Quantitative Risk Management

Topic: Market Risk Measurement

Vanessa Dietze
08 June 2018
University of Cologne

Structure

- Motivation
- Conditional and Unconditional Loss Distributions
- Various Simulations
- Monte Carlo Method
- Estimating Risk Measures
- Losses and Scaling
- Example

Motivation

- $L_{t+1} = l_{[t]}(X_{t+1})$
- 2 problems
  $\rightarrow$ finding the estimation for the distribution of X_{t+1}
  $\rightarrow$ evaluate L_{t+1} numerically

Conditional and Unconditional Loss Distributions

- conditional distribution of risk-factor changes:
  $F_{X_{t+1}|\mathcal{F}_t}$ where $\mathcal{F}_t = \sigma(\{X_s : s \leq t\})$
- conditional loss distribution:
  df $F_{L_{t+1}|\mathcal{F}_t}(l) = P(l_{[t]}(X_{t+1}) \leq l \,|\mathcal{F}_t)$ where $l_{[t]}(\cdot)$ is loss operator under $F_{X_{t+1}|\mathcal{F}_t}$
- unconditional loss distribution: process of risk factor changes $(X_s)_{s \leq t}$ is stationary multivariate time series
- if risk-factor changes are iid $\rightarrow F_{X_{t+1}|\mathcal{F}_t} = F_X$
  so it follows conditional = unconditional
- since 2007 estimations using stressed VaR data

Various Simulations

Historical Simulation:

- most popular method
- estimation of the distribution of the loss operator under empirical distribution of data $X_{t-n+1}, ..., X_t$
- construct a univariate data set and get a set of historically simulated losses
  $\{\tilde{L}_s = l_{[t]}(X_s) : s = t-n+1, ..., t\}$
- Assuming risk-factor changes are iid with df F_X:
  with the strong law of large numbers as $n \to \infty$
  $$F_n(l) = \frac{1}{n} \sum_{s=t-n+1}^{t} I_{\{\tilde{L}_s \leq l\}} = \frac{1}{n} \sum_{s=t-n+1}^{t} I_{\{l_{[t]}(X_s) \leq l\}} \to P(l_{[t]}(X) \leq l) = F_L(l)$$
  where X is generic vector of risk-factor changes with distribution F_X and $L = l_{[t]}(X)$
  $\to F_n(l)$ consistent estimator

- strengths and weaknesses:

$\rightarrow$ easy to implement
$\rightarrow$ reduces to one-dimensional problem
$\rightarrow$ no statistical estimation necessary
$\rightarrow$ no assumption about dependence

$\rightarrow$ unconditional method
$\rightarrow$ dependence on ability to collect sufficient quantities of relevant data for all risk factors
$\rightarrow$ difficult to implement for large portfolios $\rightarrow$ full revaluation

Dynamic Historical Simulation

univariate approach:

- Reminder:
  historical simulation data $\{\tilde{L}_s = l_{[t]}(X_s) : s = t-n+1, ..., t\}$
- $l_{[t]} : \mathbb{R}^d \to \mathbb{R}$ and $L_{t+1} = l_{[t]}(X_{t+1})$ next RV in process
- $(\tilde{L}_s)$ satisfies $\tilde{L}_s = \mu_s + \sigma_s Z_s$ for all s
- $VaR_\alpha^t = \mu_{t+1} + \sigma_{t+1} q_\alpha(Z)$ for the α-quantile of $F_{L_{t+1}|\mathcal{F}_t}$
- $ES_\alpha^t = \mu_{t+1} + \sigma_{t+1} ES_\alpha(Z)$ where Z is generic RV with the df F_Z
- with Gaussian innovations:
  $q_\alpha(Z) = \Phi^{-1}(\alpha)$ and $ES_\alpha(Z) = \phi(\Phi^{-1}(\alpha))/(1-\alpha)$
- 2 different possible estimation strategies:
  $\to$ weighted historical simulation
  $\to$ filtered historical simulation
- weakness: loss of information

multivariate approach:

- risk-factor change data $X_{t-n+1}, ..., X_t$ from multivariate time-series process (X_s) that satisfies $X_s = \mu_s + \Delta_s Z_s$ where $\Delta_s = diag(\sigma_{s,1}, ..., \sigma_{s,d})$
- Z_s are iid random vectors, covariance matrix = correlation matrix P
- $\mathbb{E}(X_{s,k}|\mathcal{F}_{s-1}) = \mu_{s,k}$
- $\text{var}(X_{s,k}|\mathcal{F}_{s-1}) = \sigma^2_{s,k}$
- key idea: apply simulation to unobserved innovations (Z_s)
- Step 1: compute estimates $\{\hat{\mu}_s : s = t-n+1, ..., t\}$ and $\{\hat{\Delta}_s : s = t-n+1, ..., t\}$
- Step 2: construct residuals $\{\hat{Z}_s = \Delta_s^{-1}(X_s - \hat{\mu}_s) : s = t-n+1, ..., t\}$
- Step 3: Construct $\{\tilde{L}_s = l_{[t]}(\hat{\mu}_{t+1} + \hat{\Delta}_{t+1}\hat{Z}_s) : s = t-n+1, ..., t\}$

Monte Carlo Method

- simulation of an explicit parametric model for risk-factor changes
- only evaluating $L_{t+1} = l_{[t]}(X_{t+1})$ under a given model for X_{t+1}
- already estimated $X_{t-n+1}, \ldots, X_t$, now we generate m realizations $\tilde{X}_{t+1}^{(1)}, \ldots, \tilde{X}_{t+1}^{(m)}$ from $\hat{F}_{X_{t+1}|\mathcal{F}_t}$
- apply loss operator $\rightarrow \{\tilde{L}_{t+1}^{(i)} = l_{[t]}(\tilde{X}_{t+1}^{(i)}) : i = 1, \ldots, m\}$
- estimation of VaR and ES
- strengths and weaknesses:
  $\rightarrow$ free to chose m
  $\rightarrow$ m can be larger than the number of data

  $\rightarrow$ no solution for finding a model for X_{t+1}
  $\rightarrow$ computational cost could be high

Estimating Risk Measures

- Data $L_1, ..., L_n$ from underlying F_L and estimate
  $VaR_\alpha = q_\alpha(F_L) = F_L^{\leftarrow}(\alpha)$ or $ES_\alpha = (1-\alpha)^{-1} \int_\alpha^1 q_\theta(F_L) d\theta$

L-estimators:

- upper-order statistics $L_{1,n} \geq ... \geq L_{n,n}$
- lower-order statistics $L_{(1)} \leq ... \leq L_{(n)}$
- $L_{k,n} = L_{(n-k+1)}$ for $k = 1, ..., n$
- $F_n(x) = n^{-1} \sum_{i=1}^{n} \mathbf{1}_{\{L_i \leq x\}} \ \rightarrow \ F_n^{\leftarrow}(\alpha) = L_{(k)}$ for $\frac{k-1}{n} < \alpha \leq \frac{k}{n}$
- $F_n^{\leftarrow}(\alpha) = L_{(\lceil n\alpha \rceil)}$
- $\lfloor -x \rfloor = -\lceil x \rceil$ and therefore $L_{(\lceil n\alpha \rceil)} = L_{k,n}$ where
  $k = n - \lceil n\alpha \rceil + 1 = \lfloor n(1-\alpha) \rfloor + 1$
- $\widehat{VaR}_\alpha = L_{k,n}$

L-estimator of ES:

- $$\widehat{ES}_\alpha = \frac{1}{n(1-\alpha)} \sum_{k=1}^{n} L_{(k)}((k-n\alpha)^+ - ((k-1)-n\alpha)^+)$$
  $$= \frac{1}{n(1-\alpha)}((\sum_{k=\lceil n\alpha \rceil+1}^{n} L_{(k)}) + (\lceil n\alpha \rceil - n\alpha)L_{(\lceil n\alpha \rceil)})$$
  $$= \frac{1}{n(1-\alpha)}((\sum_{k}^{\lfloor n(1-\alpha) \rfloor} L_{k,n}) + (\lceil n\alpha \rceil - n\alpha)L_{(\lfloor n(1-\alpha) \rfloor)+1,n})$$

EVT-based estimators:

- inaccurate for n modest size
- solution: use of EVT (based on generalized Pareto distribution)
- high threshold $u = L_{k+1,n}$
- ML estimation based on k exceedances of threshold $\rightarrow \hat{\beta}$ and $\hat{\xi}$
- $\frac{k}{n} > 1 - \alpha$
- $\widehat{VaR}_\alpha = u + \frac{\hat{\beta}}{\hat{\xi}}((\frac{1-\alpha}{\frac{k}{n}})^{-\hat{\xi}-1})$
- $\widehat{ES}_\alpha = \frac{\widehat{VaR}_\alpha}{1-\hat{\xi}} + \frac{\hat{\beta}-\hat{\xi}u}{1-\hat{\xi}}$

Losses and Scaling

Losses over Several Periods:

- regulatory capital purposes: 99% VaR estimation for 10 trading days
- model historical risk-factor changes over 10-day interval
- Example:
  n=1000 days $\rightarrow$ 100 10-day observations
  BUT we would need n=10.000 days of data for accuracy
  $\rightarrow$ square-root-of-time rule to estimate only one-day VaR

Scaling:

- $h \in \mathbb{Z}$, $h \geq 1$ and loss defined by $L_{t+h}^{(h)}$

$$\begin{aligned} L_{t+h}^{(h)} &= -(V_{t+h} - V_h) \\ &= -(g(\tau_{t+h}, Z_{t+h}) - g(\tau_t, Z_t)) \\ &= -(g(\tau_{t+h}, Z_t + X_{t+1} + ... + X_{t+h}) - g(\tau_t, Z_t)) \\ &=: l_{[t]}^{(h)}(\sum_{i=1}^{h} X_{t+i}) \end{aligned}$$

- for simplicity: $l_{[t]}^{(h)}(x) = l_{[t]}(x) \ \rightarrow \ l_{[t]}^{\Delta}(x) = b_t' x$

$$\rightarrow L_{t+h}^{(h)\Delta} = l_{[t]}^{\Delta}(\sum_{i=1}^{h} X_{t+i}) = \sum_{i=1}^{h} b_t' X_{t+i}$$

Example

square-root-of-time scaling:

- risk-factor change vectors are iid with $N_d(0, \sum)$
- $\sum_{i=1}^{h} X_{t+i} \sim N_d(0, h \sum)$
- $L_{t+h}^{(h)\Delta} \sim N(0, hb_t' \sum b_t)$

  $\rightarrow$ scaling according to $\sqrt{h}$
- $ES_\alpha^{(h)} = \sqrt{h}\sigma \frac{\phi(\Phi^{-1}(\alpha))}{1-\alpha}$ where $\sigma^2 = b_t' \sum b_t$

  $\rightarrow ES_\alpha^{(h)} = \sqrt{h} ES_\alpha^{(1)}$
- $VaR_\alpha^{(h)} = \sqrt{h} VaR_\alpha^{(1)}$

Monte Carlo approach:

- time-series model for risk-factor changes $(X_s)_{s \leq t}$
- future processes: $\tilde{X}^{(i)}_{t+1}, \dots, \tilde{X}^{(i)}_{t+h}$ for $i = 1, \dots, m$
- Monte Carlo simulated losses:

  $\{\tilde{L}^{(h)(i)}_{t+h} = l^{(h)}_{[t]}(\tilde{X}^{(i)}_{t+1} + \dots + \tilde{X}^{(i)}_{t+h}) : i = 1, \dots, m\}$
- statistical inference about loss distribution and associated risk measures

Literature

Used literature:

[MFE] A.McNeil R.Frey P.Embrechts, *Quantitative Risk Management: Concepts, Techniques and Tools*, University Press Group Ltd 2015, Subsection 9.2, pages 338-351

Thank you for your attention!