Market Risk Measurement

Quantitative Risk Management

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The Loss Operator

- function that relates portfolios losses to changes in the risk factors

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  $\Rightarrow$ loss operator

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- the loss:

$$L_{t+1} := -(V(\tau_{t+1}) - V(\tau_t)) = -(V_{t+1} - V_t)$$

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- mapping leads to

$$V_t = g(\tau_t, \mathbf{Z}_t) \tag{1}$$

for some measurable function $g : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ and some vectors $\mathbf{Z}_t$

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determined by X_{t+1}

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$$L_{t+1} = l_{[t]}(X_{t+1})$$

Delta and Delta-Gamma Approximations

How can we approximate the non-linear loss operator over short time intervals by linear and quadratic functions?

Delta Approximation and the linear loss operator

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- first-order Taylor series approximation:

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$$l_{[t]}^{\triangle}(x) := -(g_\tau(\tau_t, z_t)\triangle t + \sum_{i=1}^{d} g_{z_i}(\tau_t, z_t)x_i) \quad (3)$$

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- $\Gamma(\tau_t, z_t)$ denotes the matrix with (i,j)th element given by $g_{z_i z_j}(\tau_t, z_t)$

$$(4)\ \delta(\tau_t, z_t) = (g_{z_1}(\tau_t, z_t), \ldots, g_{z_d}(\tau_t, z_t))'$$
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\begin{aligned}
g(\tau_t + \triangle t, z_t + x) \approx g(\tau_t, z_t) + g_\tau(\tau_t, z_t)\triangle t + \delta(\tau_t, z_t)'x \\
+\frac{1}{2}(g_{\tau\tau}(\tau_t, z_t)(\triangle t)^2 + 2\omega(\tau_t, z_t)'x\triangle t \\
+x'\Gamma(\tau_t, z_t)x)
\end{aligned}
$$

The quadratic loss operator

$$l_{[t]}^{\triangle\Gamma}(x) := -(g_\tau(\tau_t, z_t)\triangle t + \delta(\tau_t, z_t)'x + \frac{1}{2}x'\Gamma(\tau_t, z_t)x) \tag{6}$$

Example-European call option

(1) $V_t = g(\tau_t, \mathbf{Z}_t)$

- $$V_t = S_t h_t - C^{BS}(\tau_t, S_t; r, \sigma_t, K, T) \tag{7}$$

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(3) $l_{[t]}^{\triangle}(x) := -(g_\tau(\tau_t, z_t)\triangle t + \sum_{i=1}^{d} g_{z_i}(\tau_t, z_t)x_i)$

(a) $C^{BS} = S_t\phi(d_1) - Kexp(-r(T-t)\phi(d_2)$
(b) $d_1 = \frac{ln(S-t/K)+(r+\sigma_t^2/2)(T-t)}{\sigma_t * \sqrt{T-t}}$
(c) $d_2 = d_1 - \sigma_t\sqrt{T-t}$
(d) $\phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} exp(\frac{-z^2}{2})dz$

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$\Rightarrow l_{[t]}^{\triangle}(x) = C_\tau^{BS} * (1/250) + C_\sigma^{BS} * 0.02 \approx -0.019 + 0.698 = 0.679$

$$(6)\ l_{[t]}^{\triangle\Gamma}(x) := -(g_\tau(\tau_t, z_t)\triangle t + \delta(\tau_t, z_t)'x + \frac{1}{2}x'\Gamma(\tau_t, z_t)x)$$

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- additional complexity of second-order approximation may often be warranted

Mapping Bond Portfolios

- apply the idea of the loss operator to the mapping of a portfolio

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- relate this to the concept of duration and convexity in risk management

Basic definitions of bond pricing

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  - mapping $T \rightarrow p(t, T)$ for different maturities

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- normalize the face value $p(T, T)$ of the bond to 1
- ways of discribing the term structure of interest rates at time t
  - mapping $T \to p(t, T)$ for different maturities
  - the continuously compounded yield $T \mapsto y(t, T)$ of a zero-coupon bond is $y(t, T) = -(1/(T - t)) lnp(t, T)$

$$p(t, T) = exp(-(T - t)y(t, T))$$

Detailed mapping of a bond portfolio

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- d default free zero-coupon bonds with maturities T_i and price $p(t, T_i), \ 1 \leq i \leq d$
- λ_i ist the number of bonds with maturity T_i
- $V(t) = \sum_{i=1}^{d} \lambda_i p(t, T_i) = \sum_{i=1}^{d} \lambda_i exp(-(T_i - t)y(t, T_i))$

(1) $V_t := g(\tau_t, \mathbf{Z}_t)$
$\tau_t := t(\triangle t)$

- for a discrete-time set-up

$$V_t = g(\tau_t, \mathbf{Z}_t) = \sum_{i=1}^{d} \lambda_i exp(-(T_i - \tau_t) Z_{t,i})$$

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- risk factors are the yields $Z_{t,i} = y(\tau_t, T_i),\ 1 \leq i \leq d$
- risk-factor changes $X_{t+1,i} = y(\tau_{t+1}, T_i) - y(\tau_t, T_i),\ 1 \leq i \leq d$

(3) $l_{[t]}^{\triangle}(x) := -(g_{\tau}(\tau_t, z_t)\triangle t + \sum_{i=1}^{d} g_{z_i}(\tau_t, z_t)x_i)$

- first derivatives of the mapping function
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  - $g_{z_i}(\tau_t, z_t) = -\lambda_i (T_i - \tau_t) exp(-(T_i - \tau_t) z_{t,i})$

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is the convexity of the bond portfolio

Literature

Alexander J. McNeil, Ruediger Frey, Paul Embrechts (2015): Quantitative Risk Management: Concepts, Techniques and Tools: University Press Group Ltd

Thank you for your attention!