Basics of Copula

Seminar on "Quantitatives Risikomanagement: Theorie und Praxis in der Versicherungsbranche"

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Motivation

Problem:

· given: a portfolio with d stocks

· wanted: the distribution function of the portfolio value

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→ used in quantitative finance to model and minimize tail risk and portfolio optimization applications

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Introduction to copulas

Definition (Copula)

Let $U_i \in [0,1]$ be a standard uniform distributions with $i \in \{1, ..., d\}$ and $d \in \mathbb{N}$.
The $d-$dimensional **copula** $C(u) = C(u_1, ..., u_d) : [0,1]^d \rightarrow [0,1]$ is a distribution function that maps the unit hypercube into the unit interval.
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1) If $min\{u_1,...,u_d\} = 0 \Rightarrow C(u_1,...u_d) = 0$.
2) $C(1,...1,u_i,1,...,1) = u_i \ \forall \ i \in \{1,...,d\}, u_i \in [0,1]$.
3) For all $(a_1,...,a_d),(b_1,...,b_d) \in [0,1]^d$ with $a_i \leq b_i \ \forall i \in \{1,...,d\}$ we have

$$\sum_{i_1=1}^{2} \cdots \sum_{i_d=1}^{2} (-1)^{i_1+\dots+i_d} C(u_{1i_1},...,u_{di_d}) \geq 0 \qquad (2.1)$$

with $u_{j1} = a_j$ and $u_{j2} = b_j \ \forall j \in \{1,...,d\}$.

Rectangle inequality

Example

The *rectangle inequality* with $d = 3$ is

$$\begin{aligned}
&\sum_{i_1=1}^{2}\sum_{i_2=1}^{2}\sum_{i_3=1}^{2}(-1)^{i_1+i_2+i_3}C(u_{1i_1}, u_{2i_2}, u_{3i_3}) \\
=& -C(a_1, a_2, a_3) + C(a_1, a_2, b_3) + C(a_1, b_2, a_3) - C(a_1, b_2, b_3) \\
&+ C(b_1, a_2, a_3) - C(b_1, a_2, b_3) - C(b_1, b_2, a_3) + C(b_1, b_2, b_3) \\
=& -(C(a_1, a_2, a_3) + C(a_1, b_2, b_3) + C(b_1, a_2, b_3) + C(b_1, b_2, a_3) \\
&+ (C(a_1, a_2, b_3) + C(a_1, b_2, a_3) + C(b_1, a_2, a_3) + C(b_1, b_2, b_3)) \\
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\geq& \; 0
\end{aligned}$$

Conclusion

$\rightarrow P(a_1 \leq U_1 \leq b_1, ..., a_d \leq U_d \leq b_d)$ is non-negative for a random vector $(U_1, ..., U_d)$.

Application of copulas

Proposition

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$\Rightarrow$ We can transform risks with a particular continuous distribution function to have any other continuous distribution.

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Let X have a standard normal distribution F

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$\Rightarrow F(X)$ is uniform by proposition (2).

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Let Y be the transformed variable $Y := -ln(1-F(X))$
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Let Y be the transformed variable $Y := -ln(1-F(X))$
$\Rightarrow P(Y \leq v) = P(G^{\leftarrow}(F(X)) \leq v) = G(v)$
$\Rightarrow Y$ has a standard exponential distribution by proposition (1).

Sklar's Theorem

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Theorem

Let F be a joint distribution function with margins $F_1, ...F_d$. Then there exists a copula $C : [0,1]^d \to [0,1]$ such that, for all $x_1, ..., x_d \in \overline{\mathbb{R}} = [-\infty, \infty]$,

$$F(x_1, ..., x_d) = C(F_1(x_1), ..., F_d(x_d)). \tag{3.2}$$

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If all F_i's are continuous, the copula C is unique, otherwise C is uniquely determined on $RanF_1 \times ... \times RanF_d$, with $RanF_i := F_i(\overline{\mathbb{R}})$ being the range of F_i.

Example

Revision: Bernoulli distribution:

$$F_X(X) = \begin{cases} 0, & \text{if } x < 0, \\ 1-p, & \text{if } 0 \leq x < 1, \\ 1, & \text{if } x \geq 1 \end{cases}$$

Let (X_1, X_2) have a bivariate Bernoulli distribution with

$$P(X_1 = 0, X_2 = 0) = \frac{1}{8},$$
$$P(X_1 = 0, X_2 = 1) = \frac{2}{8},$$
$$P(X_1 = 1, X_2 = 0) = \frac{2}{8},$$
$$P(X_1 = 1, X_2 = 1) = \frac{3}{8},$$

Bounds for a copula

Theorem

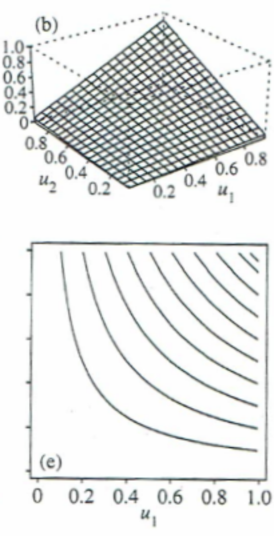
For every copula $C(u_1, ..., u_d)$ we have the bounds

$$max\left(\sum_{i=1}^{d} u_i + 1 - d,\ 0\right) \leq C(u) \leq min(u_1, ..., u_d).$$

Proof.

On board. □

Independence copula

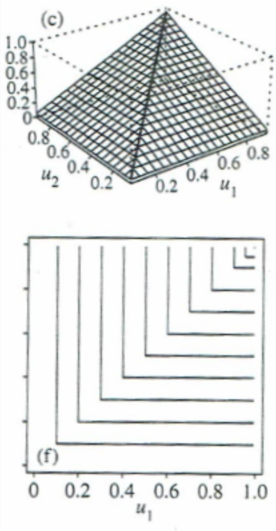


Definition

Random vectors with continuous distributions are independent if and only if their dependence structure is given by the **independence copula**

$$C(u_1, ..., u_d) = \prod_{i=1}^{d} u_i.$$

Comonotonicity copula



Definition

The **comonotonicity copula** is the Fréchet upper bound copula from the last theorem:

$$C(u_1, ..., u_d) = min(u_1, ..., u_d)$$

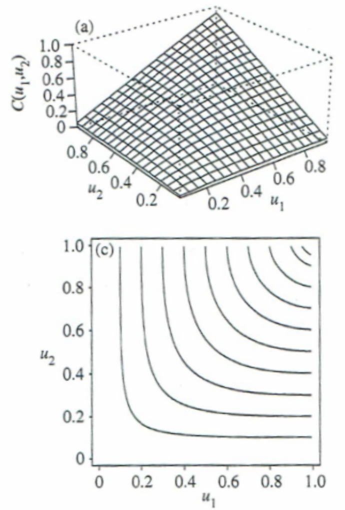
Gauss copula

Definition

If $Y \sim N_d(\mu, \Sigma)$ is a multivariate normal random vector, then its copula is a so-called **Gauss copula** which is given by

$$
\begin{aligned}
&C_P^{Ga}(u) \\
=&P(\Phi(X_1) \leq u_1, ..., \Phi(X_d) \leq u_d) \\
=&\Phi_P(\Phi^{-1}(u_1), ..., \Phi^{-1}(u_d))
\end{aligned}
$$

with P being the correlation matrix of Y.



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Thank you for your attention!