Monte-Carlo Methods in Financial Engineering

Regression-Based Methods for Pricing American Options

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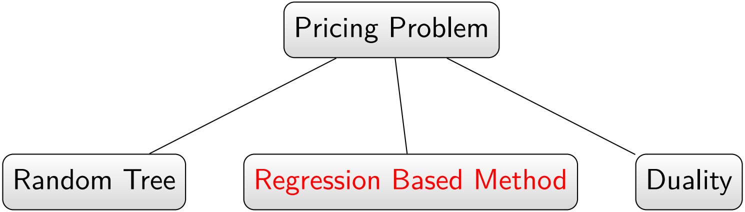
Introduction

- The pricing of an American option is a mathematical challenge in finance for several years.
- American options can be exercised at any time up to their expiration dates.
- We want to approximate American Options by considering Bermudan Options.

European $\leq$ Bermudan $\leq$ American

Introduction



Random tree method:

- Very simple to implement. Provides a confidence interval containing the true option value.
- Suitable only for options with few exercise opportunities (not more than about five).

⇒ The regression-based method is a very powerful technique for solving high-dimensional problems with many exercise opportunities.

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The continuation value of a Bermudan option is the value of holding rather than exercising the option.

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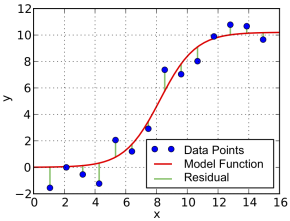
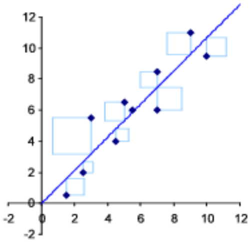
Optimal Stopping Rule

$$\hat{\tau} = \min\{i : h_i(X_i) \geq \hat{C}_i(X_i)\}$$

Least-Squares Method

- Standard approach in regression analysis that finds the line of best fit for a dataset.
- First, choose a well fitting model function (we will only use polynomial functions as model functions).
- The best fit is a function with minimal deviation to the data points. While using the least-square method, minimize the sum of squared residuals to find the best coefficients in your model function.
- Residual: The difference between the observed value and the fitted value, provided by the model.





Least-Squares Method

- Assume that our data can be well fitted by a polynomial function.
- We choose $\psi_s, r(x)$,$r = 1 \ldots, M$ basis functions and $s = 1, ..., N$.

- The given data set is approximated by a linear system of equations that looks like:

$$\underbrace{\begin{pmatrix} \psi_{1,1} & \cdots & \psi_{1,M} & \\ & \vdots & \ddots & \vdots \\ \psi_{N,1} & \cdots & \psi_{N,M} & \end{pmatrix}}_{A} \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_N \end{pmatrix} = \underbrace{\begin{pmatrix} y_1 \\ \vdots \\ y_N \end{pmatrix}}_{Y}$$

- β_i are unknown coefficients that we want to estimate by least-squares.

Least-Squares Method

- We are looking for some β_i so that the difference between the observed value and the fitted value, provided by the model, is as small as possible.
- So we want to find a β for the minimzation problem below:

$$\min_{\beta} ||Y - A \cdot \beta||_2^2$$

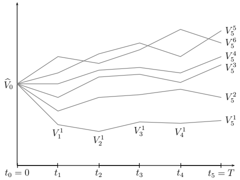
- How can you solve the linear system of equations?
  - Normal equations $A^T A\beta = A^T Y$
  - QR decomposition $A = QR$

Longstaff and Schwartz Algorithm - Overview

- The algorithm for valuing American options was developed by Francis A. Longstaff and Eduardo S. Schwartz in 2001.
- Hereafter, LSM will be the shortcut for Longstaff and Schwartz Algorithm.
- The algorithm can be divided into four sections:
  - Simulation of paths
  - Labeling the nodes at the expiration date
  - Carrying out a regression analysis
  - Valuing the option

Longstaff and Schwartz Algorithm

The intuition behind the approach is that the holder of an American option compares the payoff from exercising immediately with the expected payoff from continuation at any exercising date.



Step 1:

- Simulate b independent paths.

Step 2:

- At the final expiration date of the option, the investor exercises the option if it is in-the-money or allows it to expire if it is out of-the-money.

Longstaff and Schwartz Algorithm

Step 3a:

- We use the regression technique to approximate the conditional expectation function at each exercising date:
$$C_i(x) = \mathbb{E}[V_{i+1}(X_{i+1})|X_i = x]$$

- We assume that the continuation values C_i can be approximated by a linear combination of known functions of the current state:
$$C_i(x) = \mathbb{E}[V_{i+1}(X_{i+1})|X_i = x] = \sum_{r=1}^{M} \beta_{ir}\psi_r$$

- Then, we choose M basis functions and apply the least-squares method to estimate the coefficients in our regression function.

- We only use in-the-money paths in the estimation since the exercise decision is only relevant if the option is in-the-money.

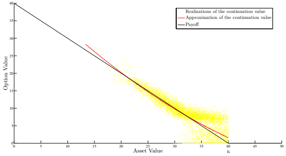
Longstaff and Schwartz Algorithm

Step 3b:

- Now, we can determine whether an early exercise or holding the option is optimal at the current time point. Repeating this procedure for each in-the-money path generates an optimal stopping rule:

$$\hat{V}_{ij} = \begin{cases} h_i(X_{ij}) & h_i(X_{ij}) \geq \hat{C}_i(X_{ij}) \\ \hat{V}_{i+1j} & h_i(X_{ij}) < \hat{C}_i(X_{ij}) \end{cases}$$

- Afterwards, we go back one point in time and repeat step 3.



Longstaff and Schwartz Algorithm

Step 4:

- Starting at time zero, we move forward along each path until the first stopping time occurs:
- Then, we discount the resulting cash flows from the different exercising dates:
- Finally, we take the average of all paths and obtain the value of the American option:

$$\hat{V}_0 = \frac{\hat{V}_{11} + \ldots + \hat{V}_{1b}}{b}$$

Example

- Bermudan put option on a share of non-dividend-paying stock
- Strike price: K= 1.10
- Exercise dates: $t = 1, t = 2, t = 3$
- Riskless rate: 6%
- 8 sample paths for the price of the stock
- Payoff-function: $h_m = (K - S(t))^+$
- All values are in Euro

Objective

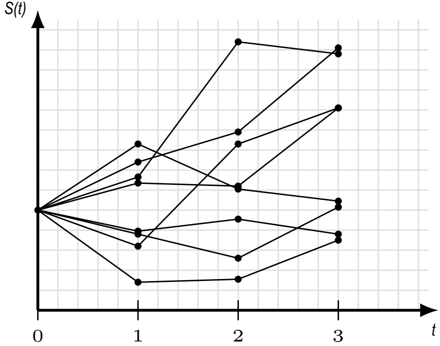
Find the stopping rule that maximizes the value of the option at each point along each path.

Step 1: Simulate 8 Independent Paths

The table below shows the simulated data for our example.

Table: Values of 8 different paths

Path	t=0	t=1	t=2	t=3
1	1.00	1.09	1.08	1.34
2	1.00	1.16	1.26	1.54
3	1.00	1.22	1.07	1.03
4	1.00	0.93	0.97	0.92
5	1.00	1.11	1.56	1.52
6	1.00	0.76	0.77	0.90
7	1.00	0.92	0.84	1.01
8	1.00	0.88	1.22	1.34



Step 2: Terminal Nodes

- At terminal nodes we set $\hat{V}_{3j} = h_3(X_{3j})$, $j = 1, ..., 8$ as option value.
- Our payoff-function is given by $h_m = (K - S(t))^+$, $m = 1, 2, 3$.

Path	t=1	t=2	t=3	$\hat{\tau}_3$
1	-	-	$1.10 - 1.34 = 0.00$	0
2	-	-	$1.10 - 1.54 = 0.00$	0
3	-	-	$1.10 - 1.03 = 0.07$	1
4	-	-	$1.10 - 0.92 = 0.18$	1
5	-	-	$1.10 - 1.52 = 0.00$	0
6	-	-	$1.10 - 0.90 = 0.20$	1
7	-	-	$1.10 - 1.01 = 0.09$	1
8	-	-	$1.10 - 1.34 = 0.00$	0

Step 3a: Regression (1)

- We have to find a function that approximates the continuation value at time 2.
- To do this, we use the regression approach.
- We only consider paths that are in-the-money.

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Path	t=0	t=1	t=2	t=3
1	1.00	1.09	1.08	1.34
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6	1.00	0.76	0.77	0.90
7	1.00	0.92	0.84	1.01
8	1.00	0.88	1.22	1.34

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6	1.00	0.76	0.77	0.90
7	1.00	0.92	0.84	1.01
8	1.00	0.88	1.22	1.34

Step 3a: Regression (1)

- We are looking for an expression like $Y_i = \beta_0 + \beta_1 \cdot X_i + \beta_2 \cdot X_i^2$
- $Y :=$ discounted cash flow at $t = 3$
- $X :=$ stock price at $t = 2$
- $i :=$ path

$$\underbrace{\begin{pmatrix} 1 & 1.08 & 1.08^2 \\ 1 & 1.07 & 1.07^2 \\ 1 & 0.97 & 0.97^2 \\ 1 & 0.77 & 0.77^2 \\ 1 & 0.84 & 0.84^2 \end{pmatrix}}_{\text{Values of path itm}} \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 0.00 \cdot 0.94176 \\ 0.07 \cdot 0.94176 \\ 0.18 \cdot 0.94176 \\ 0.20 \cdot 0.94176 \\ 0.09 \cdot 0.94176 \end{pmatrix}}_{\text{discounted payoffs of } t=3}$$

$$\Rightarrow Y_i = -1.069983 + 2.983396X - 1.813567X^2$$

Step 3b: Exercise decision (1)

Now, we try to figure out whether to exercise or to further hold the option, by comparing the exercise value and the continuation value.

Table: Early exercise decision at t=2

Path	Exercise	Continuation
1	0.02	0.0369
2	0.00	0.00
3	0.03	0.0461
4	0.13	0.1176
5	0.00	0.00
6	0.33	0.1520
7	0.26	0.1565
8	0.00	0.00

We obtain

- the exercise value by the payoff function $h_m = (K - S(t))^+$ and
- the continuation value by inserting X_{2b} in the model function that we estimated by regression.

Step 3b: Exercise decision (1)

Table: Early exercise decision at $t = 2$

Path	Exercise	Continuation
1	0.02	0.0369
2	0.00	0.00
3	0.03	0.0461
4	0.13	0.1176
5	0.00	0.00
6	0.33	0.1520
7	0.26	0.1565
8	0.00	0.00

We obtain a vector, which reflects the stopping rule for $t = 2$.

$$\hat{\tau}_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

Step 3a: Regression (2)

Determining the paths, which are in the money at time $t = 1$.

Table: Values of 8 different paths

Path	t=0	t=1	t=2	t=3
1	1.00	1.09	1.08	1.34
2	1.00	1.16	1.26	1.54
3	1.00	1.22	1.07	1.03
4	1.00	0.93	0.97	0.92
5	1.00	1.11	1.56	1.52
6	1.00	0.76	0.77	0.90
7	1.00	0.92	0.84	1.01
8	1.00	0.88	1.22	1.34

Step 3a: Regression (2)

- We are looking for an expression like $Y_i = \beta_0 + \beta_1 \cdot X_i + \beta_2 \cdot X_i^2$
- $Y :=$ discounted cash flow at $t = 2$
- $X :=$ stock price at $t = 1$
- $i :=$ path

$$\underbrace{\begin{pmatrix} 1 & 1.09 & 1.09^2 \\ 1 & 0.93 & 0.93^2 \\ 1 & 0.76 & 0.76^2 \\ 1 & 0.92 & 0.92^2 \\ 1 & 0.88 & 0.88^2 \end{pmatrix}}_{\text{Values of paths itm}} \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 0.00 \cdot 0.94176 \\ 0.13 \cdot 0.94176 \\ 0.33 \cdot 0.94176 \\ 0.26 \cdot 0.94176 \\ 0.00 \cdot 0.94176 \end{pmatrix}}_{\text{Discounted payoffs of } t=2}$$

$$\Rightarrow Y_i = 2.037503 - 3.335427X + 1.356450X^2$$

Step 3b: Exercise decision (2)

Again, we try to figure out whether to exercise or to further hold the option by comparing the exercise value and the continuation value.

Table: Early exercise decision at $t = 1$

Path	Exercise	Continuation
1	0.01	0.0139
2	0.00	0.00
3	0.00	0.00
4	0.17	0.1092
5	0.00	0.00
6	0.34	0.2866
7	0.18	0.1175
8	0.22	0.1533

We obtain

- the exercise value by the payoff function $h_m = (K - S(t))^+$ and
- the continuation value by inserting X_{1b} in the model function that we estimated by regression.

Step 3b: Exercise decision (2)

Table: Early exercise decision at $t = 1$

Path	Exercise	Continuation
1	0.01	0.0139
2	0.00	0.00
3	0.00	0.00
4	0.17	0.1092
5	0.00	0.00
6	0.34	0.2866
7	0.18	0.1175
8	0.22	0.1533

We obtain a vector, which reflects the stopping rule for $t = 1$.

$$\hat{\tau}_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

Step 4: Pricing

- Having identified the cash flows generated by the Bermudan put at each point along each path, the option can now be valued ...

Table: Stopping rule

Path	t=1	t=2	t=3
1	0	0	0
2	0	0	0
3	0	0	1
4	1	0	0
5	0	0	0
6	1	0	0
7	1	0	0
8	1	0	0

Table: Option cash flow matrix

Path	t=1	t=2	t=3
1	0.00	0.00	0.00
2	0.00	0.00	0.00
3	0.00	0.00	0.07
4	0.17	0.00	0.00
5	0.00	0.00	0.00
6	0.34	0.00	0.00
7	0.18	0.00	0.00
8	0.22	0.00	0.00

Step 4: Pricing

- ... by discounting each cash flow in the option cash flow matrix back to time zero
- and calculating the average of all paths.

- A mathematical frame for the discounted option value is given by:

$$\hat{V}_0 = \frac{\hat{V}_{11} + \ldots + \hat{V}_{1b}}{b}$$

Value of the Bermudan option

$$\hat{V}_0 = \frac{0.07 \cdot e^{-0.06 \cdot 3} + 0.17 \cdot e^{-0.06} + 0.34 \cdot e^{-0.06} + 0.18 \cdot e^{-0.06} + 0.22 \cdot e^{-0.06}}{8} = 0.11443433$$

Eurpean Put Vs. Bermudan Put

Value of the Bermudan option

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Value of the European option

$$\hat{V}_0 = \frac{0.07 \cdot e^{-0.06 \cdot 3} + 0.18 \cdot e^{-0.06 \cdot 3} + 0.20 \cdot e^{-0.06 \cdot 3} + 0.09 \cdot e^{-0.06 \cdot 3} +}{8} = 0.05638$$

Eurpean Put Vs. Bermudan Put

Value of the Bermudan option

$$\hat{V}_0 = \frac{0.07 \cdot e^{-0.06 \cdot 3} + 0.17 \cdot e^{-0.06} + 0.34 \cdot e^{-0.06} + 0.18 \cdot e^{-0.06} + 0.22 \cdot e^{-0.06}}{8} = 0.11443433$$

Value of the European option

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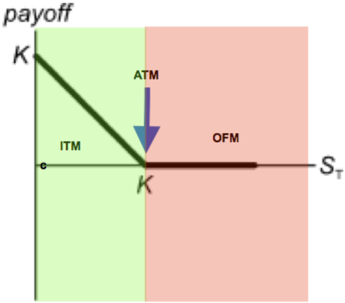
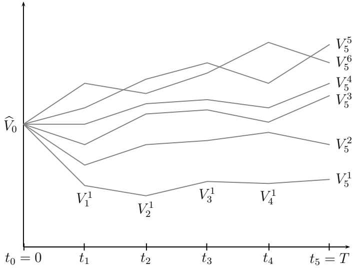
- The value of the Bermudan put is roughly twice the value of the European put.

Graphical Derivation

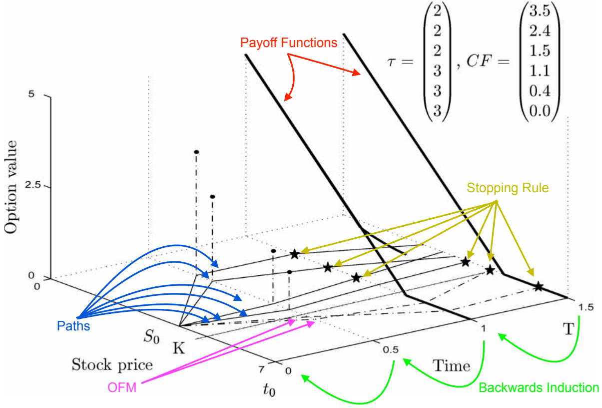




Graphical Derivation

Functionality of the algorithm:

- Six paths
- Four exercise dates including t_0



Pricing Algorithm by Longstaff and Schwartz

(1) Simulate b independent paths $\{X_{1j}, X_{2j}, \ldots, X_{mj}\}$, $j = 1, \ldots, b$

(2) At terminal nodes, set $\widehat{V}_{mj} = h_m(X_{mj})$, $j = 1, \ldots, b$

(3) Apply backward induction: for $i = m - 1, \ldots, 1$

- given the estimated values $\widehat{V}_{i+1j}$, $j = 1, \ldots, b$ use regression to calculate $\hat{\beta}_i$
- set

$$\widehat{V}_{ij} = \begin{cases} h_i(X_{i,j}) & h_i(X_{i,j}) \geq \widehat{C}_i(X_{i,j}) \\ \widehat{V}_{i+1j} & h_i(X_{i,j}) < \widehat{C}_i(X_{i,j}) \end{cases}$$

with $\widehat{C}_i(X_{ij}) = \widehat{\beta}_i^T \psi(x)$

(4) Set $\widehat{V}_0 = \frac{\hat{V}_{11} + \ldots + \hat{V}_{1b}}{b}$

Pricing Algorithm by Tsitsiklis and Van Roy

(1) Simulate b independent paths $\{X_{1j}, X_{2j}, ..., X_{m,j}\}$, $j = 1, ..., b$

(2) At terminal nodes, set $\widehat{V}_{m,j} = h_m(X_{m,j})$, $j = 1, ..., b$

(3) Apply backward induction: for $i = m - 1, ..., 1$

- given the estimated values $\widehat{V}_{i+1j}$, $j = 1, ..., b$ use regression to calculate $\widehat{\beta}_i$
- set

$$\hat{V}_{i,j} = \max\{h_i(X_{i,j}), \widehat{C}_i(X_{i,j})\}$$

with $\widehat{C}_i(X_{i,j}) = \widehat{\beta}_i^T \psi(x)$

(4) Set $\widehat{V}_0 = \frac{\widehat{V}_{11} + ... + \widehat{V}_{1b}}{b}$

Difference between Longstaff/Schwartz and Tsitsiklis/Van Roy

- **The calculation of $\widehat{V}_{ij}$**

Difference between Longstaff/Schwartz and Tsitsiklis/Van Roy

- **The calculation of $\widehat{V}_{ij}$**
  - Tsitsiklis and Van Roy: $\widehat{V}_{i,j} = \max\{h_i(X_{i,j}), \widehat{C}_i(X_{i,j})\}$
  - Longstaff and Schwartz: $\widehat{V}_{ij} = \begin{cases} h_i(X_{i,j}) & h_i(X_{i,j}) \geq \widehat{C}_i(X_{i,j}) \\ \widehat{V}_{i+1,j} & h_i(X_{i,j}) < \widehat{C}_i(X_{i,j}) \end{cases}$

Difference between Longstaff/Schwartz and Tsitsiklis/Van Roy

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- **Only in-the-money paths**

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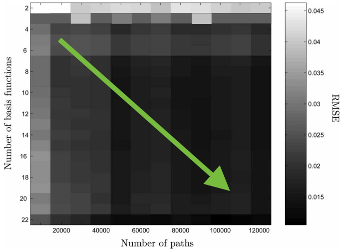
- **Only in-the-money paths**
  - To get a better approximation of the continuation value, Longstaff and Schwartz suggest to consider only paths that are in-the-money.

Measuring the Performance of a Monte-Carlo Estimator

- The estimator $\hat{V}$ for a quantity V can be measured by the root-mean-square-error (RMSE).
- $RMSE(V) := \sqrt{\mathbb{E}[(\hat{V} - V)^2]} = \sqrt{Bias(\hat{V})^2 + Variance(\hat{V})}$
- We see the joint influnece of the number of basis functions and the number of paths.
- The more basis functions/paths chosen, the lower the RMSE.



Overview

- We have introduced a method for valuing options with early exercise features.
- The main idea is to estimate the continuation value. Then, we can compare the continuation value to the corresponding exercise value. This procedure gives us a stopping rule. Due to the corresponding cash flows, we can finally value the option. The average of the discounted payoffs of each path is the price of the option.
- For a better performance you have to increase the number of paths and the number of basis functions.

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