Pricing an American Option - Duality

MAXIMILIAN DRAXLER

MONTE CARLO METHOD IN FINANCIAL ENGINEERING

Agenda

1. Upper Bound through martingales
2. Martingales from value functions
3. Martingales from stopping rules
4. Comparison
5. Numerical Example

1. Upper bound through martingales

Haugh and Kogan, and Andersen and Broadie first established dual formulations in which the price is represented through a minimization problem.

Through minimization over martingales or super martingales, the dual generates a upper bound on prices.

Connected with the Regression we have a valid upper and lower bound on prices.

1. Upper bound through martingales

Results from dynamic programming recursion:

$$V_i(X_i) = \max(h_i(X_i), E[V_{i+1}(X_{i+1})|X_i]), \quad i = 0,1,\ldots,m-1$$

One can follow that

$$V_i(X_i) \geq E[V_{i+1}(X_{i+1})|X_i], \quad i = 0,1,\ldots,m-1$$

which is the definition of a super martingale.

Also, $V_i(X_i) \geq h_i(X_i), i = 0,1,\ldots,m-1$ and $V_i(X_i), i = 0,1,\ldots,m-1$ is in fact the minimal super martingale that dominates $h_i(X_i)$.

From this characterization Haugh and Kogan formulated the pricing of American options as a minimization problem.

1. Upper bound through martingales

Let $M = \{M_i, i = 0, \dots, m\}$ be a martingale with $M_0 = 0$

From the optional sampling theorem of martingales one can follow

$$E[M_i] = 0 \ for \ all \ i = 0, .., m$$

And thus for every stopping time $\tau \ in \ \{1, \dots, m\}$ we have:

$$E[h_\tau(X_\tau)] = E[h_\tau(X_\tau) - M_\tau] \leq E[\max_{k=1,\dots,m}\{h_k(X_k) - M_k\}]$$

As this equation holds for every martingale with initial value of 0

$$E[h_\tau(X_\tau)] \leq \inf_M E[\max_{k=1,\dots,m}\{h_k(X_k) - M_k\}]$$

1. Upper bound through martingales

And as this equation holds for every stopping time τ, it also holds for the supremum over all τ

$$V_o(X_o) = \sup_{\tau} E\left[h_\tau(X_\tau)\right] \leq \inf_{M} E\left[\max_{k=1,\ldots,m}\{h_k(X_k) - M_k\}\right]$$

This minimization problem is the dual problem.

By constructing a martingale for which the right side yields in $V_o(X_o)$ one can show that the dual problem holds with equality

To proof that, we define

$$\Delta_i = V_i(X_i) - E\left[V_i(X_i)|X_{i-1}\right], i = 1, \ldots, m$$

And

$$M_i = \sum_{k=1}^{i} \Delta_k, i = 1, \ldots, m$$

1. Upper bound through martingales

With

$$E[\Delta_i|X_{i-1}] = E[V_i(X_i) - E[V_i(X_i)|X_{i-1}]|X_{i-1}] = 0$$

One can follow, that $M = \{M_i, i = 0, \dots, m\}$ is a martingale:

$$E[M_i|X_{i-1}] = \sum_{k=1}^{i} E[\Delta_k|\mathrm{X}_{\mathrm{i}-1}] = \sum_{k=1}^{i-1} \Delta_k = M_{i-1}$$

By using induction one can show that

$$V_i(X_i) = \max\{h_i(X_i), h_{i+1}(X_{i+1}) - \Delta_{i+1}, \dots, h_m(X_m) - \Delta_m - \cdots - \Delta_{i+1}\} \;\; for\ all\ i = 1, \dots, m$$

As $h_0(X_0) = 0$ the value at 0 is:

$$V_o(X_o) = E[V_1(X_1)|X_0] = V_1(X_1) - \Delta_1$$

And by rewriting $V_1(X_1)$ using the induction we get:

$$V_0(X_0) = \max_{k=1,\dots,m}(h_k(X_k) - M_k)$$

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1. Upper bound through martingales

By defining

$$M_i = \sum_{k=1}^{i} \Delta_k \,, i = 1, \dots, m$$

With

$$\Delta_i = V_i(X_i) - E\left[V_i(X_i)|X_{i-1}\right], i = 1, \dots, m$$

we found the optimal martingale $M = \{M_i, i = 0, \dots, m\}$ so that

$$V_0(X_0) = \max_{k=1,\dots,m}(h_k(X_k) - M_k)$$

holds.

1. Upper bound through martingales

To estimate the value of an American option by

$$\hat{V}_0(X_0) = \max_{k=1,\dots,m} (h_k(X_k) - \widehat{M}_k)$$

we now have to find a martingale $\widehat{M}$ that is close to the optimal martingale M.

As this $\widehat{M}$ is suboptimal, we then have a upper bound for the price of the option.

In connection with lower bound of the regression, we generated a range containing the price of the option.

1. Upper bound through martingales

Andersen and Broadie, and Haugh and Kogan developed the idea of constructing a

- Martingale from value functions (Haugh and Kogan)
- Martingale from stopping rules (Anderson and Broadie)

2. Martingales from value functions

To compute $\widehat{\Delta}_i = \hat{V}_i(X_i) - E[\hat{V}_i(X_i)|X_{i-1}]$ Haugh and Kogan use a nested simulation.

Evaluate $\hat{V}_i(X_i)$:

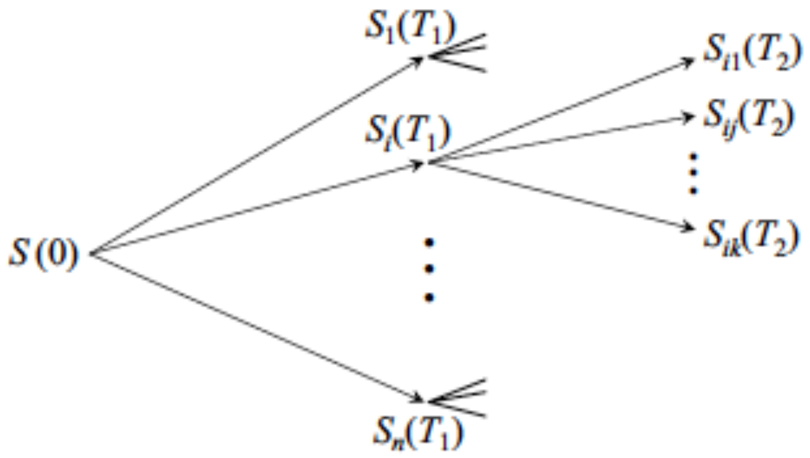
- Just evaluate $\hat{V}_i(x) = \max\{h_i(x), \hat{C}_i(x)\}$ along the Markov chain.

Evaluate $E[\hat{V}_i(X_i)|X_{i-1}]$:

- Use a nested simulation:

At each step X_{i-1} of the simulated Monte Carlo Path, simulate another n successors $X_i^{(1)}, \ldots, X_i^{(n)}$ and use the arithmetic mean to estimate the conditional expectation of $V_i(X_i)$ given X_{i-1}.

$$\frac{1}{n}\sum_{j=1}^{n} \hat{V}_i(X_i^{(j)})$$



2. Martingales from value functions

Conclusion of the algorithm for one Monte Carlo Path:

1. Simulate a path of the underlying Markov chain $X_0, \dots, X_n$
2. For each X_i , $i = 0,1, \dots n$
   i. Calculate the approximated value function $\hat{V}_i(X_i) = \max\{h_i(X_i), \hat{C}_i(X_i)\}$ with $\hat{C}_i(X_i)$ given from a previous regression or parametric approximation
   ii. Simulate m successors $X_{i+1}^{(1)}, \dots, X_{i+1}^{(n)}$ of X_i and calculate
$$\widehat{M}_i = \sum_{K=1}^{i} \widehat{\Delta}_k \text{ , with } \widehat{\Delta}_k = \hat{V}_k(X_k) - \frac{1}{n}\sum_{j=1}^{n} \hat{V}_k(X_k^{(j)})$$
3. Set $\hat{V}_0(X_0) = \max_{k=1,\dots,n}(h_k(X_k) - \widehat{M}_k)$

3. Martingales from stopping rules

Again we want to find a approximation of Δ_i.

Let $\tau_1, \dots, \tau_n$ be stopping times with τ_i interpreted as the exercise time of an option, issued at the i-th exercise date.

$$\tau_i = \min\{k = i, \dots, n: h_k(X_k) \geq \hat{C}_k(X_k)\}$$

The martingale differences are again defined as

$$\widehat{\Delta}_i = E\left[h_{\tau_i}\left(X_{\tau_i}\right)\middle|X_i\right] - E[h_{\tau_i}(\mathrm{X}_{\tau_i})|\mathrm{X}_{\mathrm{i}-1}]$$

3. Martingales from stopping rules

To calculate the expected payoffs

$$E\left[h_{\tau_{k+1}}\left(X_{\tau_{k+1}}\right)\middle|X_k\right], k = 0, \dots n-1$$

Anderson and Broadie again used a nested simulation:

1. Simulate a path $X_0, X_1, \dots, X_n$ of the underlying Markov chain.
2. At each $X_i, i = 0,1,\dots,n-1$
   i. Evaluate $h_i(X_i)$ and $\hat{C}_i(X_i)$ and check which is larger.

      Set $E\left[H_{\tau_i}(X_{\tau_i})\middle|X_i\right] = \begin{cases} h_i(X_i), & if\ h_i(X_i) \geq \hat{C}_i(X_i) \\ E\left[h_{\tau_{i+1}}\left(X_{\tau\ i+1}\right)\middle|X_i\right], & if\ h_i(X_i) < \hat{C}_i(X_i) \end{cases}$

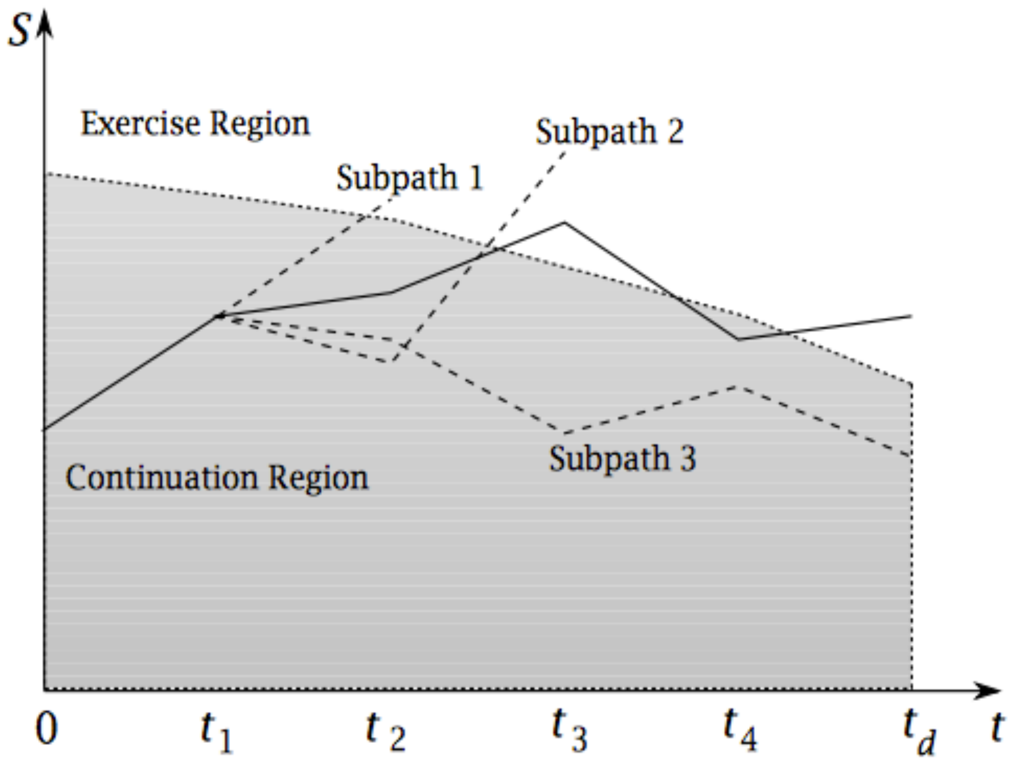
   ii. Simulate m sub paths of X_i and calculate the payoff $h_{\tau_{i+1}}\left(X_{\tau_{i+1}}\right)$, following the same exercise policy
   iii. Use the average to estimate $E[h_{\tau_{i+1}}\left(X_{\tau_{i+1}}\right)|X_i]$
3. Use these conditional expectations to estimate $\widehat{\Delta}_i$ and $\widehat{M}_i$.
4. Evaluate the maximum of $h_k(X_k) - \widehat{M}_k$ over $k = 1, \dots, n$.

3. Martingales from stopping rules

Difference in the nested simulations in the algorithms:

For martingales from value functions, the algorithm simulates exactly on step in each sub path.

Whereas for martingales from stopping rules, the algorithms simulates a random number of steps for each sub path.



4. Comparison

The following table shows a comparison of the upper bound for price of an American max option with 2 underlying assets, following a geometric Brownian motion, generated from duality with martingales from value functions and stopping times.

The approximation is based on:

- 4000 initial paths to estimate regression coefficients
- 100 independent path followed by n=100 and n=10 sub paths
- This was replicated 100 times to estimate standard errors
- 3 assets with initial values $S(0) = 100, 110, 90$

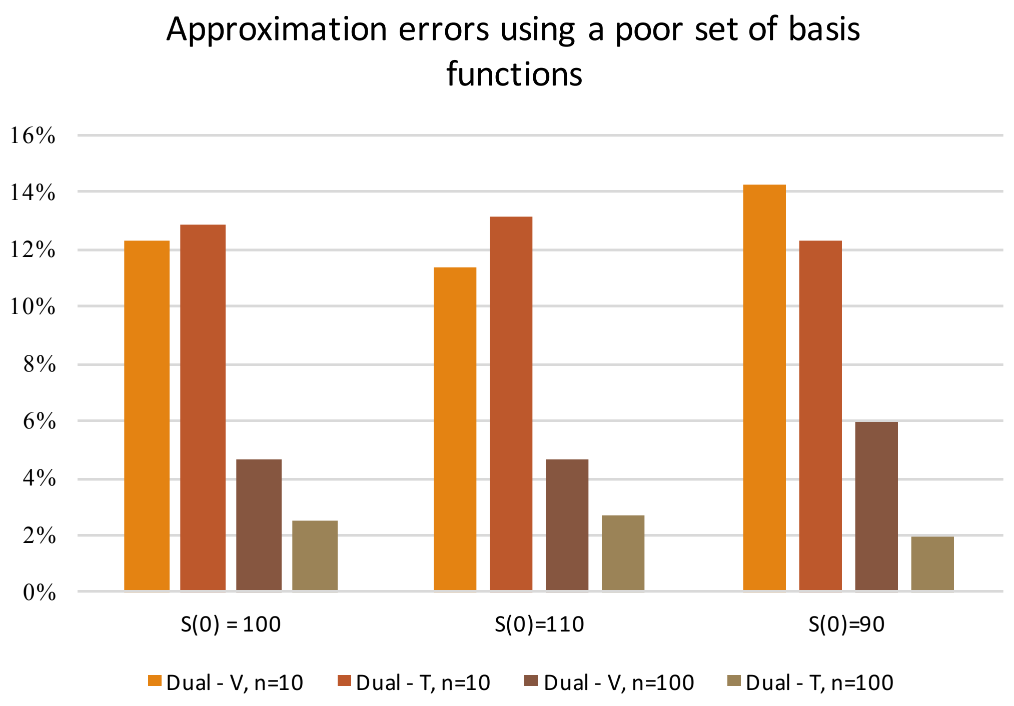
The correct prices are 13.90, 21.34 and 8.08

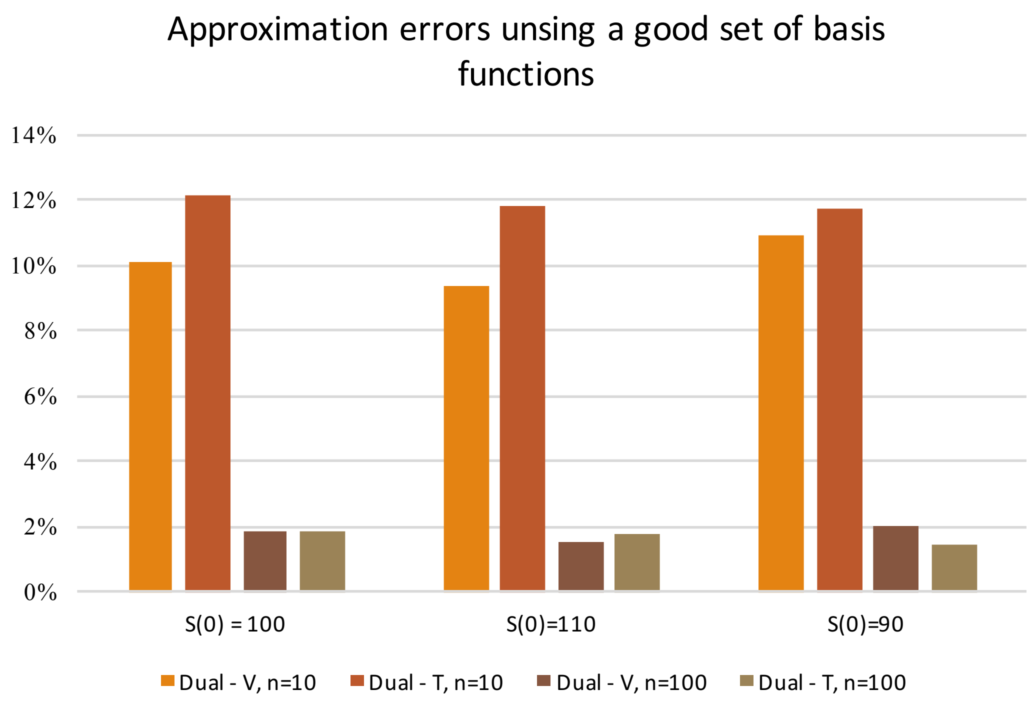
Standard errors of all estimations were between 0.02 and 0.03

4. Comparison

	n=10 Dual – V	n=10 Dual - T	n=100 Dual - V	n=100 Dual - T	correct values
poor set of basis function	15,86	15,96	14,58	14,26	13,9
	24,09	24,56	22,38	21,94	21,34
	9,43	9,21	8,59	8,24	8,08
good set of basis function	15,46	15,82	14,16	14,16	13,9
	23,55	24,21	21,68	21,72	21,34
	9,07	9,16	8,25	8,2	8,08

4. Comparison





4. Comparison

Conclusion of approximation results:

1. Increasing the number of sub paths improves results noticeably
2. Using a good set of basis functions for regression improves results
3. Only small differences between deriving martingales from value functions or stopping times

As martingales derived from stopping times are based on an implementable exercise rule and not from an approximated value function, results from stopping times usually better.

Nevertheless, martingales from stopping times requires more computing time per path.

5. Numerical Example

To confirm their theoretical results Anderson and Broadie tested two classes of problems:

- Pricing of multi asset equity options
- Pricing of interest-rate derivatives

In the first example an max-call equity option, and in the second example a Bermudan swap option will be priced.

5. Numerical Example

Pricing an American max call option with 2, 3 and 5 assets

The payoff of a max-call-option:

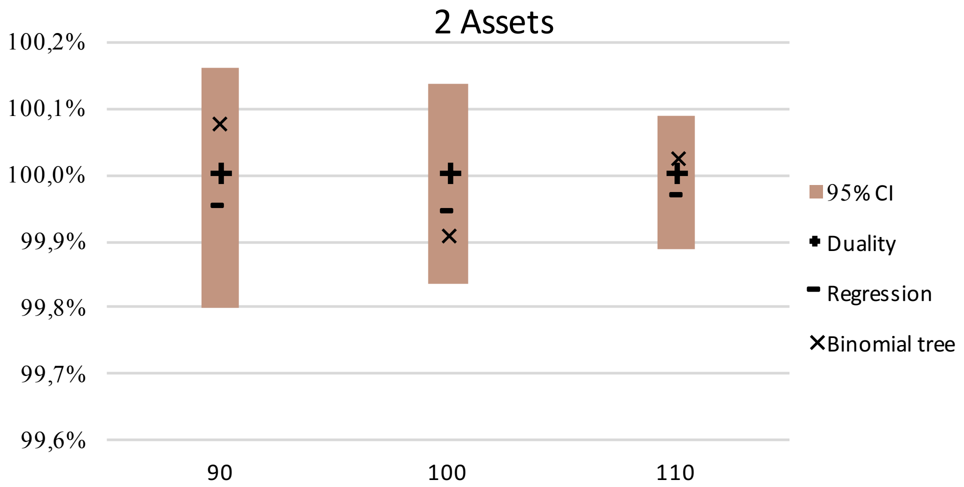
$$h_t(S_t) = (\max(S_t^1, \ldots, S_t^n) - K)^+$$

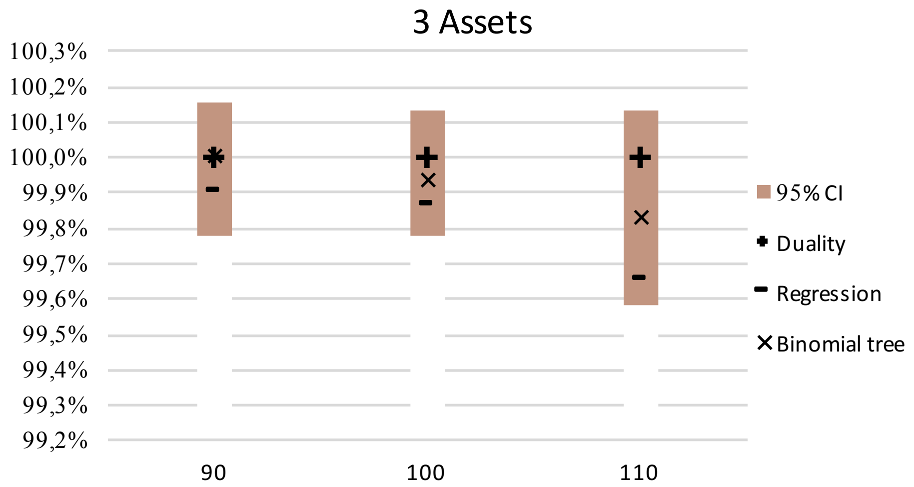
Set up:

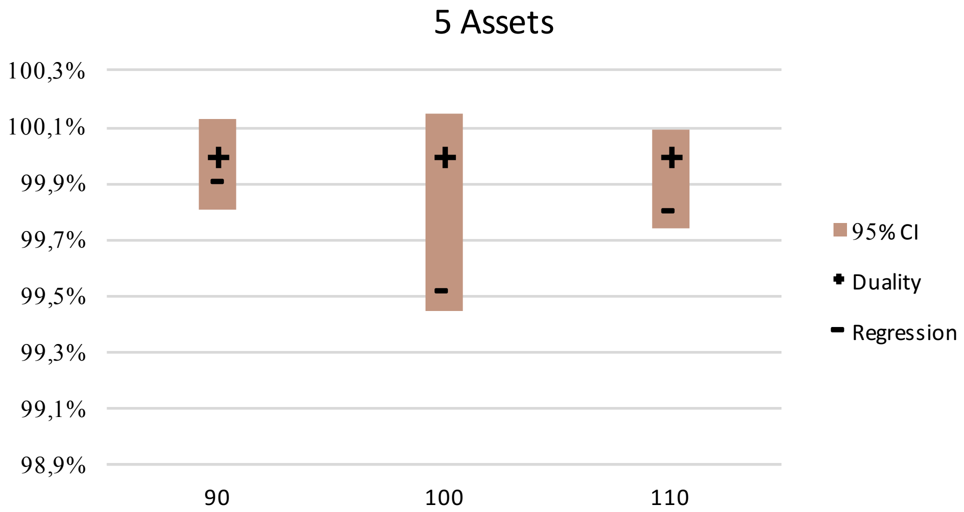
- the dynamics of the 2, 3 and 5 assets follow a geometric Brownian motion
- Continuation values are approximated with the Longstaff and Schwartz approach and linear regression
- The initial asset prices are 90, 100 and 110
- 200,000 trials to estimate regression coefficient for determining the exercise policy
- 2,000,000 trials to estimate a lower bound via regression
- 10,000 trials to estimate the upper bound via duality
- Strike price $K = 100$, expiration time $T = 3$ with exercise opportunities $t_i = \frac{iT}{d}, i = 0,1, \ldots, 9$.

5. Numerical Example

n=2	Regression	Duality	95% CI	Binomial Tree
S(0) = 90	8,065 (0,006)	8,069 (0,007)	[8,053; 8,082]	8,075
S(0) = 100	13,907 (0,008)	13,915 (0,01)	[13,892; 13,934]	13,902
S(0) = 110	21,333 (0,009)	21,34 (0,01)	[21,316; 21,359]	21,345
n=3				
S(0) = 90	11,279 (0,007)	11,29 (0,009)	[11,265; 11,308]	11,29
S(0) = 100	18,678 (0,009)	18,703 (0,013)	[18,661; 18,728]	18,69
S(0) = 110	27,531 (0,01)	27,627 (0,019)	[27,512; 27,663]	27,58
n=5				
S(0) = 90	16,618 (0,008)	16,634 (0,01)	[16,602; 16,655]	
S(0) = 100	26,128 (0,01)	26,253 (0,02)	[26,109; 26,292]	
S(0) = 110	36,725 (0,011)	26,798 (0,017)	[36,704; 36,832]	


2 Assets
100,2%
100,1%
100,0%
99,9%
99,8%
99,7%
99,6%
90
100
110
95% CI
Duality
Regression
Binomial tree


3 Assets
100,3%
100,2%
100,1%
100,0%
99,9%
99,8%
99,7%
99,6%
99,5%
99,4%
99,3%
99,2%
90
100
110
95% CI
Duality
Regression
Binomial tree


5 Assets
100,3%
100,1%
99,9%
99,7%
99,5%
99,3%
99,1%
98,9%
90
100
110
95% CI
Duality
Regression

5. Numerical Example

Pricing of a Bermudan swap option in the Libor Market

Swap:

Agreement between to parties to swap one interest payment against a an other at some specific future dates $t_1, \ldots, t_d$.

Anderson and Broadie considered a fixed-for-floating interest rate swap with a fixed rate θ changed against the LIBOR.

5. Numerical Example

Notation:

Let $P(t,T)$ represent the price of 1$ in time t, which is received with certainty at time T.

The Libor-style discrete forwards can then be defined as

$$F_i(t) = \frac{\frac{P(t,t_i)}{P(t,t_{i+1})}}{t_{i+1} - t_i}$$

According to Andersen, the dynamics of forward rates are can be assumed to satisfy

$$dF_i(t) = \mu_i(t)dt + \varphi\big(F_i(t)\big)\lambda_i^T(t)dW(t)$$

Where W is an Brownian motion.

5. Numerical Example

The value of a fixed-for-floating-interest- rate swap at time $t < t_1$, with fixed interest rate θ and floating interest rate $F_i(t)$, seen by the fixed payer is

$$s(t) = \sum_{i=1}^{d-1} P(t, t_{i+1}) * [F_i(t) - \theta] * (t_{i+1} - t_i)$$

Bermudan swap option:

The right to enter into a specific swap at several future dates $t_1, \dots, t_d$.

The value of a Bermudan swap option at an exercise opportunity $t_k, k = 1, \dots, d$ is

$$c_i(t_i) = S(t_i)^+$$

5. Numerical Example

t_1	t_d	θ	upper bound	95% CI
0,25	1,25	8%	184,62 (0,11)	[184,5; 184,7]
0,25	1,25	10%	49,12 (0,103)	[48,8; 49,4]
0,25	1,25	12%	8,904 (0,101)	[8,7; 9,1]
1	3	8%	355,67 (0,41)	[354,9; 356,4]
1	3	10%	158 (0,52)	[156,9; 158,9]
1	3	12%	61,84 (0,41)	[61,1; 62,5]
1	6	8%	807,43 (0,93)	[805,4; 809,3]
1	6	10%	418,43 (0,96)	[415,9; 420,3]
1	6	12%	213,03 (0,94)	[210,9; 214,8]
1	11	8%	1382,9 (1,7)	[1378,4; 1386,2]
1	11	10%	814,2 (1,5)	[810; 817]
1	11	12%	496,5 (1,6)	[492,7; 499,6]
3	6	8%	493,28 (0,81)	[491,6; 494,9]
3	6	10%	294,25 (1,6)	[291,8; 296,1]
3	6	12%	170,83 (0,86)	[168,6; 172,5]

Thank you for your Attention

References

- Paul Glassermann, “Monte Carlo Methods in Financial Engineering“ (2003)
- Andersen and Broadie, “Pricing Multidimensional American Options”
- Jonen, “Efficient Princing of High-Dimensional American-style Derivatives: A Robust Regression Monte Carlo Method”