Seminar

Monte-Carlo Methods in Finance Practice

Geometric Brownian Motion

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Contents

I. Introduction

Stochastic Process

- Collection of random variables on $(\Omega, \mathcal{F}, \mathrm{P})$
- For given $(\Omega, \mathcal{F}, \mathrm{P})$ and (S, Σ) : $\{ X(t) : t \in T \}$
- Applications in many disciplines
- Random changes in financial market $\longrightarrow$ stochastic process in finance
- Examples : Bernoulli Process, Poisson Process, Brownian Motion,... etc

Brownian Motion

- Historical connection with physical process “Brownian Movement‘‘
- Often used in pure and applied mathematics, physics, biology
- Important role in finance modeling and simulating path
- continuous-time stochastic process, called Wiener Process
- Louis Bachelier modeled price changes in early 1900

Properties

i. $W(0) = 0$

ii. $\forall 0 \leq t < T \leq s < S$, $W(T) - W(t), W(S) - W(s)$ independent

iii. $\forall 0 \leq t < s$, $W(s) - W(t)$ normal random variable

iv. $\forall \omega \in \Omega$, path $t \mapsto W(t)(\omega)$ is a continuous function

v. For each $t > 0$, $W(t)$ normally distributed with

- zero mean
- variance t
- density $f(t,x) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t}$

Related Process

- Brownian Motion with Drift

  - Definition: A Brownian Motion $X(t)$ is the solution of an SDE with constant drift and diffusion coefficients

$$dX(t) = \mu dt + \sigma dW(t)$$

with initial value $X(0) = x_0$

  - By direct Integration:

$$X(t) = x_0 + \mu t + \sigma W(t)$$

mean: $x_0 + \mu t$ variance : $\sigma^2 t$ density: $\frac{1}{\sigma\sqrt{2\pi t}} e^{-(x-x_0-\mu t)^2/2\sigma^2 t}$

Related Process

- Geometric Brownian motion

A stochastic process, which is used to model processes

<u>that can never take on negative values</u>,

such as the values of stocks.

II. Geometric Brownian Motion

Geometric Brownian Motion

- Definition: Suppose W is a standard Brownian Motion. A stochastic process $S(t)$ is said to follow a GBM if it satisfies the following SDE:

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t)$$

with drift parameter μ and volatility parameter σ

$$\Longrightarrow \qquad S(t) = \exp\left[\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W(t)\right], \qquad t \in [0, \infty)$$

- $S(t)$ is Geometric Brownian Motion: $S \sim GBM(\mu, \sigma^2)$

- If $S(t)$ has initial value $S(0)$, then

$$S(t) = S(0)\exp\left[\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W(t)\right]$$

more generally, if $u < t$

$$S(t) = S(u)\exp\left[\left(\mu - \frac{\sigma^2}{2}\right)(t-u) + \sigma\big(W(t) - W(u)\big)\right]$$

- Increments of W are independent and normally distributed, reverse procedure

$$S(t_{i+1}) = S(t_i)\exp\left[\left(\mu - \frac{\sigma^2}{2}\right)(t_{i+1} - t_i) + \sigma\sqrt{t_{i+1} - t_i}Z_{i+1}\right]$$

for simulating values of S at $0 = t_0 < t_1 < \cdots < t_n$ with $Z_1, Z_2, \ldots, Z_n$ independent standard normals.

Basic Properties

- Return Values $\{\frac{S_{t_{i+1}}}{S_{t_i}}\}$ are independent for $0 \leq t_i \leq t_{i+1} \leq T$
- Simply an exponentiated Brownian Motion $\Longrightarrow$ only positive values
- Mean: (using moment generating function)

$$\mathbb{E}[S(t)] = \mathbb{E}[S_0\, exp(\mu t + \sigma W(t))]$$

$$= S_0 \exp\left(\mu t + \frac{\sigma^2}{2} t\right)$$

- Variance: $$Var[S(t)] = \mathbb{E}[(S(t) - \mathbb{E}[S(t)])^2]$$

$$= S_0^2 \exp(2\mu t + \sigma^2 t)(\exp(\sigma^2 t) - 1)$$

- $\mu S(t)$ represent the drift, the deterministic portion

  ↪ If $\mu > 0$, generally assume a positive growth

  ↪ If $\mu < 0$, generally assume a fall off

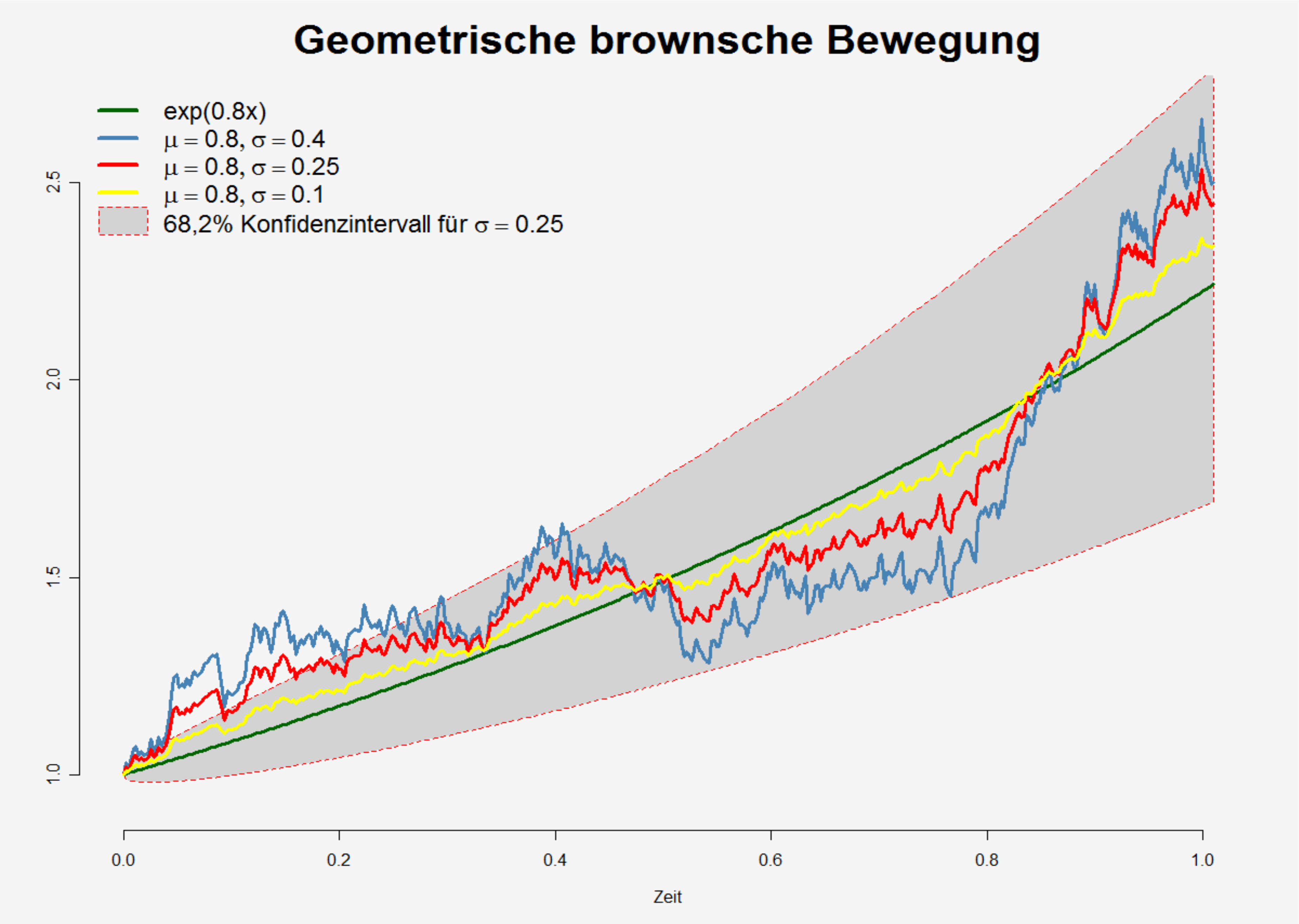
- $\sigma S(t)$ represent the diffusion term, the stochastic portion

  ↪ If $\sigma = 0 \Longrightarrow$ Deterministic differential equation

- Parameter $\mu - \frac{\sigma^2}{2}$ determines the asymptotic behavior of GBM

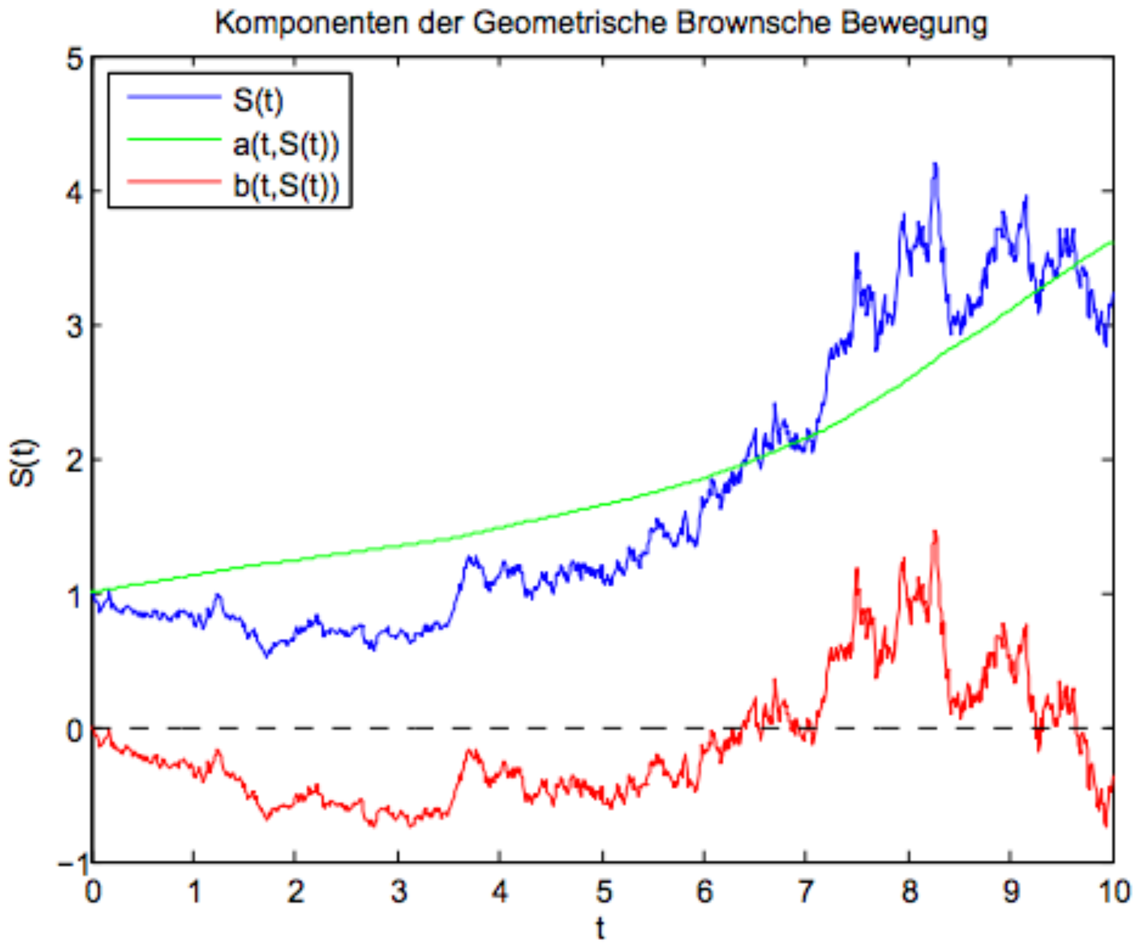
  - If $\mu > \frac{\sigma^2}{2}$ then $X_t \longrightarrow \infty$ as $t \longrightarrow \infty$ with probability 1
  - If $\mu < \frac{\sigma^2}{2}$ then $X_t \longrightarrow 0$ as $t \longrightarrow \infty$ with probability 1
  - If $\mu = \frac{\sigma^2}{2}$ then X_t has no limit as $t \longrightarrow \infty$ with probability 1

- If drift parameter μ is 0 $\Longrightarrow$ GBM is a martingale



Geometrische brownsche Bewegung
exp(0.8x)
μ = 0.8, σ = 0.4
μ = 0.8, σ = 0.25
μ = 0.8, σ = 0.1
68,2% Konfidenzintervall für σ = 0.25
1.0
1.5
2.0
2.5
0.0
0.2
0.4
0.6
0.8
1.0
Zeit

- GBM and his components (a deterministic and b stochastic)
  $S(0) = 1, \quad \mu = 0.15, \quad \sigma = 0.3, \quad T = 10$



Multiple Dimensions

- Multidimensional GBM specified through system of SDEs

$$\frac{dS_i(t)}{S_i(t)} = \mu_i dt + \sigma_i X_i(t) \text{ , } i = 1, \ldots, d$$

  where X_i is standard one-dimensional BM with

- $X_i(t)$ and $X_j(t)$ have correlation ρ_{ij}
- Define Σ as $d \times d$ Matrix with $\Sigma_{ij} = \sigma_i \sigma_j \rho_{ij}$ (covariance Matrix of S)
- $S = (S_1, \ldots, S_d)$ with $\mu = (\mu_1, \ldots, \mu_d) \Longrightarrow S \sim GBM(\mu, \Sigma)$

- Let $\Sigma = AA^T$, $BM(0, \Sigma)$ can be written as $AW(t)$ with $W \sim BM(0, \mathrm{I})$

$$\Longrightarrow \frac{dS_i(t)}{S_i(t)} = \mu_i dt + a_i dW(t), i = 1, \ldots, d$$

with a_i the i-th Row of A

- More explicitly

$$\frac{dS_i(t)}{S_i(t)} = \mu_i dt + \sum_{j=1}^{d} A_{ij} dW(t), i = 1, \ldots, d$$

Simulating multiple Dimensions GBM

- For one-dimensional GBM, it is

$$S(t_{i+1}) = S(t_i)\exp\left[\left(\mu - \frac{\sigma^2}{2}\right)(t_{i+1} - t_i) + \sigma\sqrt{t_{i+1} - t_i}Z_{i+1}\right]$$

- For multiple dimensional GBM, it is

$$S_i(t_{k+1}) = S_i(t_k)\exp\left[\left(\mu_i - \frac{{\sigma_i}^2}{2}\right)(t_{k+1} - t_k) + \sqrt{t_{k+1} - t_k}\sum_{j=1}^{d} A_{ij}\, Z_{k+1,j}\right]$$

$$i = 1, \dots, d; \quad k = 0, \dots, n-1; \quad Z_k = (Z_{k1}, \dots, Z_{kd}) \sim N(0, I)$$

III. Application in Financial Modeling

Geometric Brownian Motion in Finance

- model stock prices, for example in the Black-Scholes Model
- most widely used
- Question: How realistic is GBM regarding finance modeling?

Advantages

- Expected returns are independent of the value of the stock price
- GBM process only assumes positive values
- GBM process shows the same kind of ‘roughness‘ in paths
- Calculations with GBM are relatively easy

Disadvantages

- Volatility is assumed constant
- In GBM the path is continuous

  ↳ in reality, stock prices often jumps caused by unpredictable events and news

Extensions

- Attempt to make GBM more realistic
- Volatility σ has to be inconstant

I. Local Volatility model : volatility a deterministic function

II. Stochastic Volatility model : volatility has randomness , described by different equation by different BM

Application in Finance - stock price

- The increase of the stock price in the next intervall is ΔS
- It is

$$\Delta S = S * \left(\mu \Delta t + \ \sigma \sqrt{\Delta t} * Z \right)$$

where $Z \sim N(0,1)$ random

- Example 1: $\mu = 15\ \%,\ \sigma = 30\ \%$, inital stock price is 100 €, consider time intervall of one week (0.0192 year)

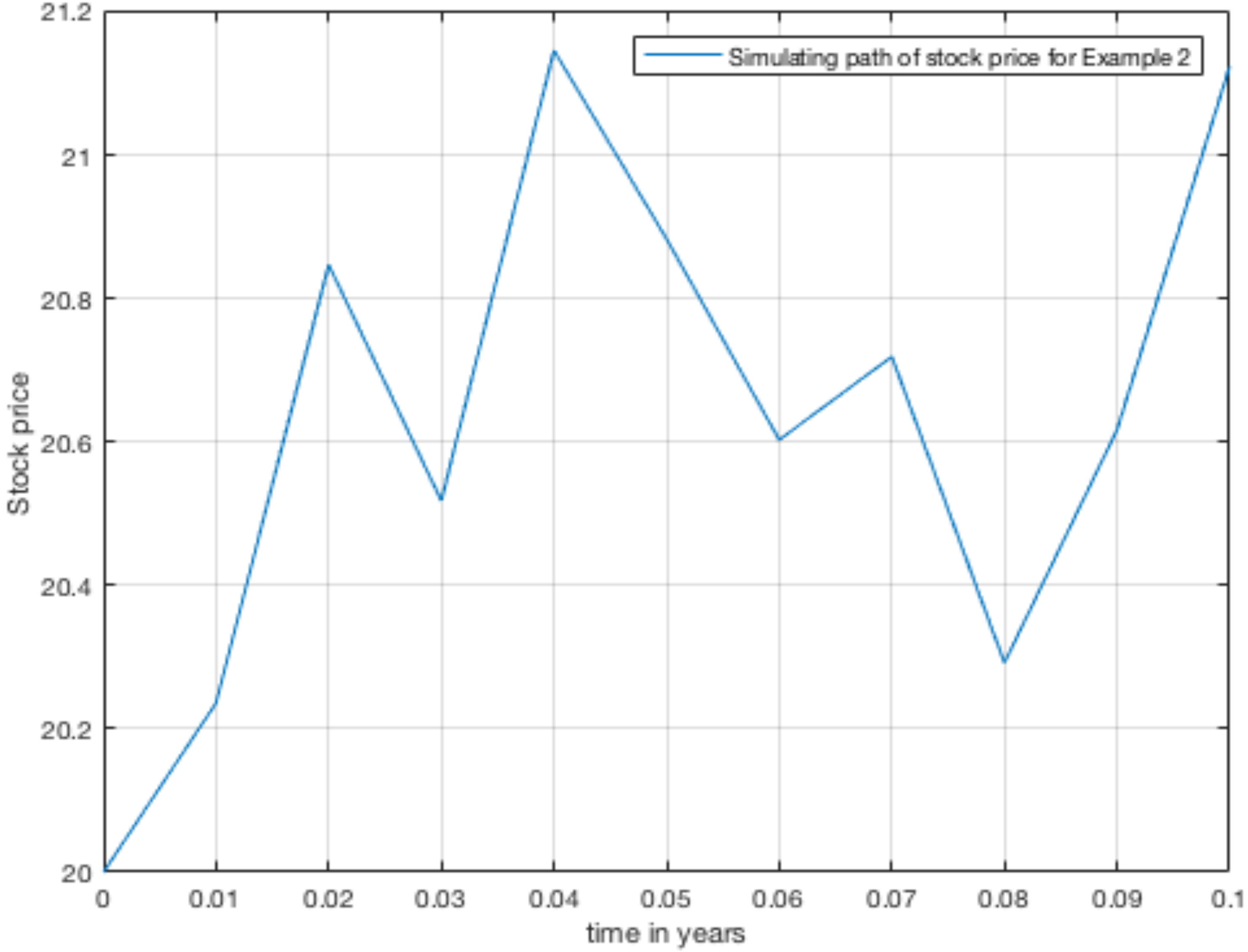
$$\Delta S = 100 * (0.15 * 0.0192 + 0.3 * \sqrt{0.0192} * Z)$$
$$= 100 * (0.00288 + 0.0416Z) = 0.288 + 4.16Z$$

$\Longrightarrow$ price increase with mean 0.288 € and standard deviation of 4.16 €

- Example 2: $\mu = 14\ \%,\ \sigma = 20\ \%$, suppose $\Delta t = 0.01$

$$\Delta S = 0.14 * 0.01 * S + 0.2\sqrt{0.01}SZ$$
$$= 0.0014S + 0.02SZ$$

Stock Price at Start of Period	Random Sample for Z	ΔS during Period
20.000	0.52	0.236
20.236	1.44	0.611
20.847	- 0.86	- 0.329
20.518	1.46	0.628
21.146	- 0.69	-0.262
20.883	- 0.74	- 0.280
20.603	0.21	0.115
20.719	- 1.10	- 0.427
20.292	0.73	0.325
20.617	1.16	0.507
21.124	2.56	1.111



Simulating path of stock price for Example 2
Stock price
time in years
21.2
21
20.8
20.6
20.4
20.2
20
0
0.01
0.02
0.03
0.04
0.05
0.06
0.07
0.08
0.09
0.1

Application in Finance - Path-Dependent Options

- Focus on pricing options
- Not simply value $S(T)$, but the path
- Essential is the choice of drift parameter μ
- Assume existence of constant continuously compounded interest rate r
- The growth rate is $\beta(t) = e^{rt}$

- Suppose S pays no dividends
- Under risk-neutral-measure, discounted price process

$$\frac{S(u)}{\beta(u)} = \mathbb{E}[\frac{S(t)}{\beta(t)} \mid \{S(\tau), 0 \leq \tau \leq u\}]$$

is a martingale

- GBM is lognormal distribution, so $\mathbb{E}[S(t) | \{S(\tau), 0 \leq \tau \leq u\}] = e^{\mu(t-u)}S(u)$

$\Longrightarrow$ If S is a GBM under risk-neutral measure, then $\mu = r$

$$\Longrightarrow \quad \frac{dS(t)}{S(t)} = rdt + \sigma dW(t)$$

- Suppose S pays dividends
- Let $D(t)$ be value of any dividends and interest earned on those dividends
- Suppose asset pays a continous dividend yield δ
- Then $$\frac{dD(t)}{dt} = \underbrace{\delta S(t)}_{influx\ of\ new\ dividends} + \underbrace{rD(t)}_{\substack{interest\ earned\ on\ accumulated \\ dividends}}$$
- The Martingale property requires $\mu + \delta = r;\ i.e.,\quad \mu = r - \delta$

Specific Settings

- Pricing index options (Equity Indices)
  - Level of index modeled as GBM
  - Index not an asset, not paying dividends
  - Individual stocks pays dividends
  - Effect often approx. by continous dividend yield

- Pricing currency options (Exchange Rates)
  - exchange rate S as price of foreign currency
  - A unit of foreign currency has risk-free rate r_f
  - $\mu = r - r_f$

Financial Models using GBM

- Simply GBM not 100% realistic
- GBM essential in many models simulating stock prices
- Many other models more fitting, but using GBM principle

  I. Black Scholes Model
  II. Merton Model
  III. Heston Model
  IV. Bates Model

Black-Scholes-Model

- Fischer Black and Myron Scholes 1973 in a seminal paper
- Estimates price of the option over time
- Main Idea: buy and sell asset in the ‘‘right way‘‘ (hedge option)

  $\Longrightarrow$ eliminate risk
- BS-equation: $\frac{dV}{dt} + \frac{1}{2}\sigma^2 S^2 \frac{d^2V}{dS^2} + rS\frac{dV}{dS} - rV = 0$
- Used for the European call option price
- Underestimation of extreme moves
- Useful approximation, robust basis for refined models

Merton Model

- Robert Merton 1976, revised BSM Model
- Allows discontinuous trajectories
- Adding jumps to the stock price dynamics
- Equation:

$$\frac{dS}{S} = rdt + \sigma dW + dZ$$

- Z is a compound Poisson process with lognormal-distributed jumps

Heston Model

- Volatility paramter not constant anymore
- Stochastic Volatility Models
- σ is another GBM

Bates Model

- Bates 1996
- Combination of both
- Stochastic volatility and jumps

IV. Summary

Summary

- GBM essential for simulating path for stock prices/options
- ‘‘relatively‘‘ easy calculation
- Less data is needed to forecast future than some other forecasting models
- More accurate for short-term
- Due to obvious reasons $\Longrightarrow$ Revision via more complex models
- No models are absolutely right, have their own risk

References

- Paul Glassermann, “Monte Carlo Methods in Financial Engineering“ (2003), p.93-107
  Springer Verlag

Thank you for your attention!