Martingale Measure

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What is a martingale?

- Stochastic process
- Defined by conditional mean
- Models a fair `game`
- Knowledge of past events do not help to predict the expectation of the future
- Unbiased random walk is a martingale
  - At any point in the observed sequence the expectation of the next value only depends on the current value
- Drawing from a card Deck is not a martingale
  - Prior events can in fluence the probability of drawing certain card in the future and therefore change the expectation

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There are three diffrent definitions that are important for martingales

- Def. of Filtration
- Def. of a measurable function
- Def. of almost surely convergence

Definition of a Filtration

Requirement

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, $\mathcal{T} \subset \mathbb{R}$ an index set and for every $t \in \mathcal{T}$ let $\mathcal{F}_t$ be a Sub-σ-Algebra of $\mathcal{A}$.

Then the set of σ-Algebra $\mathbb{F} = (\mathcal{F}_t)_{t \in \mathcal{T}}$ is called a filtration on $(\Omega, \mathcal{A}, \mathbb{P})$ or in $\mathcal{A}$ if

for every $s, t \in \mathcal{T}$ with $s \leq t$ implies $\mathcal{F}_s \subseteq \mathcal{F}_t$

A filtration models the available informations at different times in a random process.

Definiton of a measurable function

<u>Requirement</u>

Let $(\Omega',\ \mathcal{F}',\ \mathbb{P}')$ be a probability Space and $(\Omega,\ \mathcal{F})$ a measurable space.

A function $X:\ \Omega' \to \Omega$ is called measurable if

for all $A \in \mathcal{F}$ the following is true: $X^{-1}(A) := \{\omega' \in \Omega' : X(\omega') \in A\} \in \mathcal{F}$

Definiton of $\mathbb{P}$ −almost surely convergence

<u>Requirement</u>

Let $(X_n)_{n\in\mathbb{N}},\ X$ real random variables in the probability space $(\Omega, \mathcal{A}, \mathbb{P})$.

X_n converges $\mathbb{P}$ −almost surely to X $(X_n \to X)$ if

$$\mathbb{P}[\{\omega :\ lim_{n\to\infty} X_n(\omega) = X(\omega)\}] = 1$$

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Definition of a Martingale (discrete case)

<u>Requirement</u>

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be an probability space, $\mathbb{F} = (\mathcal{F}_n)_{n\in\mathbb{N}}$ is a filtration in $\mathcal{A}$ and $X = (X_n)_{n\in\mathbb{N}}$ a stochastic process on $(\Omega, \mathcal{A})$.

<u>It applies</u>

- $\mathbb{E}[|X_n|] < \infty$ for all $n \in \mathbb{N}$ (integrable process)
- X_n is $\mathcal{F}_n$-measurable for all $n \in \mathbb{N}$ (the process is adapted to $\mathbb{F}$)

Then X is a martingale regarding $\mathbb{F}$ if

$\mathbb{E}[X_{n+1} \mid \mathcal{F}_n] = X_n$ $\mathbb{P}$-almost surely for all $n \in \mathbb{N}$.

Whereby $\mathbb{E}[Y \mid \mathcal{B}]$ is the conditional mean of the random variable Y given the σ-Algebra $\mathcal{B}$.

The expectation of the next step is equal to the current value given the available information $\mathcal{F}_n$.

Definiton of a Martingale (general case)

<u>Requirement</u>

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be an probability space, $\mathcal{T}$ an ordered index set and $\mathbb{F} = (\mathcal{F}_t)_{t \in \mathcal{T}}$ a filtration in $\mathcal{A}$.

$X = (X_t)_{t \in \mathcal{T}}$ is a stochastic process on $(\Omega, \mathcal{A})$.

<u>It applies</u>

- $\mathbb{E}[|X_t|] < \infty$ for all $t \in \mathcal{T}$ (integrable process)
- X_t is $\mathcal{F}_t$-measurable for all $t \in \mathcal{T}$ (the process is adapted to $\mathbb{F}$)

Then X is a martingale regarding $\mathbb{F}$ if for all $t \in \mathcal{T}$

$\mathbb{E}[X_s \mid \mathcal{F}_t] = X_t$ $\mathbb{P}$-almost surely for all $s > t$.

Not only for the next step rather for all further steps (and all time points between those steps) the expactation is equal to the current value given the available information $\mathcal{F}_t$.

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Supermartingale

An integrable and to a filtration $\mathbb{F}$ adapted discrete stochastic process is called a supermartingale if

$\mathbb{E}[X_{n+1} \mid \mathcal{F}_n] \leq X_n$ $\mathbb{P}$-almost surely for all $n \in \mathbb{N}$.

In the continuous case it is $\mathbb{E}[X_s \mid \mathcal{F}_t] \leq X_t$ $\mathbb{P}$-almost surely for all $s > t$.

That implies that supermartingales tend to decline over time.

Submartingale

An integrable and to a filtration $\mathbb{F}$ adapted discrete stochastic process is called a submartingale if

$\mathbb{E}[X_{n+1} \mid \mathcal{F}_n] \geq X_n$ $\mathbb{P}$-almost surely for all $n \in \mathbb{N}$.

In the continuous case it is $\mathbb{E}[X_s \mid \mathcal{F}_t] \geq X_t$ $\mathbb{P}$-almost surely for all $s > t$.

That implies that submartingales tend to increase over time.

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Example of a martingale

<u>A fair gambling game is an example of a martingale.</u>

Let K_0 be the starting capital of a player

X_i is the random profit of round i (can be negative, positive and zero)

This implies the capital of the player after round one is $K_1 = K_0 + X_1$

In general after round n the capital is $K_n = K_0 + \sum_{i=1}^{n} X_i$

In a fair gambling game the expected profit is zero $\Longrightarrow \mathbb{E}[X_i] = 0$ for all $i \in \mathbb{N}$

Example of a martingale

After n rounds the values $K_0, K_1, \dots, K_n$ are known

If the profit in round $n+1$ is independent of the observed game then the expected capital is $K_{n+1} = K_n + X_{n+1}$

This implies

$$\mathbb{E}[K_{n+1} \mid K_0, \dots, K_n] = \mathbb{E}[K_n \mid K_0, \dots, K_n] + \mathbb{E}[X_{n+1} \mid K_0, \dots, K_n] = K_n + \mathbb{E}[X_{n+1}] = K_n$$

$\Longrightarrow K \coloneqq (K_i)_{i \in \mathbb{N}}$ is a martingale

This shows a martingale models a fair gambling game.

In the real world gambling games are not fair.

The doubling strategy for roulette is a supermartingale, because the probability to hit black or red ist smaller than ½.

$\Longrightarrow \mathbb{E}[X_i] < 0 \Longrightarrow \mathbb{E}[K_{n+1} \mid K_0, \dots, K_n] \leq K_n$

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Multi-period financial market model

The n-dimensional multi-period financial market model $M = ((\Omega, \mathcal{A}, \mathbb{F}, \mathbb{P}), (S_t)_{t\in\mathcal{T}})$ with:

1. $(\Omega, \mathcal{A}, \mathbb{P})$ a probability space

2. $\mathcal{T} = \{0, \dots, T\}$ set of the time period (Note: tradig times are $\frac{1}{2}, \dots, T - \frac{1}{2}$)

3. $\mathbb{F} = (\mathcal{F}_t)_{t\in\mathcal{T}}$ filtration in $\mathcal{A}$

4. Stochastic pricing process $(\bar{S}_t)_{t\in\mathcal{T}}: (\Omega, \mathcal{A}, \mathbb{P}) \longrightarrow \mathbb{R}^{n+1}_{\geq 0}$ adapted to $\mathbb{F}$ with $\bar{S}_t = (S_t^0, \dots, S_t^n)^T$ where S_t^i is the price of asset i at time t.

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Numeraire

$(S_t^0)_{t\in\mathcal{T}}$ is called a numeraire when it is used as relative reference value for all other assets $(S_t^i)_{t\in\mathcal{T}}$ as follows:

The process $(\bar{X}_t)_{t\in\mathcal{T}}$ with the vectors $\bar{X}_t$ and the entries $X_t^i = \; S_t^i \;/\; S_t^0$ is called the relative asset process.

X_t^i is the value of asset i at time t in units of S^0.

This means all other asset prices are discounted by the growth rate of asset S^0.

Note: Any asset can be the numeraire.

The things that change are the underlying probability measure $\mathbb{P}$ and the drift of all prices.

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Risk-neutral measure

- Among those $n+1$ assets can be a risk-free asset $\beta(t)$ with a risk-free rate $r(t)$.

- Price at time t for the risk-free asset is $\beta(t) = \exp\{\int_0^t r(u)du\}$ (Accumulation of the growth rate $r(t)$ until time t)

- Using such an asset as numeraire means changing the probability measure to $\mathbb{P}_\beta$ through
  $(d\mathbb{P}_\beta \ / \ d\mathbb{P}_0)_t = (\beta(t)/Z(t))/(\beta(0)/Z(0))$ with
  $Z(t)$ is a stochastic discount factor and $\mathbb{P}_0$ the original probability measure.

- **Important note:** $\beta(t)/Z(t)$ is a martingale under $\mathbb{P}_0$.

Risk-neutral measure

- $\mathbb{P}_\beta$ is called the **risk-neutral measure**.

- A **risk-neutral measure** is a probability measure such that each asset price is exactly equal to the discounted expectation of the asset price under $\mathbb{P}_\beta$.

- $\mathbb{P}_\beta$ is equivalent to $\mathbb{P}_0$ in the sense of measures ($\mathbb{P}_\beta[B] = 0 \Leftrightarrow \mathbb{P}_0[B] = 0$).

Risk-neutral measure

- After changing to the new measure $\mathbb{P}_\beta$ it is possible to discount any asset S^i with $\beta(t)$ which means $\rightsquigarrow$ $(S_t^i \,/\, \beta(t))$.

- The corresponding pricing formula for the other assets is $V(t) = \mathbb{E}_\beta \left[\exp\left\{-\int_t^T r(u)du\right\} V(T) \middle| \mathcal{F}_t\right]$.
- This means expressing the current price $V(t)$ of an asset as the expectation of the terminal value $V(T)$ discounted by the risk-free rate $r(t)$ under $\mathbb{P}_\beta$.

- The drift of all assets is then equal to the risk-free rate $r(t)$ meaning in a world of risk-neutral investors, the rate of return on risky assets is the same as the risk-free rate.

Risk-neutral measure

- It is possible to choose any of the assets S^i as the numeraire instead of $\beta(t)$.

- This means changing the probability measure to $\mathbb{P}_{S^i}$ through $(d\mathbb{P}_{S^i} \mathbin{/} d\mathbb{P}_{\beta})_t = (S_t^i \mathbin{/} \beta(t)) \mathbin{/} (S_0^i \mathbin{/} \beta(0))$.

- **Important note:** $S_t^i \mathbin{/} \beta(t)$ is a martingale under $\mathbb{P}_{\beta}$.

- $\mathbb{P}_{S^i}$ is equivalent to $\mathbb{P}_{\beta}$ in the sense of measures.

Risk-neutral measure

- After changing to the new measure $\mathbb{P}_{S^i}$ it is possible to discount any other asset S^d with S_t^i which means $\leadsto$ $(S_t^d \ / \ S_t^i)$.

- The pricing formula becomes $V(t) = \ S_t^i \mathbb{E}_{S^i}\left[\left(V(T)/S_T^i\right)\middle|\mathcal{F}_t\right]$
  with $\mathbb{E}_{S^i}[Y] = \ \mathbb{E}_\beta\left[Y(d\mathbb{P}_{S^i} \ / \ d\mathbb{P}_\beta)_t\right] = \ \mathbb{E}_\beta[Y \cdot (S_t^i \beta(0)) \ / \ (\beta(t) S_0^i)]$

- This means expressing the current price $V(t)$ of an asset as the expectation of the terminal value $V(T)$ divided by the terminal value of the numeraire under $\mathbb{P}_{S^i}$ and multiply this with the current value of the numeraire.

Risk-neutral measure

To summarize:

- Changing the numeraire also changes the probability measure and the drift of the prices.

- $S_t^i \,/\, \beta(t)$ is a martingale under $\mathbb{P}_\beta$ and $S_t^d \,/\, S_t^i$ is a martingale under $\mathbb{P}_{S^i}$, because $\beta(t)$ and S_t^i are attainable prices which means they are $\mathcal{F}_t$-measureable.
- The **Girsanov theorem** gives the proof of exactly that. If you start with a martingale you can change the numeraire and therefore the price process and get an equivalent probability measure under which you have an equivalent martingale.

- $\mathbb{P}_\beta \sim \mathbb{P}_{S^i}$ for all $i \in \{0, \dots, n\}$, this means all ratios $S_t^d \,/\, S_t^i$ are martingales.

Risk-neutral measure

To summarize:

- If there is a risk-free asset with a risk-free rate it is possible to change the original probability measure to a new equivalent one the **risk-neutral measure**. And then use this asset as numeraire. You get a martingale under this **risk-neutral measure**.

- After that it is possible (Girsanov theorem) to find other equivalent probability measures for using other assets as numeraire. You get equivalent martingales.

- This means it is possible to change the numeraire without losing the martingale properties and therefore it is possible to use any asset or discount factor for discounting the price process.

Example with a stochastic interest rate

- Considering an interest rate derivative with a payoff of $V(T)$ at time T.

- When using the **risk-neutral measure** the price can be expressed as $V(0) = \mathbb{E}_{\beta}[\exp\left\{-\int_0^T r(u)du\right\}V(T)]$.

- Now taking a zero-coupon bond as numeraire.
- (gives no periodic interest payments instead it is bought at a price lower than its face value, with the face value of 1 repaid at time of maturity T_M)

- Denote the value of the bond at time t with $B(t, T_M)$ so that $B(T_M,\ T_M) = 1$.

- The associated measure is $\mathbb{P}_{T_M}$.

Example with a stochastic interest rate

- Using this measure the price becomes $V(0) = B(0,T_M)\mathbb{E}_{T_M}[V(T) \,/\, B(T,T_M)]$.

- If choosing $T_M = T$ the price becomes $V(0) = B(0,T)\mathbb{E}_T[V(T)]$.

Observation:

- The discount factor (initial bond price) is deterministic even though the interest rate $r(t)$ may be stochastic.

- This leads to a simplification of the pricing formula just by changing the measure and numeraire.

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Market price of risk

- Considering a stochastic discount factor $Z(t)$ with $\mathbb{P}_0$.

- $Z(t)$ may not correspond to an asset price. It just models an additional discount factor.

- We find that $\left(d\mathbb{P}_0 \;/\; d\mathbb{P}_\beta\right) = Z(t) \;/\; \beta(t)$ and therefore $\exp\left\{-\int_0^t r(u)du\right\} Z(t)$ is a positive martingale under $\mathbb{P}_\beta$.

- This time we discounted the discount factor with the risk-free rate.

- Switchig between $\mathbb{P}_\beta$ and $\mathbb{P}_0$ is equivalent to switching the numeraire between $Z(t)$ and $\beta(t)$. This is a general change of measure identity.

Market price of risk

- The existence of a stochastic discount factor implies that the drifts of the asset prices also have an additional factor.

- This factor determines the amount by which the drift of an asset exceeds the risk-free rate $r(t)$.

- This factor can be interpreted as a risk premium which decribes the additional value of an asset price by taking a higher risk.

- This value is called the **market price of risk**.

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First fundamental theorem of finance mathematics

<u>Given:</u>

1. Probability space $(\Omega, \mathcal{A}, \mathbb{P})$

2. Discrete time period $1, \dots, T$

3. Random pricing process S

4. All possible trading strategies $\mathcal{P}$

First fundamental theorem of finance mathematics

<u>Notations:</u>

1. $\mathbb{F} = (\mathcal{F}_n)_{t=0,\dots,T}$ filtration in $\mathcal{A}$

2. $S = \ (S_t)_{t=0,\dots,T}$ stochastic price process adapted to $\mathbb{F}$

3. $\mathcal{P} = \left\{(H_t)_{t=0,\dots,T} \,\middle|\, H_t \ is \ \mathcal{F}_t - measurable\right\}$ set of trading strategies

4. $H \cdot S_T \coloneqq \ \sum_{t=1}^{T} H_t(S_t - S_{t-1})$ trading profits

5. $R_T \ \coloneqq \{\xi \mid \xi = H \cdot S_T, \ H \in \mathcal{P}\}$ set of all realisable profits

6. $B_T \ \coloneqq \ R_T \setminus L_+^0$ with L_+^0 set of all positive random variables

First fundamental theorem of finance mathematics

There is no risk-free profit (arbitrage) if and only if the pricing process is a martingale under an equivalent probability measure.

This means following statements are equivalent:

1. $B_T \cap L^0_+ = \{0\}$ (there are no realisable profits)

2. There is an equivalent probability measure $\mathbb{P}' \sim \mathbb{P}$ with $d\mathbb{P}'/d\mathbb{P} \in L^\infty$ so that S is a $\mathbb{P}'$-martingale

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Consequences

What have we seen?

- The pricing process discounted by a risk-free rate is a martingale under the risk-neutral measure.
- If there is a risk-neutral measure you find other equivalent probability measures under which the pricing process discounted by other rates or assets is an equivalent martingale. (Change of Numeraire)

The theorem implies

- A risk-neutral measure, which is equivalent to the original probability measure, only exists if the market is arbitrage-free.

Or the other way around

- The market ist arbitrage-free only if a risk-free rate exists.

$\Longrightarrow$ Only a arbitrage-free market is a fair market!

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Risk-free rate in the real world

- In order to use martingales to describe arbitrage-free markets a risk-free rate is necessary.
- Other market models like the Black-Scholes-Model also need a risk-free rate and a arbitrage-free market.

➢ <u>But in the real world are no 100% risk-free rates</u>

<u>Solution:</u>

Use as well as risk-free rates

Long-term

- Federal bonds from Germany (for the Euro)
- Governments bonds from the United States (for the US-Dollar)

Short-term

- EURIBOR (Euro InterBank Offered Rate), the rate at which banks loan each other money
- Three months treasury bonds from the US-Goverment

<u>Sources:</u>

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Thanks for your attention!