)$

- But $E\left[\hat{C}_n^{(1)}\right] = E[e^{-rT_1}(\hat{C}_m^{(2)}\big(S_i(T_1)\big) - K_1)^+]$

$$= E[E[e^{-rT_1}(\hat{C}_m^{(2)}\big(S_i(T_1)\big) - K_1)^+] \,|S_i(T_1)]$$

$$\geq E[e^{-rT_1}(E[\hat{C}_m^{(2)}\big(S_i(T_1)\big)|S_i(T_1)] - K_1)^+]$$

$$= E[e^{-rT_1}(C_m^{(2)}\big(S_i(T_1)\big) - K_1)^+]$$

$$= C^{(1)}$$

- Bias vanishes as $m \to \infty$

Balance Bias/Variance

- Reducing Bias at the expanse of computing time results in less replications per given computational budget → increase estimator variance

- MSE($\hat{C}_n$) $= Bias^2[\hat{C}_n]$ + Var[$\hat{C}_n$]
  - Calculating exact MSE generally impractical
  - Compare estimators through asymptotic MSE

- Biased estimator $\hat{C}(n,\delta) = \frac{1}{n}\sum_{i=1}^{n} C_i^{\delta}$ with C_i^{δ} i.i.d. $E[C_i^{\delta}] \neq C$

  and δ the bias determining parameter, $E\left[\hat{C}(n,\delta)\right] \to C$ as $\delta \to 0$

Balance Bias/Variance

- *s computational budget*

  τ_δ (const.) computing time per replication at parameter δ

- Assume there are constants $\beta, \eta, b, c > 0$ such that, as $\delta \to 0$

$$\text{Bias}[\hat{C}(n,\delta)] = b\delta^{\beta} + o(\delta^{\beta})$$

$$\tau_\delta = c\delta^{-\eta} + o(\delta^{-\eta})$$

- *Smaller* $\delta \to$ *greater* τ_δ
  - *less replications → increased Variance*
  - *higher effort to reduce Bias*

Balance Bias/Variance

- $s \longmapsto \delta(s) = as^{-\gamma} + o(s^{-\gamma})$ an allocation rule for δ with a,$\gamma > 0, \gamma < 1/\eta$
- $\hat{C}(s) \equiv \hat{C}(N(s), \delta(s))$ N(s) = $\left\lfloor \frac{s}{\tau_{\delta(s)}} \right\rfloor$

- $$Bias^2\left[\hat{C}(s)\right] = b^2\delta(s)^{2\beta} + o\left(\delta(s)^{2\beta}\right)$$
$$= b^2a^{2\beta}s^{-2\beta\gamma} + o(s^{-2\beta\gamma})$$
$$= O(s^{-2\beta\gamma})$$

- $$Var[\hat{C}(s)] = \frac{\sigma^2 c\delta(s)^{-\eta}}{s} + o\left(\frac{\delta(s)^{-\eta}}{s}\right)$$
$$= \sigma^2 ca^{-\eta}s^{\gamma\eta-1} + o(s^{\gamma\eta-1})$$
$$= O(s^{\gamma\eta-1})$$

Balance Bias/Variance

- $\text{MSE}\left(\hat{C}(s)\right) = O(s^{-2\beta\gamma}) + O(s^{\gamma\eta-1})$
- Choose $\gamma = \frac{1}{2\beta+\eta}$

- $\text{MSE}\left(\hat{C}(s)\right) = \left(b^2 a^{2\beta} + o^2 c a^{-\eta}\right) s^{-\frac{2\beta}{2\beta+\eta}} + o(s^{-\frac{2\beta}{2\beta+\eta}})$

$\sqrt{\text{MSE}\left(\hat{C}(s)\right)} = O(s^{-\frac{\beta}{2\beta+\eta}})$

- Unbiased case:

$\sqrt{\text{MSE}(\hat{C}_{N(s)})} = \sqrt{Var[\hat{C}_{N(s)}]} = O(s^{-\frac{1}{2}})$

Balance Bias/Variance

- Large β $\rightarrow$rapidly vanishing bias $\lim_{\beta \rightarrow \infty} \frac{\beta}{2\beta+\eta} \rightarrow \frac{1}{2}$
- Small η $\rightarrow$small costs of reducing bias $\lim_{\eta \rightarrow 0} \frac{\beta}{2\beta+\eta} \rightarrow \frac{1}{2}$