Extreme Value Theory - Maxima

Seminar on Quantitative Risk Management

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Convergence of sums

Let $(X_i)_{i \in \mathbb{N}}$ be a sequence of independent and identically distributed (iid) random variables (rvs) representing financial losses with finite variance.

Further: $S_n = \sum_{i=1}^{n} X_i$

From the central limit theorem (CLT) :

$$\lim_{n \to \infty} P\left(\frac{S_n - a_n}{b_n} \leq x\right) = \phi(x), x \in \mathbb{R} \tag{1.1}$$

Generalized Extreme Value Distribution

Definition 1:
The df of the (standard) GEV distribution:

$$H_\xi(x) = \begin{cases} exp(-(1+\xi x)^{-1/\xi}) & \text{if } \xi \neq 0 \\ exp(-e^{-x}) & \text{if } \xi = 0 \end{cases}$$

where $1 + \xi x > 0$, ξ is called the shape parameter.

$\xi > 0$: Fréchet distribution
$\xi = 0$: Gumbel distribution
$\xi < 0$: Weibull distribution.

The location parameter $\mu \in \mathbb{R}$ simply shifts the graph left or right on the horizontal axis.

The scale parameter $\sigma > 0$ has the effect to stretch out the graph if $\sigma > 1$ and to squeeze the graph if $\sigma < 1$.

Exponential distribution

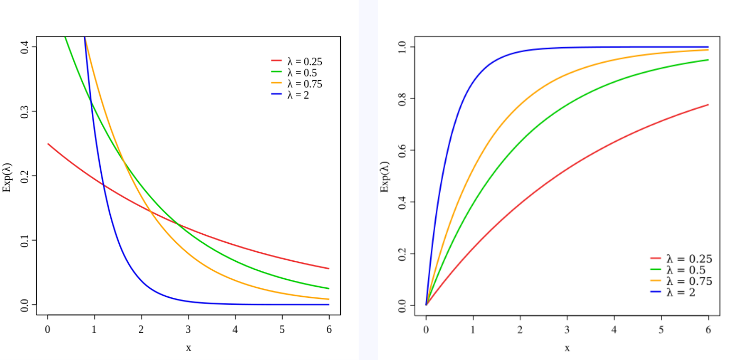

Figure: probability density function(left), cumulative distribution function(right)

source: wikipedia.org

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Example 1

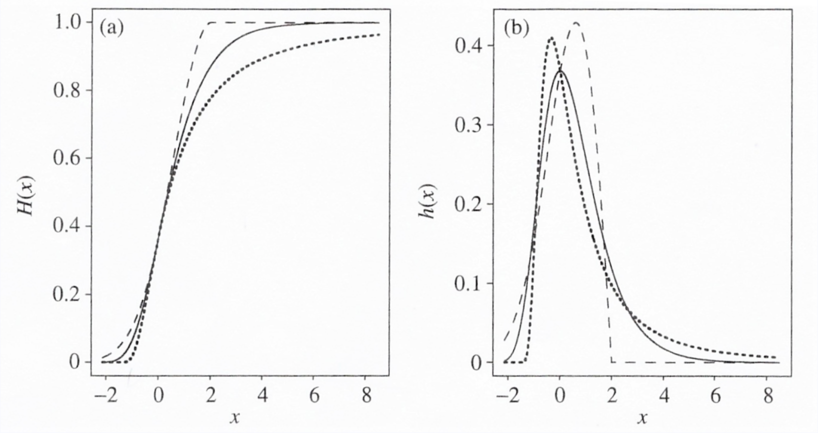

Figure: a) df of the GEV distribution. b) density

Convergence of Maxima

Suppose: block maxima M_n of iid rvs converge in distribution.

We know: $P(M_n \leq x) = F^n(x)$ converges.
$\Rightarrow$ there exist sequences (d_n) and (c_n) with $c_n > 0 \ \forall \ n$ such that

$$\lim_{n\to\infty} P\left(\frac{(M_n - d_n)}{c_n} \leq x\right) = \lim_{n\to\infty} F^n(c_n x + d_n) = H(x) \quad (1.2)$$

for some non-degenerate df $H(x)$.

Definition 4: maximum domain of attraction
If (1.2) holds for some non-degenerate df H, then F is said to be in the maximum domain of attraction of H, written F $\in MDA(H)$.

Theorem 1: Fisher-Tippett,Gnedenko
If $F \in MDA(H)$ for some non-degenerate df H then H must be a distribution of type H_ξ, i.e. a GEV distribution.

Example 2:The exponential distribution

df: $F(x) = 1 - exp(-\beta x)$ for $\beta > 0$ and $x \geq 0$.
Choose normalizing sequences $c_n = 1/\beta$ and $d_n = ln(n/\beta)$ to calculate the limiting distribution of maxima using 1.2.
We get:

$$F^n(c_n x + d_n) = \left(1 - \frac{1}{n} exp(-x)\right)^n, \ x \geq -ln(n)$$

$$\lim_{n\to\infty} F^n(c_n x + d_n) = exp(-e^{-x}), \ x \in \mathbb{R}$$

$\Rightarrow F \in \mathrm{MDA}(H_0)$

Convergence of minima

By using the identity $min\left(X_1, \ldots, X_n\right) = -max\left(-X_1, \ldots, -X_n\right)$ we see that normalized minima of iid samples with df F will convergence in distribution df $\tilde{F}\left(x\right) = 1 - F\left(-x\right)$.
$M_n^* = max\left(-X_1, \ldots, -X_n\right)$

Suppose: $\tilde{F} \in MDA\left(H_\xi\right)$

We get: $\lim_{n\to\infty} P\left(\frac{M_n^* - d_n}{c_n} \leq x\right) = H_\xi\left(x\right)$

$\Rightarrow lim_{n\to\infty} P\left(\frac{min(X_1,\ldots,X_n) + d_n}{c_n} \leq x\right) = 1 - H_\xi\left(-x\right)$

The Block Maxima Method

Suppose: unknown data from unknown distribution F with F $\in MDA(H_\xi)$.

j-th block of the block maximum: M_{nj}, so our data are $M_{n1}, \ldots, M_{nm}$.

To fit the GEV distribution we use maximum likelihood.

We write $h_{\xi,\mu,\sigma}$ for the density of the GEV distribution.

The log-likelihood:

$$l(\xi, \mu, \sigma; M_{n1}, \ldots, M_{nm}) = \sum_{i=1}^{m} \ln\left(h_{\xi,\mu,\sigma}(M_{ni})\right)$$

$$= -m\, ln\sigma - \left(1 + \frac{1}{\xi}\right) \sum_{i=1}^{m} ln\left(1 + \xi \frac{M_{ni} - \mu}{\sigma}\right) - \sum_{i=1}^{m} \left(1 + \xi \frac{M_{ni} - \mu}{\sigma}\right)^{-1/\xi}$$

which must be maximized subject to the parameter constraints that $\sigma > 0$ and $1 + \xi(M_{ni} - \mu)/\sigma > 0 \; \forall \; i$.

Example 3

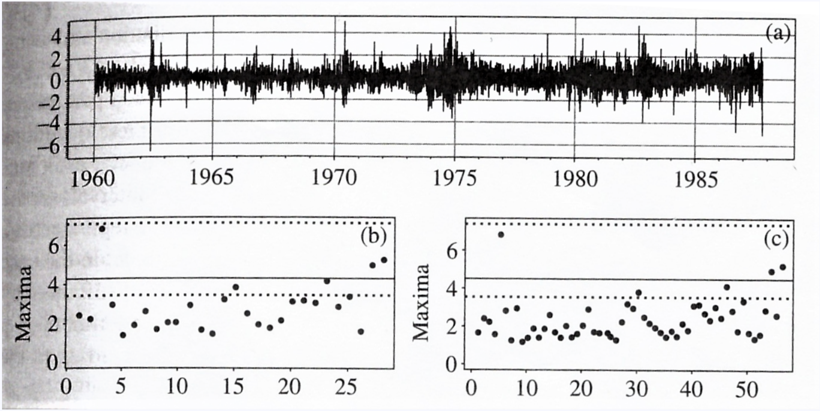

Figure: block maxima analysis of S and P return data

Definition 5 (return level)
Let H denote the df of the true distribution of the n-block maximum. The k n-block return level is $r_{n,k} = q_{1-1/k}(H)$, i.e. the $(1 - 1/k)$-quantile of H.

Definition 6 (return period)
Let H denote the df of the true distribution of the n-block maximum. The return period of the event $\{M_n > u\}$ is given by $k_{n,u} = 1/\bar{H}(u)$.

Literature

Quantitative Risk Management: McNeil, Frey, Embrechts

Uni Muenster Extremwerttheorie, Statistik der Extremwertverteilungen