GIT II: Stratifications & VGIT

<u>Recall</u>: $\underset{\text{red}}{G} \curvearrowright \underset{\text{proj}}{X}$ w.r.t $\begin{matrix} L \\ \downarrow \\ X \end{matrix}$ linearisation (G-line bundle)

<u>often today</u>: L is ample

Last time: $X \subseteq \mathbb{P}^n$ $G \to GL(n+1)$ <u>i.e.</u> $L = \mathcal{O}_{\mathbb{P}^n}(1)|_X$

$X^{s}(L) \subseteq X^{ss}(L) \subseteq X$ (to determine ss locus look for G-inv sections)

$X^{sss} = X^{ss} - X^{s}$, $X^{us} = X - X^{ss}$

<u>Hilbert-Mumford</u> (L ample)

$x \in X$ is G-(s)s wrt $L \iff \forall$ 1-PS $\lambda: \mathbb{G}_m \to G$ $\mu^L(x, \lambda) \geq 0$

<u>Recall</u>:

If $x \in X \subseteq \mathbb{P}^n$ $\lambda: \mathbb{G}_m \to G \curvearrowright V = \mathbb{A}^{n+1} \rightsquigarrow V = \bigoplus_{r \in \mathbb{Z}} V_r$

Pick $\hat{x} \in \tilde{X}$ = affine cone over X $\subseteq \mathbb{A}^{n+1}$ so $\hat{x} = \sum_{r \in \mathbb{Z}} x_r$

Then $\mu^L(x, \lambda) := -\min\{r : x_r \neq 0\}$

$= -$ wt of $\mathbb{G}_m$-action on $\tilde{x}_0$ $\left[x_0 = \lim_{t \to 0} \lambda(t) \cdot x \in X \right]$

$= -$wt of $\mathbb{G}_m$-action on $\mathcal{O}_{\mathbb{P}^n}(-1)_{x_0}$

$=$ wt of $\mathbb{G}_m$-action on $\mathcal{O}_{\mathbb{P}^n}(1)_{x_0}$

($\mathcal{O}_{\mathbb{P}^n}(-1)$ = taut. line bdle $\to$ fibre over x is pts $\tilde{x} \in \mathbb{A}^{n+1}$ lying over x)

<u>Defn</u>: $\mu^L(x, \lambda) :=$ wt of $\mathbb{G}_m$-action on L_{x_0} where $x_0 = \lim_{t \to 0} \lambda(t) \cdot x$ is fixed by $\lambda(\mathbb{G}_m)$

<u>Torus version of Hilbert-Mumford</u>

$\boxed{G = \mathbb{G}_m} \curvearrowright X \subseteq \mathbb{P}^n \rightsquigarrow \mathbb{A}^{n+1} = \bigoplus_{r \in \mathbb{Z} = \mathcal{X}^*(G)} V_r$

"wts of action" ↑ character lattice

As $\mathcal{X}_*(G) = \mathbb{Z}$ for HM criterion we only have two 1-PSs to check for: $\lambda(t) = t$ & λ^{-1}

$x \in X^{ss} \iff \begin{cases} \mu^L(x, \lambda) \geq 0 \iff \tilde{x} \neq \sum_{r \in \mathbb{Z}_{>0}} x_r \\ \mu^L(x, \lambda^{-1}) \geq 0 \iff \tilde{x} \neq \sum_{r \in \mathbb{Z}_{<0}} x_r \end{cases}$ (Can't have all +ve wts or all −ve wts)

$\hat{x} = \sum_r x_r$

The $\mathbb{G}_m$-state of x $\mathrm{st}_{\mathbb{G}_m}(x) := \{r \in \mathcal{X}^*(\mathbb{G}_m) : x_r \neq 0\}$

<u>HM</u>: x is $\mathbb{G}_m$-ss $\iff 0 \in \overline{\mathrm{st}_{\mathbb{G}_m}(x)}$ = convex hull of wts

x is $\mathbb{G}_m$-s $\iff 0 \in \mathrm{Int}(\overline{\mathrm{st}_{\mathbb{G}_m}(x)})$

$\boxed{G = T \cong (\mathbb{G}_m)^r}$ $\curvearrowright$ $X \subseteq \mathbb{P}^n \rightsquigarrow \mathbb{A}^{n+1} = \bigoplus V_\chi$

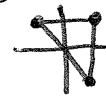
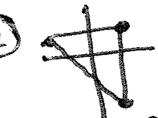
$\chi \in \mathcal{X}^*(G) = \mathbb{Z}^r$

HM : Check $\mu^L(x, \lambda) = -\min\limits_{\chi \in st_T(x)} \langle \chi, \lambda \rangle$

x is T-ss $\iff$ $0 \in \overline{st_T(x)}$

x is T-s $\iff$ $0 \in \mathrm{Int}(\overline{st_T(x)})$

Eg $r = 2$

①  , ② 

$x \in X^{sss}$ $\quad$ $x \in X^s$

③  $x \in X^{us}$

$\boxed{G \text{ reductive}}$ Fix $T \subseteq G$ max^e torus

If $\lambda : \mathbb{G}_m \to G$, $\exists$ g s.t $g\lambda g^{-1} \in \mathcal{X}_*(T)$

(any 2 max^e tori are conjugate)

& $\mu^L(x, \lambda) = \mu^L(g \cdot x, g\lambda g^{-1})$

HM : x is G-ss $\iff$ $\forall g \in G$ $\quad 0 \in \overline{st_T(g \cdot x)}$

x is G-s $\iff$ $\forall g \in G$ $\quad 0 \in \mathrm{Int}(\overline{st_T(g \cdot x)})$

Moreover $X^{us}_G(L) = \bigcup\limits_{g \in G} g \cdot X^{us}_T(L)$

Hesselink Stratification of X^{us} (still L is ample)

Idea: if $x \in X^{us}$ $\exists$ λ (by HM) s.t $\mu^L(x, \lambda) < 0$

We want to find λ which is most responsible for the instability of x.

But as $\mu^L(x, \lambda^n) = n\mu^L(x, \lambda)$ is unbounded we can't use μ to measure this.

Instead if we have an invariant norm $\|-\|$ on 1-PS

$\|\lambda^n\| = n\|\lambda\|$ so :

Def^n: λ is optimal for x if $\dfrac{\mu^L(x, \lambda)}{\|\lambda\|} = \inf\limits_{\lambda' \in \mathcal{X}_*(G)} \dfrac{\mu^L(x, \lambda')}{\|\lambda'\|} =: M^L(x)$

To define $\|-\|$ we choose a norm on $\mathcal{X}_*(T)$ (inv. under Weyl gp)

eg. $T \subseteq GL_n$ diag matrices

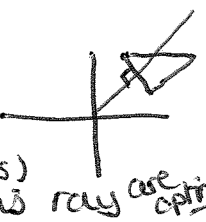
$\lambda(t) = \begin{pmatrix} t^{r_1} & & \\ & \ddots & \\ & & t^{r_n} \end{pmatrix} \rightsquigarrow \|\lambda(t)\|^2 = \sum r_i^2$ standard dot product

(Note $GL_n / GL_n \cong T/W$)

acts by conj

Eg $G = T$ draw $st_T(x)$

$\exists !$ ray which meets $st_T(x)$ orthogonally $\to$ the 1-PS(s) assoc to this ray are optimal



Why? As $\mu^L(x,\lambda) = -\min_{\chi \in st_T(x)} \langle \chi, \lambda \rangle$

Rmk: x is G-pss $\iff$ $M^L(x)(\geq)0$

Let $\Lambda^L(x) = \{\lambda \in X_*(G) : \lambda$ is non-divisible & optimal for $x\}$

Properties: 1) $\Lambda^L(gx) = g\Lambda^L(x)g^{-1}$ (i.e. $M^L(x)$ is G-inv)

2) if $\lambda \in \Lambda^L(x)$ & $x_0 = \lim_{t\to 0} \lambda(t)\cdot x$, then $\lambda \in \Lambda^L(x_0)$ & $M^L(x) = M^L(x_0)$.

Idea: stratify X^{ss} by optimal 1-PS & wt of 1-PS)

For $d > 0$ & $\langle\lambda\rangle \in X_*(G)/G$ conj class of 1-PSs define:

$$S^L_{d,\langle\lambda\rangle} = \{x \in X : M^L(x) = -d \ \& \ \langle\lambda\rangle \cap \Lambda^L(x) \neq \phi\}$$

Then $X - X^{ss}(L) = \bigcup_{d,\langle\lambda\rangle} S^L_{d,\langle\lambda\rangle}$ G-inv stratification

(At this point it is not obvious that there are only finitely many strata - we'll see this is the case)

Fix $T \subseteq G$ max^e torus & pick $\lambda \in X_*(T)$ representing $\langle\lambda\rangle$

then let:

$$Z^L_{d,\lambda} = \{x \in X : \lambda(\mathbb{G}_m) \subseteq \underset{\text{stabiliser}}{G_x} \ \& \ \lambda \in \Lambda^L(x) \ \& \ M^L(x) = -d\}$$

$$\underset{\text{closed}}{X \supseteq} X_d^{\lambda(\mathbb{G}_m)}$$ ∩ open

$\leftarrow$ components of fixed pt locus $X^{\lambda(\mathbb{G}_m)}$ on which $\frac{\lambda}{\|\lambda\|}$ acts with weight d

In fact $Z^L_{d,\lambda}$ = GIT-ss set for smaller red gp acting on $X_d^{\lambda(\mathbb{G}_m)}$

$\lim_{t\to 0} \lambda(t)\cdot x$

$$x \in S^L_{d,\lambda} = \{x \in X : \lambda \in \Lambda^L(x) \ \& \ M^L(x) = -d\}$$

Then $\boxed{S^L_{d,\langle\lambda\rangle} = G S^L_{d,\lambda}}$ loc. closed $\subseteq X$

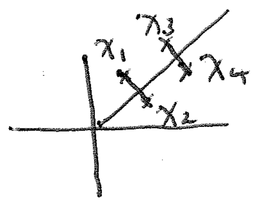
There are a finite no. of T-wts so a finite no. of poss $st_T(x)$ which determine λ, d

Thm (Hesselink)

① There are only finitely many strata

② There is an ordering of the strata so the closure of a stratum is the union of itself with higher strata

Rmk: To see why we need d suppose we have T wts



If $st_T(x) = \{x_1, x_2\}$
$st_T(y) = \{x_3, x_4\}$
then λ is optimal for x & y
but y is more unstable

$$\mathrm{Dist}(0, \overline{st_T(x)}) = \frac{\mu^L(x,\lambda)}{\|\lambda\|} = d$$

Finiteness Results (Dolgachev-Hu, Thaddeus ...)

Q: There are (in general) an infinite number of ample G-line bundles L, but how many different GIT quotients $X/\!/_L G$ are there?

Thm (Dolgachev-Hu)
$\{ S^L_{d,\langle\lambda\rangle} : L \text{ ample } \; d>0 \;\&\; \lambda \in X_*(G)\}$ is finite

Cor ① $\{X^{ss}(L) : L$ ample G-line bdle$\}$ is finite
② $\{X/\!/_L G : L$ " $\}$ is finite

Proof of thm: Induction on rk of G

Take $T \subseteq G$ max^e torus

Claim: $\{X_d^{\lambda(\mathbb{G}_m)} : d>0 \;\; \lambda \in X_*(T)\}$ is finite

Pf: Pick $X \hookrightarrow \mathbb{P}^n = \mathbb{P}(V)$ G-equiv embedd

$T \curvearrowright V \rightsquigarrow V = \bigoplus_{\chi} V_\chi$ only finitely many wts

Then $X_d^{\lambda(\mathbb{G}_m)} = \mathbb{P}(\bigoplus_{\substack{\chi \\ \text{s.t. } \langle\chi,\lambda\rangle = d}} V_\chi) \cap X$

But there are only a finite no. of ~~distinct~~ non trivial direct summands of V

As $Z^L_{d,\lambda}$ = GIT ss scheme for smaller red gp acting on $X_d^{\lambda(\mathbb{G}_m)}$ (by induction) there are only finitely many

& $S^L_{d,\langle\lambda\rangle}$ is determined by $Z^L_{d,\lambda}$ //

Example $G = SL(2,\mathbb{C}) \curvearrowright (\mathbb{P}^1)^3 \xhookrightarrow{\text{Segre}} \mathbb{P}^{2^3-1}$ ③

$\cup\!\!\!\mid$ & $L = \mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathcal{O}_{\mathbb{P}^1}(1)$

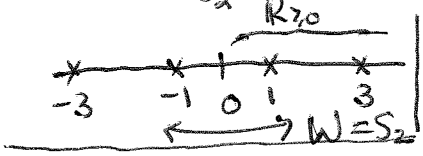
$T = \left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} : t \in \mathbb{C}^\times \right\}$

T-wts of $T \curvearrowright \mathbb{P}^1$: ± 1 $\qquad (\mathbb{P}^1)^T = \{[1:0],[0:1]\}$

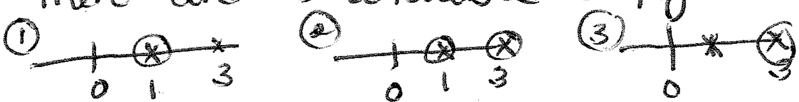
T-wts on $(\mathbb{P}^1)^3$: $\pm1 \pm1 \pm1 = \{-3,-1,1,3\}$

Only 2 non-divisible 1-PSs of T: $\lambda(t) = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$ & $\lambda^{-1}(t)$

As Weyl gp $\overset{\parallel}{S_2}$ maps λ to λ^{-1} & interchanges ~~reflects~~ +ve Weyl chambers we can just focus on 1 +ve Weyl chamber & λ



There are 3 unstable configurations:



As $\|\lambda\| = 1$ we see $d = 1$ for ① & ② & $d = 3$ for ③

$Z_{1,\lambda} \subseteq X_1^T = \{(p_1,p_2,p_3)$: exactly 2 p_i's are $[1:0]$ & the third is $[0:1]\}$ → picture ①

3 pts

In fact $Z_{1,\lambda} = X_1^T$ (its clear λ is optimal for these pts)

(picture ② ←) $S_{1,\lambda} = \{(p_1,p_2,p_3)$: 2 out of 3 are $[1:0]$ & third is $[*:1]\}$

$S_{1,\langle\lambda\rangle} = \{(p_1,p_2,p_3)$: exactly 2 p_i agree$\}$

(3 components)

Similarly $Z_{3,\lambda} = \{(p_1,p_2,p_3) : p_i = [1:0]\}$

$S_{3,\langle\lambda\rangle} = \{(p,p,p)\} \cong \mathbb{P}^1$

So $X^{ss} = X - (S_{1,\langle\lambda\rangle} \cup S_{3,\langle\lambda\rangle}) = \{(p,q,r)$ distinct pts$\}$

Variation of GIT : G-ample cone (Dolgachev & Hu, Thaddeus) [Assume X is proper & normal]

$NS^G(X) = \operatorname{Pic}^G(X)/\sim$ ← $L_1 \sim L_2$ if $\exists$ family $\begin{matrix} L \\ \downarrow \\ G \curvearrowright X \times T \end{matrix}$ of G-line bdles param. by conn^d var. T and $t_i \in T$ closed s.t $L_i = L|_{X \times \{t_i\}}$

" Néron-Severi group of G-line bdles "

Lemma: If $L_1 \sim L_2$, then $\forall\, x, \lambda$ $\mu^{L_1}(x,\lambda) = \mu^{L_2}(x,\lambda)$.

In particular, $X^{ss}(L_1) = X^{ss}(L_2)$

Pf: Recall $\mu^{L_i}(x,\lambda) =$ wt of $\mathbb{G}_m$-action on $L_{(t_i,x_0)}$

where $x_0 = \lim_{t\to 0} \lambda(t)\cdot x$

We get $\begin{aligned} T &\longrightarrow \mathbb{Z} \\ t &\longmapsto \mu^{L_t}(x,\lambda) \end{aligned}$ which is locally constant & as T is connected its constant. //

Prop: $M^\cdot(x) : \mathrm{Pic}^G(X) \longrightarrow \mathbb{Z}$ factors through $NS^G(X)$ & can be extended to a cts function $M^\cdot(x) : NS^G(X)_{\mathbb{R}} \to \mathbb{R}$

s.t 1) $M^{\alpha L}(x) = \alpha M^L(x)$

2) $M^{L_1 \otimes L_2}(x) \geq M^{L_1}(x) + M^{L_2}(x)$

"Pf": The first statement follows from the lemma above

1) follows from: $\mu^{L^n}(x,\lambda) = n\mu^L(x,\lambda)$ $\quad \forall n \in \mathbb{Z}$

2) follows from: $\inf_\lambda \dfrac{\mu^{L_1}(x,\lambda) + \mu^{L_2}(x,\lambda)}{\|\lambda\|} \geq \inf_\lambda \dfrac{\mu^{L_1}(x,\lambda)}{\|\lambda\|} + \inf_\lambda \dfrac{\mu^{L_2}(x,\lambda)}{\|\lambda\|}$

Cor: $\Omega(x) = \{\ell \in NS^G(X)_{\mathbb{R}} : M^\ell(x) \geq 0\}$ is a convex cone.

$$\Updownarrow$$

$$x \in X^{ss}(\ell)$$

Def^n: An ample linear L is G-effective if $X^{ss}(L) \neq \emptyset$

Let $C^G(X) \subseteq NS^G(X)$ be the cone gen by G-effective L

We say L_1 & L_2 are GIT equivalent if $X^{ss}(L_1) = X^{ss}(L_2)$

Idea / Aim: Cut $C^G(X)$ up into pieces on which the GIT equiv class is constant

Ideally want a fan structure on $C^G(X)$ s.t the GIT-equiv classes are the rel. interiors of the cones.

Def^n: For $x \in X$ with $\dim G_x > 0$ we define

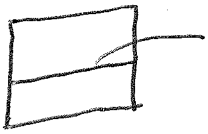
$H(x) = \{\ell : M^\ell(x) = 0\} \subseteq C^G(X)$

D&H call these walls (but for BFK they only say $H(x)$ is a wall if it has codim 1.)

(Note all $H(x)$ are contained in a $H(y)$ of codim 1)

Cells = connected components of
$H^\bullet(x)^\circ = \{\ell \in H(x) : \text{if } \ell \in H(y) \text{ then } H(x) \subseteq H(y)\}$

Eg

$H(x) =$  $H(y)$

plane

$H(x)$ has two cells $H^\bullet(x)^\circ = H(x) - H(y)$

$H(y)^{\circ\bullet} = H(y)$

Chambers = connected components of $C^G(X) - \bigcup_{\substack{x \text{ s.t.} \\ \dim G_x > 0}} H(x)$

Lemma i) $\ell \in H(x) \iff X^{sss}(\ell) \neq \emptyset$ Pf: ($\Rightarrow$) obv ($\Leftarrow$) find closed orbit in X^{sss}

ii) $\forall\, \ell, \ell' \in C =$ chamber $X^{ss}(\ell) = X^s(\ell) = X^s(\ell') = X^{ss}(\ell')$

"Pf": ii) Essentially use the fact that $M^\cdot(x)$ doesn't change sign on C

Thm (Dolgachev–Hu, Thaddeus)

i) There are a finite number of walls, chambers & cells

ii) The closure of each chamber is a rat^{e} polyhedral cone

iii) Each GIT equiv class is a chamber or a union of cells in the same wall.

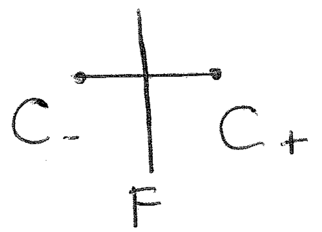
In particular we can write for a chamber C or cell F
$$X^{ss}(C) \quad \text{or} \quad X^{ss}(F)$$

Comments on the proof: i) Relies on finiteness results above & nice properties of $M^{\bullet}(x)$

ii) Essentially uses properties of $M^{\bullet}(x)$.

Wall crossing: (we still assume X is normal)

Suppose we have a cell F contained in the closure of 2 adjacent chambers $C_{\pm}$ s.t we can draw a line segment joining these:



As $-M^{\circ}(x)$ is convex: ① $X^{s}(C_{\pm}) = X^{ss}(C_{\pm}) \subseteq X^{ss}(F)$

② $X^{s}(C_-) \cap X^{s}(C_+) \subseteq X^{s}(F)$

In fact as M° is cts ② is an equality.

The morphism

$$\begin{array}{ccc} X^{s}(C_{\pm}) & \hookrightarrow & X^{ss}(F) \\ \downarrow & & \downarrow \\ X/\!/_{C_{\pm}} G & \xrightarrow{\ f_{\pm}\ } & X/\!/_{F} G \end{array}$$

induces $f_{\pm} : X/\!/_{C_{\pm}} G \longrightarrow X/\!/_{F} G$ (by the universal property of cat quotients)

which is birat^{e} & an isomorphism over the complement of $\Sigma = X^{sss}(F)/\!/G$ in $X/\!/_F G$.

Fact: For any cell F we have that (as $F \subseteq \overline{C_{\pm}}$) $\exists$ $\ell_{\pm} \in C_{\pm}$ which lie on the line segment s.t the GIT stratification assoc. to ℓ_0 refines the GIT stratification associated to $\ell_{\pm}$ (where ℓ_0 is the point lying on $F \cap$ line segment). In particular:

$$X^{ss}(\ell_-) \cup \bigcup_{i=1}^{p} S^{-}_{\lambda_i(-), d_i(-)} = X^{ss}(\ell_0) = X^{ss}(\ell_+) \cup \bigcup_{i=1}^{q} S^{+}_{\lambda_i(+), d_i(+)}$$

For "nice" cells F we'd like to describe the GIT flip in greater depth.

What do we mean by nice F?

GIT flip

$$\begin{tikzcd} X/\!/_{C_-}G \arrow[dr, "f_-"] & & X/\!/_{C_+}G \arrow[dl, "f_+"'] \\ & X/\!/_F G & \end{tikzcd}$$

Defⁿ: A codim 1 wall H is truly faithful if $\forall \{x : H = H(x) \ \&\ \forall y \in \overline{G\cdot x} - G\cdot x \ \ H \neq H(y)\}$ we have $G_x \cong \mathbb{G}_m$

(These pts x are called pivotal points for H)

Prop: if X is smooth (& proj as always) & $G = T$ then all codim 1 walls are truly faithful.

Lemma: (X smooth) In the decomposition of $X^{ss}(\ell_0)$ above in terms of strata for $\ell_\pm$ we have:

i) $p = q$ ii) $\lambda_i(-) = (\lambda_i(+))^{-1}$

"Pf" $X^{ss}(\ell_0)_{\substack{\text{closed}\\\text{orbits}}} = X^s(\ell_0) \cup \bigcup_{i=1}^{p} (G\cdot Z^-_{\lambda_i(-), d_i(-)})$

(If $G\cdot x$ is closed in $X^{ss}(\ell_0)$ & not stable wrt ℓ_0 then it is not stable wrt ℓ_+ also (as $\dim G_x > 0$))

We get the same for ℓ_+ and so we see $p = q$ & $\lambda_{i(-)} = (\lambda_{i(+)})^{\pm 1}$ (The case where $\lambda_{i(-)} = \lambda_{i(+)}$ leads to a contradiction $\rightsquigarrow$ we get $S^-_{\lambda_i(-)} = S^+_{\lambda_i(+)}$)

Thm (X smooth) Let Σ_i = connected component of $\Sigma = X^{sss}(F)/\!/G$ then the fibres of $f_\pm$ over Σ_i are weighted proj spaces of dim $d^i_\pm$ & $d^i_+ + d^i_- + 1 = \operatorname{codim} \Sigma_i$.

Rmk (VGIT for torus gps): $G \subseteq (\mathbb{G}_m)^n \curvearrowright \mathbb{A}^n = X$

$NS^G(X) = \chi^*(G)$ (A character χ can be used to twist the trivial linear $\begin{smallmatrix}\mathcal{O}_X\\\downarrow\\X\end{smallmatrix}$)

$C :=$ Cone spanned by images of standard basis $e_i \in \mathbb{Z}^n$ under the surj $\mathbb{Z}^n \twoheadrightarrow \chi^*(G)$ ($\mathbb{Z}^n = \chi^*(\mathbb{G}_m^n)$)

Then: $C = C^G(X)$ & the GIT fan = secondary fan. Moreover, for wall crossing we have $p = q = 1$ and so

$X^{ss}(\ell_-) \cup S^-_{\lambda^{-1}, d_-} = X^{ss}(\ell_0) = X^{ss}(\ell_+) \cup S^+_{\lambda, d_+}$