Derivates Pricing and Arbitrage

What are Derivatives?

- Derivatives are complex financial products which come in many different forms.
- They are, simply said, a contract between two parties, which specify payments between those parties.
- Different derivatives have different conditions.

What are Derivatives?

- They usually get traded „Over the Counter“ (OTC)
- They *derive* their value from an underlying asset.

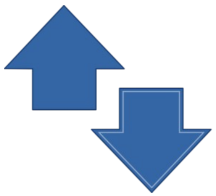
  → they do not have any inherent value

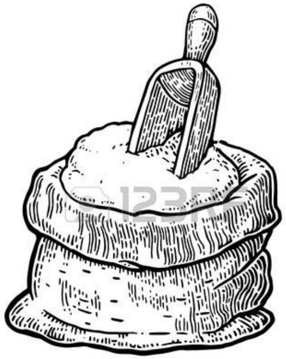
What are Derivatives?

- Different kinds of derivatives: Forwards, Swaps, Options etc.
- Option: One party sells another party the „right“ to buy a certain asset at a certain price at one point in the future (European Option) or at a time frame in the future (American Option)
- Forward: Two parties agree to a certain deal in the future and sign a contract

Derivatives: An Example

- Two parties: A farmer and a cookie factory
- Agree to trade the flour of the farmer to a fixed price after the harvest (e.g. 200€/t)
- If the marketprice rises over that amout → profit for the factory otherwise → profit for the farmer
- Security for both parties.





Why do we need to find a way to price them?

- Derivatives are no product per se, they do not have any inherent value
- The value of a derivative is subject to changes in the market price of the underlying asset
- We need to develop a strategy to find a price for the contract, which is fair on both parties

How can we achieve this?

Principles of Derivatives Pricing

1. If a derivative security can be replicated by trading in other assets, the price of this security is the price of using that trading strategy.
2. One can „*discount*" or „*deflate*" asset prices, such that the resulting prices are martingales under a probability measure.
3. In a complete market any payoff can be synthesized through a trading strategy.

Mathematical Theory

- Lets consider d assets with prices Si(t) (i = 1…..d)
- They can be described as a System of Stochastic Differential Equations:

$$dS_i(t)/S_i(t) = \mu_i(S(t), t)\, dt + \sigma_i(S(t), t)^\top dW(t)$$

- With: $\sigma_i \in R^k$, $\mu_i \in R$ and W(t) k-dimensional Brownian Motion
  - A Brownian Motion can be imagined as a sort of „random walk“, a function, which changes its value randomly

Mathematical Theory

- Let a vector $\theta \in R^d$ be the portfolio with θ_i representing the number of units held of the ith asset. The value of the portfolio at time t is:

$$\theta_1 S_1(t) + \cdot \cdot \cdot + \theta_d S_d(t) = \theta^T S(t)$$

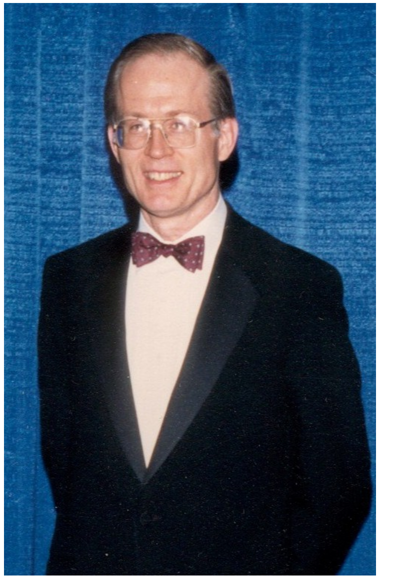
- A trading strategy can be written as a stochastical process $\theta(t)$. This is called *self-financing* if:

$$\theta(t)^\top S(t) - \theta(0)^\top S(0) = \int_0^t \theta(u)^\top \, dS(u)$$

and, equivalently:

$$\theta(t)^\top S(t) = \theta(0)^\top S(0) + \int_0^t \theta(u)^\top \, dS(u)$$

Black-Scholes Model

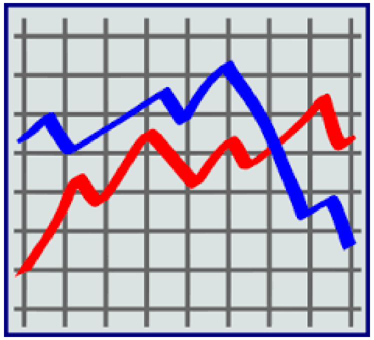


- Developed by Fisher Black and Myron Scholes in 1973
- A mathematical model to price options in the financial market
- It is widely used and describes the actual behaviour and pricing fairly well

Black-Scholes Model

- General idea: We want to find a risk-less strategy to get the same profit of the option and then price the option accordingly

- In the model there are 2 assets:
  - 1. Risky: $dS(t)/S(t)=\mu dt+\sigma * dW(t)$
    - Stock
  - 2. Riskless: $d\beta(t)/\ \beta(t)=r\ dt$
    - Savings account



Black-Scholes Model

- We now want to price a derivative with a payoff $f(S(t))$ at time t and want to find the value V
- If we insert that in the formulas derived from Ito's Lemma, regarding the price of derivatives, we can get:

$$(\partial V/\partial t) + (1/2)\sigma^2 S^2(\partial^2 V/\partial S^2) = 0$$

And

$$V(S,\beta,t) = (\partial V/\partial S)*S + (\partial V/\partial \beta)*\beta$$

- Also the boundary condition $V(S,\beta,t) = f(S(t))$ holds.

Black-Scholes Model

- Since we know that $\beta(t) = e^{rt}$ we can write $V(S,\beta,t)$ as

$$V^*(S,t) = V(S, e^{rt}, t)$$

- Now, using

$$(\partial V^*/\partial t) = (\partial V/\partial \beta)^* r\beta + (\partial V/\partial t)$$

We can get:

$$\boxed{(\partial V^*/\partial t) + rS(\partial V^*/\partial S) + (1/2)\sigma^2 S^2 \partial(\partial^2 V/\partial S^2) - rV = 0}$$

What is Arbitrage?

- Arbitrage is the possibility of generating risk-less profit.
- Often by taking advantage of price differences in different markets.
- „Making money out of nothing“

What is Arbitrage?

- In mathematics, we call a self-financing strategy θ *arbitrage*, if one of these two conditions hold:

  i. $\theta(0)^\top S(0) < 0$ and $P_0(\theta(t)^\top S(t) \geq 0) = 1$; or

  ii. $\theta(0)^\top S(0) = 0$, $P_0(\theta(t)^\top S(t) \geq 0) = 1$, and $P_0(\theta(t)^\top S(t) > 0) > 0$

- negative initial investment → non-negative final wealth with probability 1 (i.)
- initial net investment of 0 → non-negative final wealth, which is positive with positive probability (ii.)

What is Arbitrage?

- $: € Rate in Europe → 1 : 0.8
- But in the USA → 1 : 0.7
- It is possible to exchange US-$ in Europe and then exchange the resulting Euros in the USA
- 1000$ -> 800€ -> 1143 $ => 143$ profit!

What is a martingale?

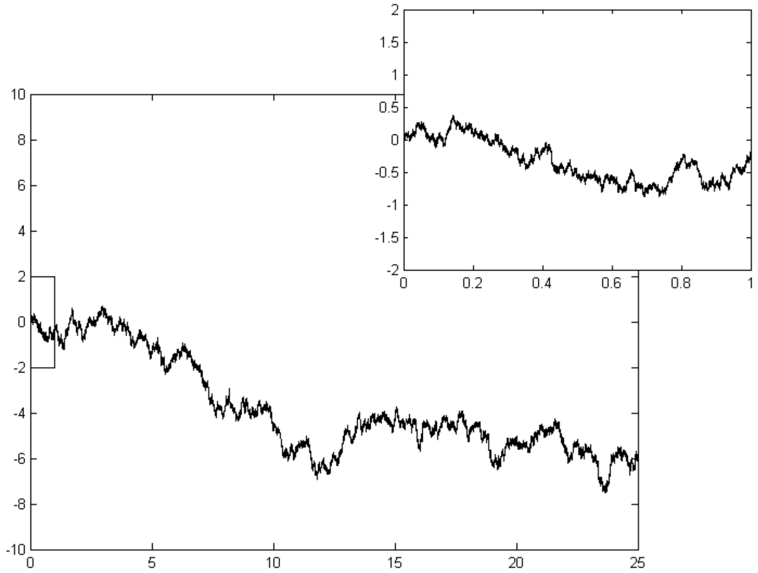
- Definition: A Martingale is a stochastical process M(t), for which:

$$M(t) = E[M(t+1)]\ \forall t$$

- So the expectation at the next step of the martingale is always the current value!

Examples of Martingales

- The budget of a gambler when playing a fair game: e.g. tossing a fair coin
- A wiener process (standard brownian motion) is a random walk with normal distributed increments → also a martingale



The Stochastical Discount Factor

- The „*stochastic discount factor*“ (or *deflator*) is a non-negative stochastic process Z(t), for which the ratio of the price and the discount factor (V(t)/Z(t)) is a martingale:

$$V(t)/Z(t) = Eo[V(T)/Z(T)|Ft]$$

(where Ft is the history of the Brownian Motion up to t)

$$\Leftrightarrow V(t) = Eo[V(T)*(Z(t)/Z(T))|Ft]$$

→ The price V(t) is the *expectation* of the price V(T) (at some point in the future) discounted by (Z(t)/Z(T)).

Fundamental Theorem of Asset Pricing

- We can normalize the stochastical discount factor by setting Z (0) ≡ 1 and therefore

$$V(0)=E_0[V(T)/Z(T)]$$

- If we now insert a self-financing strategy θ with the price process $\theta(t)^\top S(t)$, by the definition of the SDF

$$\theta(0)^\top S(0)=E_0[\theta(T)^\top S(T)/Z(T)]$$

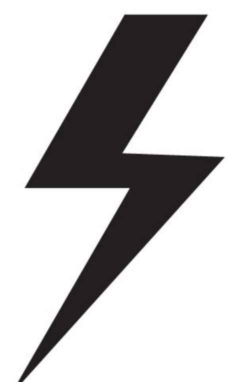
has to hold

Fundamental Theorem of Asset Pricing

$$\theta(0)^\top S(0) = E_o[(\theta(T)^\top S(T))/Z(T)]$$

- If we now compare that to the definition of arbitrage
  i. $\theta(0)^\top S(0) < 0$ *and* $P_o(\theta(t)^\top S(t) \geq 0) = 1$; *or*
  ii. $\theta(0)^\top S(0) = 0$, $P_o(\theta(t)^\top S(t) \geq 0) = 1$, *and* $P_o(\theta(t)^\top S(t) > 0) > 0$

  we can see that this can not hold, given that Z(T) is non-negative → definition



Fundamental Theorem of Asset Pricing

- There can be no arbitrage if there is a stochastical discount factor and vice versa

  → mutually exclusive
- It can also be shown that the absence of arbitrage implies the existence of a stochastical discount factor under certain conditions
- Known as „*The Fundamental Theorem of Asset Pricing*"

Any Questions?