CAT(0) 4-MANIFOLDS ARE EUCLIDEAN

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Abstract. We prove that a topological 4-manifold of globally non-positive curvature is homeomorphic to Euclidean space.

1. Introduction

1.1. **Main result.** This paper concerns the topology of *CAT(0) manifolds*. These are synthetic generalizations of complete, simply connected Riemannian manifolds of non-positive sectional curvature. By the classical theorem of Cartan–Hadamard, any such Riemannian manifold is diffeomorphic to the Euclidean space $\mathbb{R}^n$. In his seminal paper [Gro81], Gromov asked if there exist simply connected topological manifolds other than Euclidean space, which admit a metric of non-positive curvature in a synthetic sense. (See Section 1.6 for further discussion.) The most important synthetic notion of non-positive curvature is the one due to Alexandrov. The corresponding spaces were named *CAT(0)* by Gromov. Any CAT(0) space is contractible. Thus, any CAT(0) 2-manifold is homemorphic to $\mathbb{R}^2$ by the classification of surfaces. In dimensions strictly greater than two, contractible manifolds are Euclidean precisely when they are simply connected at infinity, thanks to classical topological results [Fre82, HP70, Sta62]. In dimension three, CAT(0) manifolds are indeed Euclidean [Bro61, Rol68, Thu96b]. In dimensions strictly greater than four, Davis and Januszkiewicz [DJ91] constructed examples of non-Euclidean CAT(0) manifolds, see also [DJL12]. In this paper, we deal with the remaining open case:

Theorem 1.1. *Let X be a CAT(0) space which is topologically a 4-dimensional manifold. Then X is homeomorphic to $\mathbb{R}^4$.*

1.2. **Related statements: the simplical case.** Examples of Davis and Januszkiewicz [DJ91] mentioned above are simplicial complexes with piecewise Euclidean metric. On the other hand, Stone [Sto76] verified that a *PL* n-manifold which is CAT(0) with respect to a piecewise Euclidean metric is homeomorphic to the Euclidean space $\mathbb{R}^n$. By the resolution of the Poincaré conjecture, any simplicial complex homeomorphic to a 4-manifold is a PL-manifold. Consequently, any CAT(0) 4-manifold with a piecewise Euclidean metric is homeomorphic to $\mathbb{R}^4$.

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More recently, the question as to which manifolds carry a piecewise Euclidean CAT(0) metric has been reduced to a purely topological problem in [AB20, AF19]. Namely, a manifold carries such a metric if and only if it is homeomorphic to a collapsible simplicial complex. Building on this result and previous work of Ancel–Guilbault [AG97], it has been verified in [AF19] that for all $n > 4$, the interior of any compact contractible n-manifold with boundary carries a piecewise Euclidean CAT(0) metric. Thus, in dimensions five and above, there is an abundance of contractible CAT(0) manifolds. Some necessary conditions for the existence of a piecewise Euclidean CAT(0) metric (hence for the existence of a collapsible triangulation, which is not PL) are given in [AF19, Theorem 1]. Motivated by Gromov's question, it is natural to ask the following, compare [AF19, Question 3].

Question 1.2. Are there CAT(0) topological manifolds which do not carry a piecewise Euclidean CAT(0) metric?

Question 1.3. What are necessary and sufficient conditions for the existence of a CAT(0) metric on a contractible manifold?

1.3. **Related statements: the cocompact case.** Gromov's question has been thoroughly studied in the *cocompact* setting. (Recall, that universal coverings of locally CAT(0) spaces are CAT(0) [AB90]). By [DJ91], in all dimensions strictly greater than four, there exist compact, locally CAT(0) manifolds which either have no PL structure at all [DJL12, Section 3.2], or whose universal coverings are not Euclidean. An 8-dimensional locally CAT(0) PL-manifold which is non-smoothable is constructed in [DJL12], based on previous work [DH89].

In dimension four, several classes of smoothable compact topological manifolds carrying a locally CAT(0) metric, yet not admitting a smooth metric of non-positive curvature, have been constructed in [DJL12, Sat17, Sta15]. In these examples, the universal covering is homeomorphic to $\mathbb{R}^4$ (as follows from our main theorem), and identifying the obstructions to the existence of smooth metrics relies on an intricate analysis of the group actions involved.

1.4. **Distance spheres.** Our proof depends upon an important contribution by Thurston [Thu96b]. He showed that if all distance spheres to some fixed point $o \in X$ of a 4-dimensional CAT(0) manifold X are topological 3-manifolds, then X is homeomorphic to $\mathbb{R}^4$. Using a finer analysis of the metric structure of the space, we verify this latter condition in the more general setting of *homology manifolds*; see Sections 1.5 and 3.4 for the relevant definition and properties.

Theorem 1.4. *Let X be a CAT(0) space which is a homology 4-manifold. Let $o \in X$ and $R > 0$ be arbitrary. Then the distance sphere $S_R(o)$ of radius R around o is a topological 3-manifold.*

We remark that this result does not hold true in dimensions $n \geq 5$, even for piecewise Euclidean topological n-manifolds. Indeed, this can be seen in the aforementioned examples of Davis and Januszkiewicz; compare [DJ91, Proposition 3d.3].

If X is a CAT(0) 4-manifold (and not just a homology 4-manifold), then the resolution of the Poincaré conjecture together with [Thu96b] implies that all distance spheres are homeomorphic to $\mathbb{S}^3$. Moreover, the homeomorphism in Theorem 1.1 is not completely abstract, but rather has the following geometric feature.

Corollary 1.5. *Let X be a 4-dimensional CAT(0) manifold and let $o \in X$ be an arbitrary point. Then the distance function $d_o : X \setminus \{o\} \to (0, \infty)$ is a trivial fiber bundle with fiber $\mathbb{S}^3$.*

On the other hand, for a general CAT(0) homology 4-manifold, the topology of the distance spheres may depend on the radius despite the fact that all of the spheres involved are manifolds. This can already be seen in the Euclidean cone $X = C(\Sigma)$ over the Poincaré homology sphere Σ. The fine topological analysis of [Thu96b], using the fact that the ambient space is a manifold, is therefore indispensable for the conclusion of our main theorem.

1.5. **Related statements: homology manifolds.** A homology n-manifold (without boundary) is a locally compact metric space X of finite topological dimension such that, for all $x \in X$, the local homology $H_*(X, X \setminus \{x\})$ equals $H_*(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$. The structure theory of homology manifolds has been a central topic in geometric topology for many decades and is of fundamental importance in the topological *manifold recognition* [Can78, CHR16, Rep94].

A homology n-manifold for $n \leq 2$ is always a topological n-manifold by a theorem of Moore [Wil49, Chapter IX]. On the other hand, in dimensions $n \geq 3$, a homology n-manifold may not have any manifold points at all [DW81]. While there are deep and rather robust results allowing one to recognize when a homology manifold is a manifold in dimensions five and up, much less is known in dimensions three and four [Rep94]. Even though the tools of algebraic topology allow us to recognize homology manifolds in many instances, in particular, in the situation of Theorem 1.4 (in all dimensions), passing from homology manifolds to topological manifolds is difficult and requires some geometric insight.

In the situation of Theorem 1.4, we achieve the needed local control of the topology of large spheres, by slicing them and verifying that slices are 2-dimensional spheres. Subsequently, these slices can be controlled uniformly with the help of Jordan's curve theorem. The control of the slices allows us to recover the local topology.

In contrast to the situation for general homology n-manifolds, CAT(0) homology n-manifolds are not too far from being manifolds. More precisely, a CAT(0) homology n-manifold is a topological n-manifold on the complement of a discrete subset [LN21, Theorem 1.2]; see [Wu97] for corresponding statements on spaces with lower curvature bounds.

We mention in passing a question of Busemann [Bus55, BHR11], which is related in spirit to the origins of this paper.

Question 1.6. Let X be a locally compact geodesic metric space. Assume that X is geodesically complete and that there are no branching geodesics. Does X have finite dimension? Is any such finite-dimensional X a topological manifold?

If such a space X has finite dimension n, then X is a homology n-manifold and if $n \leq 4$ then X is a manifold [Bus55, Kra68, Thu96a, BHR11]. For $n \geq 5$, the question remains open. Finally, we mention that an answer to Busemann's question would follow from a purely topological conjecture of Bing and Borsuk [HR08].

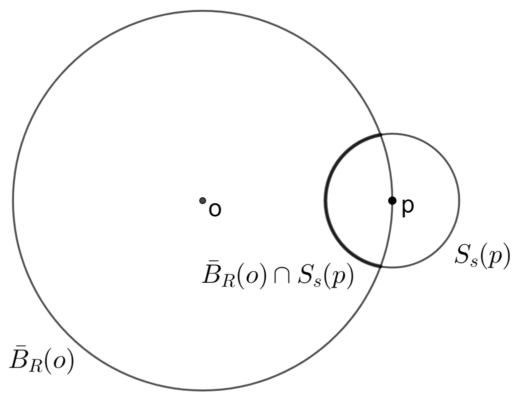
1.6. **Minor generalizations.** An application of [LS20, Theorem 1.1] extends Theorem 1.1 and Corollary 1.5 to other curvature bounds:

Corollary 1.7. *Let X be a 4-dimensional topological manifold which is a $CAT(\kappa)$ space. Let $R > 0$ be a real number with $R < \frac{\pi}{2\sqrt{\kappa}}$ if $\kappa > 0$. Then, for any $o \in X$, the open ball $B_R(o)$ is homeomorphic to $\mathbb{R}^4$ and the distance sphere $S_R(o)$ is homeomorphic to $\mathbb{S}^3$.*

There are several notions of non-positive curvature for metric spaces. In all works cited above, Gromov's question has been studied for CAT(0) spaces, even though the original question was posed for Busemann convex spaces, that is, geodesic spaces with a convex distance function. Any CAT(0) space is Busemann convex. Conversely, examples of Busemann convex spaces that are neither CAT(0) nor normed spaces are extremely rare, compare [IL19]. Our ideas will apply to this more general setting, once some structural results developed in [LN19] for CAT(0) spaces will be generalized to Busemann convex spaces.

1.7. **Comment on strategy and technique.** Our proof relies on the structural theory of geodesically complete spaces with upper curvature bounds developed in [LN19, LN21]. Since any CAT(0) manifold is geodesically complete, this theory applies to the present situation. So-called 'strainer maps', first appearing in [BGP92] and defined by distance functions to points, are particularly useful. It has been verified [LN21] that for any point $o \in X$ as in Theorem 1.4, all sufficiently small distance spheres around o are (pairwise homeomorphic) 3-manifolds. In order to get sufficient control of remote spheres, an extension of the theory of strainer maps by one additional 'orthogonal but non-straining' coordinate is required. This extension may be useful beyond the present work. We refer the reader to Section 5 and Section 7 for more details. Here, we only formulate a special case of Proposition 5.5, essential for the proof of Theorem 1.4.

Theorem 1.8. *Let X be a locally compact, geodesically complete CAT(0) space, let $o \in X$, and let $p \in X$ be at distance $R > 0$ from o. Then there exists $\epsilon > 0$ such that for all $0 < s < \epsilon$, the intersection of the distance sphere $S_s(p)$ with the closed ball $\bar{B}_R(o)$ is contractible.*



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2. Preliminaries

2.1. **Metric spaces.** We refer the reader to [AKP19, BBI01, BH99] for general background. We denote by d the distance in a metric space X. For $x \in X$, we denote by d_x the distance function $d_x(\cdot) = d(x, \cdot)$. For $x \in X$ and $r > 0$, we denote by $B_r(x)$ and $\bar{B}_r(x)$ the open, respectively, closed r-ball around x. Similarly, $B_r(A)$ denotes the open r-neighborhood of a subset $A \subset X$. Moreover, $S_r(x)$ denotes the r-sphere around x and by $\dot{B}_r(x)$ we denote the punctured r-ball $B_r(x) \setminus \{x\}$. For $\lambda > 0$, we denote by $\lambda \cdot X$ the metric space resulting from X by rescaling the metric by λ. A *geodesic* is an isometric embedding of an interval. A *triangle* is a union of three geodesics connecting three points. X is *a geodesic metric space* if any pair of points of X is connected by a geodesic. It is *geodesically complete* if every isometric embedding of an interval extends to a locally isometric embedding of $\mathbb{R}$.

A map $F : X \to Y$ between metric spaces is called *L-Lipschitz* if $d(F(x), F(\bar{x})) \leq L \cdot d(x, \bar{x})$, for all $x, \bar{x} \in X$.

The map F is called *L-open* if the following condition holds. For any $x \in X$ and any $r > 0$ such that the closed ball $\bar{B}_{Lr}(x)$ is complete, we have the inclusion $B_r(F(x)) \subset F(B_{Lr}(x))$.

An *ANR* will denote an *absolute neighborhood retract.* For finite dimensional metric spaces, the only case relevant here, being an *ANR* is equivalent to being locally contractible [Hu65].

2.2. **Spaces with an upper curvature bound.** For $\kappa \in \mathbb{R}$, let $R_\kappa \in (0, \infty]$ be the diameter of the complete, simply connected surface M^2_κ of constant curvature κ. A complete metric space is called a *CAT(κ) space* if any pair of its points with distance $< R_\kappa$ is connected by a geodesic and if all triangles with perimeter $< 2R_\kappa$ are not thicker than the *comparison triangle* in M^2_κ. A metric space is called a *space with curvature bounded above by* κ if any point has a CAT(κ) neighborhood. We refer to [AKP19, BBI01, BH99] for basic facts about such spaces.

For any CAT(κ) space X, the angle between each pair of geodesics starting at the same point is well defined. The *space of directions* $\Sigma_x X$ at $x \in X$, equipped with the angle metric, is a CAT(1) space. The Euclidean cone over Σ_x is a CAT(0) space. It is denoted by $T_x X$ and called the *tangent space* at x of X. Its tip will be denoted by o_x.

Let x, y, z be three points at pairwise distance $< R_\kappa$ in a CAT(κ) space X. Whenever $x \neq y$, the geodesic between x and y is unique and will be denoted by xy. For $y, z \neq x$, the angle at x between xy and xz will be denoted by $\angle yxz$.

In a CAT(κ) space X, all balls of radii smaller $\frac{R_\kappa}{2}$ are convex, hence X is locally contractible. In fact, X is an ANR [Kra11, Theorem 3.2].

3. Geometric topology

3.1. **Homology manifolds.** Denote by D^n the closed unit ball in $\mathbb{R}^n$.

Let M be a locally compact, separable metric space of finite topological dimension. We say that M is a *homology n-manifold with boundary* if for any $p \in M$ we have a point $x \in D^n$ such that the local homology $H_*(M, M \setminus \{p\})$ at p is isomorphic to $H_*(D^n, D^n \setminus \{x\})$. The *boundary* ∂M of M is defined as the set of all points at which the n-th local homologies are trivial. In the case where the boundary of M is empty, we simply say that M is a *homology n-manifold.*

If M is a homology n-manifold with boundary then ∂M is a closed subset of M and it is a homology $(n-1)$-manifold by [Mit90].

Any homology n-manifold (with boundary) has dimension n. For $n \leq 2$, we have the following theorem of Moore [Wil49, Chapter IX].

Theorem 3.1. *Any homology n-manifold with $n \leq 2$ is a topological manifold.*

A *homology n-sphere* is a homology n-manifold X with the same homology as the n-sphere: $H_*(X) = H_*(\mathbb{S}^n)$.

3.2. **Uniform local contractibility.** A function $\rho : [0, r_0) \to [0, \infty)$ is called a *contractibility function* if it is continuous at 0 with $\rho(0) = 0$ and for all $t \in [0, r_0)$ holds $\rho(t) \geq t$ [Pet97].

Definition 3.2. We say that a family $\mathcal{F}$ of metric spaces is *uniformly locally contractible* if there exists a contractibility function $\rho : [0, r_0) \to [0, \infty)$ such that for any space X in the family $\mathcal{F}$, any point $x \in X$ and any $0 < r < r_0$, the ball $B_r(x)$ is contractible within the ball $B_{\rho(r)}(x)$.

For example, the family of all CAT(κ) spaces is uniformly locally contractible with $\rho : [0, \frac{\pi}{\sqrt{\kappa}}) \to [0, \infty)$ being the identity map.

Here is a special case of [Pet90, Theorem A], [Pet97, Theorem 9]:

Theorem 3.3. *For any natural number n and any family $\mathcal{F}$ of uniformly locally contractible metric spaces of dimension at most n, there exists some $\delta > 0$ such that any pair of spaces $X, Y \in \mathcal{F}$, with Gromov–Hausdorff distance at most δ is homotopy equivalent.*

The homotopy equivalences and the corresponding homotopies in Theorem 3.3 can be chosen arbitrary close to the identity [Pet97].

When dealing with a family of fibers of some map, we will use the following more convenient variant of Definition 3.2 [Ung69].

Definition 3.4. Let $F : X \to Y$ be a map between metric spaces. We say that F *has uniformly locally contractible fibers* if the following condition holds true for any point $x \in X$ and every neighborhood U of x in X. There exists a neighborhood $V \subset U$ of x in X such that for any fiber $F^{-1}(y)$ with non-empty intersection $F^{-1}(y) \cap V$, this intersection is contractible in $F^{-1}(y) \cap U$.

For X compact, a map $F : X \to Y$ has uniformly locally contractible fibers in the sense of Definition 3.4 if and only if the family of the fibers is uniformly locally contractible in the sense of Definition 3.2.

3.3. **Fibrations and fiber bundles.** A map $F : X \to Y$ between metric spaces is called a *Hurewicz fibration* if it satisfies the homotopy lifting property with respect to all spaces [Hat02, Section 4.2], [Ung69].

The map F is called open if the images of open sets are open.

We will use the following result to recognize Hurewicz fibrations.

Theorem 3.5 ([Fer78, Theorem 2]; [Ung69, Theorem 1])**.** *Let X, Y be finite-dimensional, compact metric spaces and let Y be an* ANR. *Let $F : X \to Y$ be an open, surjective map with uniformly locally contractible fibers. Then X is an ANR and F is a Hurewicz fibration.*

In some situations, Hurewicz fibrations turn out to be fiber bundles. We will rely on the following.

Theorem 3.6 ([Fer91, Theorems 1.1-1.4]; [Ray65, Theorem 2])**.** *Let X, Y be finite-dimensional locally compact* ANR*s. Let $F : X \to Y$ be a Hurewicz fibration. If all fibers of F are closed n-manifolds then F is a locally trivial fiber bundle.*

3.4. **CAT(0) (homology) manifolds.** Following [LN19], we will denote a locally compact, locally geodesically complete, separable space with an upper curvature bound as GCBA. In this paper we are concerned with CAT(0) spaces which are homeomorphic to (homology) manifolds. We will call such spaces *CAT(0) homology manifolds* and *CAT(0) manifolds* respectively. Every CAT(0) homology manifold is geodesically complete [LS07, Theorem 1.5] and therefore GCBA. Hence, we can rely on the results from [LN19, LN21]. For the local arguments of [LN19, LN21], the notion of a *tiny ball* played a role. We point out that in a CAT(0) homology manifold, a tiny ball is any ball of radius at most one. After rescaling, the bound of 1 becomes irrelevant.

From [LN21, Lemma 3.1, Corollary 3.4, Theorem 6.4] we infer:

Proposition 3.7. *Let X be a CAT(0) homology n-manifold. Then any space of directions $\Sigma_x X$ is a homology $(n-1)$-sphere. If $n \leq 4$, then $\Sigma_x X$ is a topological manifold.*

Any CAT(0) homology n-manifold is a topological n-manifold on the complement of a discrete subset [LN21, Theorem 1.2]. For $n \leq 3$, a CAT(0) homology n-manifold is a manifold homeomorphic to $\mathbb{R}^n$ [LN21, Theorem 6.4], [Thu96b]. The Euclidean cone over the Poincaré sphere is a CAT(0) homology 4-manifold which is not a manifold.

Any CAT(0) homology n-manifold is locally bilipschitz equivalent to $\mathbb{R}^n$ away from a closed set of Hausdorff dimension at most $n-2$, as follows from [LN19, Theorem 1.2 and Section 10.2].

4. Strainer maps

We recall the definition and basic properties of strainer maps in the framework of CAT(0) spaces from [LN19] and [LN21]. Originally, strainer maps were introduced in [BGP92] to study Alexandrov spaces with curvature bounded below.

4.1. **Almost spherical directions.** Let X be a locally compact and geodesically complete CAT(0) space. Let v be a direction at a point $x \in X$. An *antipode* of v is a direction $\hat{v} \in \Sigma_x X$ at distance at least π from v. If v has a unique antipode $\hat{v}$, then $\Sigma_x X$ splits isometrically as a spherical join $\Sigma_x X \cong \{v, \hat{v}\} * Z$. More generally, the subset Σ^0 of points with unique antipodes in $\Sigma_x X$ is isometric to $\mathbb{S}^k$, for some k, and Σ^0 is a spherical join factor of $\Sigma_x X$ [Lyt05, Corollary 4.4].

A quantitative version is provided by the notion of δ-spherical points and tuples. The direction $v \in \Sigma_x X$ is called *δ-spherical*, if there exists some $\bar{v} \in \Sigma_x X$ such that for any $w \in \Sigma_x X$

$$d(v, w) + d(w, \bar{v}) < \pi + \delta.$$

Moreover, we say that v and $\bar{v}$ are *opposite* δ-spherical points. A δ-spherical direction v has a set of antipodes of diameter at most 2δ, [LN19, Lemma 6.3]. Therefore, if δ is small, this 'almost leads to a splitting' of $\Sigma_x X$ [LN19, Proposition 6.6].

A k-tuple $(v_1, \dots, v_k)$ of points in $\Sigma_x X$ is called *δ-spherical* if there exists another k-tuple $(\bar{v}_i)$ in $\Sigma_x X$ with the following two properties.

- For $1 \le i \le k$, the directions v_i and $\bar{v}_i$ are opposite δ-spherical.
- For $1 \le i \ne j \le k$, the distances $d(v_i, \bar{v}_j), d(v_i, v_j), d(\bar{v}_i, \bar{v}_j)$ are less than $\frac{\pi}{2} + \delta$.

Moreover, $(\bar{v}_i)$ and (v_i) are called *opposite δ-spherical k-tuples.*

4.2. **Strainers and strainer maps.** A k-tuple $(p_1, \dots, p_k)$ is called a *(k,δ)-strainer* at a point $x \in X \setminus \{p_1, \dots, p_k\}$ if the starting directions $v_i \in \Sigma_x X$ of the geodesics xp_i constitute a δ-spherical k-tuple in $\Sigma_x X$.

Two (k,δ)-strainers (p_i) and (q_i) at x are *opposite* if the corresponding δ-spherical k-tuples (v_i) and (w_i) are opposite in $\Sigma_x X$.

A k-tuple (p_i) in X is a *(k,δ)-strainer* in $A \subset X \setminus \{p_1, \dots, p_k\}$ if (p_i) is a (k,δ)-strainer at every point $x \in A$.

The set of all points $U \subset X \setminus \{p_1, \dots, p_k\}$ at which (p_i) is a (k,δ)-strainer is open in X [LN19, Corollary 7.9].

Each k-tuple (p_i) yields a *distance map* $F : X \to \mathbb{R}^k$ via $F = (d_{p_1}, ..., d_{p_k})$. If (p_i) is a (k,δ)-strainer on a subset $A \subset X$, then the associated distance map F is called a *(k,δ)-strainer map* on A.

4.3. **Properties of strainer maps.** For $\delta \le \frac{1}{4\cdot k}$ and $L = 2\sqrt{k}$, every (k,δ)-strainer map $F : U \to \mathbb{R}^k$ on an open subset $U \subset \mathbb{R}^k$ is L-open and L-Lipschitz [LN19, Lemma 8.2].

The building blocks for straining maps are the following two observations. First, for any $\delta > 0$ and any $x \in X$ the function $d_x : \dot{B}_r(x) \to (0, r)$ is a $(1,\delta)$-strainer map if r is chosen small enough [LN19, Proposition 7.3]. Secondly, let $F : U \to \mathbb{R}^k$ be a (k,δ)-strainer map and let p be a point in a fiber Π of F. Then there exists $r > 0$ and a neighborhood W of $\dot{B}_r(p) \cap \Pi$ in U such that the map $\hat{F} = (F, d_p) : W \to \mathbb{R}^{k+1}$ is a $(k+1, 4\delta)$-strainer map [LN19, Proposition 9.4].

Any (k,δ)-strainer map on an open subset of a k-dimensional CAT(0) space X provides a bilipschitz chart [LN19, Corollary 11.2]. In general, we have the following topological structure.

Theorem 4.1 ([LN21, Theorem 5.1 and Corollary 5.2])**.** *Let U be an open subset of a GCBA space X. Let $F : U \to \mathbb{R}^k$ be a (k,δ)-strainer map, for some k and some $\delta < \frac{1}{20\cdot k}$. Then any $x \in U$ has arbitrary small open contractible neighborhoods V, such that the restriction $F : V \to F(V)$ is a Hurewicz fibration with contractible fibers.*

If a fiber $F^{-1}(b)$ is compact, then there exists an open neighborhood V of $F^{-1}(b)$ in U such that $F : V \to F(V)$ is a Hurewicz fibration.

If U is a homology n-manifold, then any fiber $F^{-1}(b)$ is a homology $(n-k)$-manifold.

5. Extended strainer maps

5.1. **Definition and basic properties.** Throughout this section, X will denote a locally compact and geodesically complete CAT(0) space.

Let $(p_1, ..., p_k)$ be a k-tuple in X and let $q \in X$ be an additional point. We say that $(p_1, ..., p_k, q)$ is an *extended* (k, δ)*-strainer* in a subset $A \subset X \setminus \{p_1, ..., p_k, q\}$, if the following holds true for all $x \in A$:

The k-tuple (p_i) is a (k, δ)-strainer at x and any continuation qq' of the geodesic qx beyond x is such that, for all $1 \leq i \leq k$,

$$\angle qxp_i < \frac{\pi}{2} + \delta \ \text{ and } \ \angle q'xp_i < \frac{\pi}{2} + \delta \,.$$

By the semi-continuity of angles, the set U of all points at which $(p_1, ..., p_k, q)$ is an extended (k, δ)-strainer is open in $X \setminus \{p_1, ..., p_k, q\}$ [LN19, Section 3.3 and Corollary 7.9].

Let $(p_1, ..., p_k, q)$ be an extended (k, δ)-strainer in an open set $U \subset X$. Then we call the map

$$\hat{F} = (d_{p_1}, ..., d_{p_k}, d_q) = (F, d_q) : U \to (0, \infty)^{k+1}$$

an *extended* (k, δ)*-strainer map*.

By definition, an extended (k, δ)-strainer map $\hat{F} : U \to \mathbb{R}^{k+1}$ is also an extended (k, δ')-strainer map for any $0 < \delta' < \delta$.

5.2. **Basic properties.** Let $(p_1, ..., p_k, q)$ be an extended (k, δ)-strainer at a point $x \in X$ and let qq' be an extension of the geodesic qx. Since $\angle qxq' = \pi$, for $1 \leq i \leq k$, we have

$$\angle p_i xq > \frac{\pi}{2} - \delta \ \text{ and } \ \angle p_i xq' > \frac{\pi}{2} - \delta \ .$$

We fix an opposite (k, δ)-strainer $(p_1', ..., p_k')$ to (p_i) at the point x. The definition of opposite strainers implies:

$$\angle p_i' xq < \frac{\pi}{2} + 2\delta \ \text{ and } \ \angle p_i' xq' < \frac{\pi}{2} + 2\delta \ .$$

Therefore, we also get

$$\angle p_i' xq > \frac{\pi}{2} - 2\delta \ \text{ and } \ \angle p_i' xq' > \frac{\pi}{2} - 2\delta \ .$$

Applying [LN19, Lemma 8.1] (compare [LN19, Lemma 8.2]) we get:

Lemma 5.1. *For $\delta \leq \frac{1}{20 \cdot k}$ and $L = 2\sqrt{k+1}$, any extended (k, δ)-strainer map $\hat{F} : U \to \mathbb{R}^{k+1}$ is L-Lipschitz and L-open.*

Remark 5.2. The argument in [LN19, Lemma 8.3] allows to choose the constant L above arbitrary close to 1, if only δ is sufficiently small.

By definition, any point q is an extended $(0, \delta)$-strainer in $X \setminus \{q\}$ for any $\delta > 0$ and the distance function $d_q : X \setminus \{q\} \to (0, \infty)$ is an extended $(0, \delta)$-strainer map. We are interested in distance spheres, thus fibers of such $(0, \delta)$-strainer maps. As in [BGP92, LN19], we approach the structure of these fibers by finding more strainers:

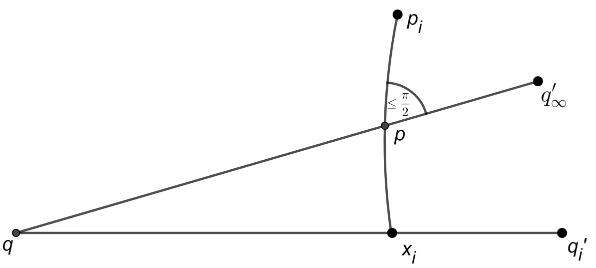
Lemma 5.3. *Let $\hat{F} = (d_{p_1}, ..., d_{p_k}, d_q) : U \to \mathbb{R}^{k+1}$ be an extended (k, δ)-strainer map for some $k \geq 0$ and $\delta < \frac{1}{20 \cdot k}$. Let $p \in U$ be arbitrary and let $\hat{\Pi}_p := \hat{F}^{-1}(\hat{F}(p))$ be the fiber of $\hat{F}$ through p.*

Then there exists $r > 0$ such that $(p, p_1, ..., p_k, q)$ is an extended $(k+1, 4 \cdot \delta)$-strainer in the intersection of $\hat{\Pi}_p$ and the punctured ball $\dot{B}_r(p)$.

Proof. We apply [LN19, Proposition 9.4] and find some $r > 0$ such that $(p, p_1, ..., p_k)$ is a $(k+1, 4\delta)$-strainer in $\hat{\Pi}_p \cap \dot{B}_r(p)$.

For any $x \in \hat{\Pi}_p \cap \dot{B}_r(p)$, the points x and p are at equal distance to q, hence $\angle pxq < \frac{\pi}{2}$. It remains to prove that, for sufficiently small $r > 0$, $\angle pxq' < \frac{\pi}{2} + \delta$, for all $x \in \hat{\Pi}_p \cap \dot{B}_r(p)$ and all points q' with $\angle qxq' = \pi$. Suppose for contradiction that we find $x_i \in \hat{\Pi}_p \setminus \{p\}$ converging to p and points q_i' lying on extensions of the geodesics qx_i such that $\angle px_iq_i' \geq \frac{\pi}{2} + \delta$. We may assume that $d(x_i, q_i') = 1$ and that q_i' converges to a point q_∞'. Using geodesic completeness, we extend x_ip to a geodesic x_ip_i of length one and such that

$$\angle p_ipq_\infty' = \pi - \angle q_\infty' px_i \leq \frac{\pi}{2} .$$



Taking a subsequence, we may assume that p_i converges to a point p_∞. Hence, $\angle p_\infty pq_\infty' \leq \frac{\pi}{2}$. But by semi-continuity of angles

$$\angle p_\infty pq_\infty' \geq \limsup_{i \to \infty} \angle p_ix_iq_i' \geq \frac{\pi}{2} + \delta .$$

This contradiction finishes the proof. ☐

5.3. **Halfspaces.** Let $\hat{F} = (F, d_o)$ be an extended strainer map on an open set U. We denote the F-fiber and $\hat{F}$-fiber through a point x by Π_x and $\hat{\Pi}_x$, respectively. We define the $\hat{F}$*-halfspace* through x by

$$\hat{\Pi}_x^+ = \{y \in \Pi_x | \ d_o(y) \leq d_o(x)\}.$$

The proof of our main results will rely on the following structural results about the fibers and halfspaces of extended strainer maps. The proofs of these results are postponed to Section 7.

The first result generalizes [Thu96b, Proposition 2.7] and [LN21, Corollary 5.2]:

Proposition 5.4. *Let U be an open subset of X. Then for any extended (k,δ)-strainer map $\hat{F} : U \to \mathbb{R}^{k+1}$ with $\delta < \frac{1}{64\cdot k}$ the following holds true for any $x \in U$.*

1. *The halfspace $\hat{\Pi}^+_x$ is an ANR.*
2. *If U is a homology n-manifold, then $\hat{\Pi}^+_x$ is a homology $(n-k)$-manifold with boundary $\hat{\Pi}_x$. The fiber $\hat{\Pi}_x$ is a homology $(n-k-1)$-manifold without boundary.*

The second statement is an extension of Theorem 1.8 which constitutes the special case where $k = 0$ and $y = x$. In the proof of our main results we will only need the cases $k = 0$ and $k = 1$.

Proposition 5.5. *For every relatively compact set $V \subset X$ there exists $\delta_0 > 0$ with the following property. Let $\hat{F} : X \to \mathbb{R}^{k+1}$ be a distance map which is an extended (k,δ_0)-strainer map at a point $x \in V$. Then there exist $\epsilon_0, s_0 > 0$ such that for any $0 < \epsilon < \epsilon_0$ and any point y with $d(x,y) < s_0 \cdot \epsilon$ the 'hemisphere' $S_\epsilon(x) \cap \hat{\Pi}^+_y$ is contractible and locally contractible.*

6. Proof of the Main Theorem

6.1. **Topology of intersecting spheres.** We begin the proof of Theorem 1.4. Thus, we fix a CAT(0) homology 4-manifold X, a point $o \in X$ and some radius $R > 0$. We denote by S the distance sphere $S = S_R(o)$ and we are going to verify that S is a topological 3-manifold.

We fix an arbitrary $p \in S$ for the rest of the proof. We need to find a neighborhood of p in S which is homeomorphic to $\mathbb{R}^3$. For this we aim to show that the restriction of d_p to S is a fiber bundle on a punctured neighborhood of p in S. The proof boils down to understanding how distance spheres intersect in our CAT(0) homology 4-manifold X.

We apply Proposition 5.5 to the relatively compact set $V := B_{2R}(o)$. Note that $\hat{F} := d_o : V \to \mathbb{R}$ is an extended $(0,\delta)$-strainer map for any $\delta > 0$. The halfspace $\hat{\Pi}^+_p$ is exactly the ball $\bar{B}_R(o)$ and $\hat{\Pi}_p = S$.

Corollary 6.1. *There exists a radius $r_p > 0$ such that $S_r(p) \cap S$ is homeomorphic to $\mathbb{S}^2$ for every $0 < r \leq r_p$.*

Proof. By [LN19, Proposition 9.4] and Lemma 5.3, we can choose r_p such that p is a $(1,\delta)$-strainer in $\dot{B}_{r_0}(p)$ and (p,o) is an extended $(1,\delta)$-strainer on $\dot{B}_{r_0}(p) \cap S$. In addition, we choose r_p smaller than the constant $\epsilon_0 = \epsilon_0(p)$ from Proposition 5.5. Then by Proposition 5.4 and Proposition 5.5, $S_r(p) \cap \bar{B}_R(o)$ is a contractible homology 3-manifold with boundary $S_r(p) \cap S$, for all $r < r_p$. Thus, $S_r(p) \cap S$ is a homology 2-manifold and therefore a 2-manifold by Theorem 3.1. By Poincaré duality, $S_r(p) \cap S$ is a homology 2-sphere, see [Thu96b, Proposition 2.8]. Due to the classification of surfaces, $S_r(p) \cap S$ is homeomorphic to $\mathbb{S}^2$. $\square$

Lemma 6.2. *There exists $r_0 = r_0(p) > 0$ such that the distance function $d_p : \dot{B}_{r_0}(p) \cap S \to (0, r_0)$ has uniformly locally contractible fibers.*

Proof. Let δ_0 be the constant from Proposition 5.5 and set $\delta = \delta_0/4$. Let r_p be as in Corollary 6.1. By [LN19, Proposition 9.4] and Lemma 5.3, we can choose $r_0 < r_p$ such that p is a $(1, \delta)$-strainer in $\dot{B}_{r_0}(p)$ and (p, o) is an extended $(1, \delta)$-strainer on $\dot{B}_{r_0}(p) \cap S$.

We fix an arbitrary $x \in \dot{B}_{r_0}(p) \cap S$ and set $t_0 := d_p(x)$. In addition, we fix a positive number $\rho_0 < r_0 - t_0$.

Sublemma. There exists a positive $\epsilon_0 < \rho_0$ and a positive $s_0 < 1$ such that for all t with $|t - t_0| < s_0 \cdot \epsilon_0$ the intersection of spheres $S_{\epsilon_0}(x) \cap S_t(p) \cap S$ is homeomorphic to $\mathbb{S}^1$.

We apply [LN19, Proposition 9.4] and Lemma 5.3 and find $\epsilon_0 < \rho_0$ small enough, so that (p, x) is a $(2, \delta_0)$-strainer in $\dot{B}_{2\epsilon_0}(x) \cap S_{t_0}(p)$ and (p, x, o) is an extended $(2, \delta_0)$-strainer in $\dot{B}_{2\epsilon_0}(x) \cap S_{t_0}(p) \cap S$.

Using the openness of the strainer property, we find some small $s_0 > 0$, such that for all t with $|t - t_0| < s_0 \cdot \epsilon_0$, the pair (p, x) is a $(2, \delta_0)$-strainer in $\dot{B}_{2\epsilon_0}(x) \cap S_t(p)$ and the triple (p, x, o) is an extended $(2, \delta_0)$-strainer in $\dot{B}_{2\epsilon_0}(x) \cap S_t(p) \cap S$.

We apply Proposition 5.4 and deduce that for all such t the intersection $S_{\epsilon_0}(x) \cap S_t(p) \cap \bar{B}_R(o)$ is a homology 2-manifold with boundary $S_{\epsilon_0}(x) \cap S_t(p) \cap S$. By Theorem 3.1, these intersections $S_{\epsilon_0}(x) \cap S_t(p) \cap \bar{B}_R(o)$ are 2-manifolds with boundary $S_{\epsilon_0}(x) \cap S_t(p) \cap S$.

By our choice of $\delta_0 = 4 \cdot \delta$, we may apply Proposition 5.5. By possibly making ϵ_0 and s_0 even smaller, we deduce that all intersections $S_{\epsilon_0}(x) \cap S_t(p) \cap \bar{B}_R(o)$ are contractible and therefore homeomorphic to closed discs. Hence their boundaries $S_{\epsilon_0}(x) \cap S_t(p) \cap S$ are circles. This finishes the proof of the sublemma.

Now we can easily finish the proof of the lemma. By the choice of r_0, ρ_0 and Corollary 6.1, any fiber $S_t(p) \cap S$ is homeomorphic to $\mathbb{S}^2$.

In order to verify the uniform local contractibility of the fibers of the restriction of d_p, we will argue that for every t with $|t - t_0| < s_0 \cdot \epsilon_0$, the set $B_{\epsilon_0}(x) \cap S_t(p) \cap S$ is contractible inside $B_{\rho_0}(x) \cap S_t(p) \cap S$.

In the same parameter range as above, $B_{\epsilon_0}(x) \cap S_t(p) \cap S$ is an open subset of the 2-sphere $S_t(p) \cap S$ whose topological boundary inside $S_t(p) \cap S$ is contained in the circle $S_{\epsilon_0}(x) \cap S_t(p) \cap S$. Therefore, by the Jordan curve theorem, $\bar{B}_{\epsilon_0}(x) \cap S_t(p) \cap S$ is either a topological disc or all of $S_t(p) \cap S$. In both cases, $B_{\epsilon_0}(x) \cap S_t(p) \cap S$ is contractible inside $\bar{B}_{\epsilon_0}(x) \cap S_t(p) \cap S$. Therefore, $B_{\epsilon_0}(x) \cap S_t(p) \cap S$ is contractible inside the larger set $B_{\rho_0}(x) \cap S_t(p) \cap S$. □

6.2. **The main results.** We can now finish:

Proof of Theorem 1.4. Let $p \in S = S_R(o)$ be arbitrary. Choose r_p as in Corollary 6.1 and $r_0 < r_p$ as in Lemma 6.2. The distance function

$d_p : \dot{B}_{r_0}(p) \cap S \to (0, r_0)$ has uniformly locally contractible fibers homeomorphic to $\mathbb{S}^2$ by Corollary 6.1 and Lemma 6.2. By Lemma 5.1 and Theorem 3.5, d_p is a fiber bundle. Hence $\dot{B}_{r_0}(p) \cap S$ is homeomorphic to $\mathbb{S}^2 \times (0, r_0)$. Therefore, $B_{r_0}(p) \cap S$ is homeomorphic to a 3-ball. Since p was arbitrary, $S = S_R(o)$ is a 3-manifold as required. □

Now the main result of [Thu96b] implies Theorem 1.1.

Next we turn to the proof of Corollary 1.5. The statement is essentially contained in the proof of [Thu96b, Theorem 4.3]. Therefore, we will only present a sketch and we will freely use some vocabulary from geometric topology, for which we refer the reader to [Thu96b].

Proof of Corollary 1.5. By Theorem 1.1 and [Thu96b, Theorem 4.3] all distance spheres $S_R(o)$ are homotopy equivalent to $X \setminus \{o\}$, hence to $\mathbb{S}^3$. By Theorem 1.4 and the resolution of the Poincaré conjecture, any sphere $S_R(o)$ is homeomorphic to $\mathbb{S}^3$.

Combining this observation with [Thu96b, Corollary 2.10 and Theorem 2.13] and the subsequent remark, we deduce that for any $R > R_0 > 0$ the geodesic contraction $S_R(o) \to S_{R_0}(o)$ is a *cell-like map*. This implies that the family of spheres $S_R(o)$ is uniformly locally contractible for all $0 < R_0 \le R \le R_1 < \infty$.

Hence an application of Theorem 3.6 completes the proof. □

7. Structure of fibers of extended strainer maps

7.1. **Generalized distance functions.** In this final section, we want to use information on limits of distance maps to conclude topological properties of their fibers. This requires a slight generalization of the notion of distance functions and strainer maps. For this purpose we make the following definitions, see also [Nag21, Section 5]. Recall that a convex function on a CAT(0) space attains its minimal value on a closed convex set or doesn't attain a minimum at all. A *generalized distance function* on a CAT(0) space X is a convex function $b : X \to \mathbb{R}$ whose (negative) gradient has unit norm on the complement of its minimal set:

$$\|\nabla_x(-b)\| := \max\left\{0, \limsup_{y\to x} \frac{b(x) - b(y)}{d(x,y)}\right\} \equiv 1.$$

This definition unifies the concept of distance functions to convex subsets and Busemann functions. Adding a constant to a generalized distance function results in a generalized distance function. On every bounded open set, a generalized distance function equals the distance function to a convex set up to a constant. In particular, the integral curves of the 'negative gradient' of a generalized distance function are geodesics and the *negative gradient* $\nabla_x(-b) \in \Sigma_x X$ is well-defined.

A map $F : X \to \mathbb{R}^k$ will be called a *generalized distance map*, if all coordinates f_i of F are generalized distance functions.

A *generalized distance map* $F : X \to \mathbb{R}^k$ with components f_i will be called a generalized (k, δ)-strainer map in a subset $A \subset X$ if the minimum sets of any f_i is disjoint from A and the following holds true. For any $x \in A$, the negative gradients $\nabla_x(-f_i) \in \Sigma_x X$ form a δ-spherical k-tuple of directions in $\Sigma_x X$.

Two generalized distance maps $F, \bar{F} : X \to \mathbb{R}^k$ with coordinates $f_i, \bar{f}_i$ are *opposite generalized* (k, δ)*-strainer maps* on $A \subset X$, if for all $x \in A$, the corresponding k-tuples of negative gradients are opposite (k, δ)-strainers.

As for (non-generalized) distance maps, the set of points $x \in X$, at which a generalized distance map $F : X \to \mathbb{R}^k$ is a generalized (k, δ)-strainer map is open. Similarly, the set of points at which a pair of distance maps $F, \bar{F}$ are opposite generalized (k, δ)-strainers is open.

Let $F : X \to \mathbb{R}^k$ be a generalized distance map with coordinates f_i and denote by b another generalized distance function. Suppose that F is a generalized (k, δ)-strainer map on a subset $A \subset X$ and b does not attain its minimum on A. Then the map $\hat{F} = (F, b) : X \to \mathbb{R}^{k+1}$ is called a *generalized extended* (k, δ)*-strainer map* on A, if at all points $x \in A$ and for every antipode $w_x \in \Sigma_x X$ of $\nabla_x(-b)$ the following holds, for all $1 \leq i \leq k$:

$$\angle_x(\nabla_x(-b), \nabla_x(-f_i)) < \frac{\pi}{2} + \delta \quad \text{and} \quad \angle_x(w_x, \nabla_x(-f_i)) < \frac{\pi}{2} + \delta.$$

The set of points where a given generalized distance map is a generalized extended (k, δ)-strainer map is open, again due to the semi-continuity of angles.

All statements about (extended) strainer maps transfer to the generalized setting. For instance, the concept of 'straining radius' introduced in [LN19, Section 7.5] generalizes as follows. Let $F : X \to \mathbb{R}^k$ be a generalized distance map with coordinates f_i. Suppose that F is a generalized (k, δ)-strainer map at a point x. Then the *straining radius* is the largest radius σ_x with the following property. For every $y \in B_{\sigma_x}(x)$ and $1 \leq i \leq k$ let $\bar{p}_i$ be any point with $d(x, \bar{p}_i) = 1$ and such that the direction at x of the geodesic $x\bar{p}_i$ is antipodal to $\nabla(-f_i)$. Then, F and $\bar{F} = (d_{\bar{p}_1}, \ldots, d_{\bar{p}_k})$ are opposite generalized $(k, 2\delta)$-strainer maps on $B_{\sigma_x}(y)$. The proof of positivity of σ_x is identical to [LN19, Lemma 7.10]. Similarly, we define an 'extended straining radius'. If $\hat{F} = (F, b)$ is an extended generalized (k, δ)-strainer map at x, then the *extended straining radius* is the largest radius $\hat{\sigma}_x \leq \sigma_x$ such that for all $y \in B_{\hat{\sigma}_x}(x)$ the map $\hat{F}$ is an extended generalized $(k, 2\delta)$-strainer map on $B_{\hat{\sigma}_x}(y)$.

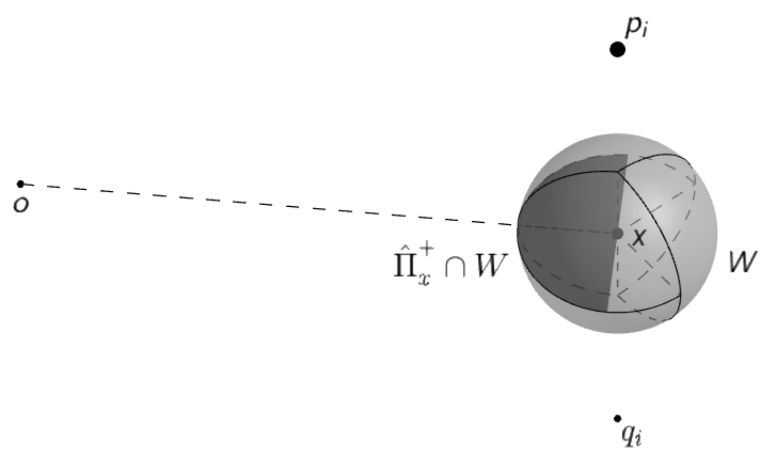
Let (X_n, x_n) be a sequence of pointed locally compact CAT(0) spaces converging in the pointed Gromov–Hausdorff topology to a space (X, x), Then, for any sequence of generalized distance functions $f_n : X_n \to \mathbb{R}$

with uniformly bounded $f_n(x_n)$, we find a subsequence converging to a generalized distance function $f : X \to \mathbb{R}$.

Let $F_n : X_n \to \mathbb{R}^k$ be a sequence of generalized distance maps converging to a generalized distance map $F : X \to \mathbb{R}$. The semi-continuity of angles under convergence implies the following, as in [LN19, Lemma 7.8]. If F is a generalized (extended) (k, δ)-strainer at x then F_n is a generalized (extended) (k, δ)-strainer at x_n, for all n large enough. Moreover, for all n large enough, the (extended) straining radius σ_{x_n} of F_n at x_n is bounded from below by half of the (extended) straining radius σ_x.

7.2. **Local topology of halfspaces.** The following result on strainer maps translates to the generalized setting as well. But since we only apply it in the non-generalized setting, and since this allows us to directly rely on [LN19, Theorem 9.1], we refrained from formulating a generalized version even though proofs extend literally.

Proposition 7.1. *Let $\hat{F} = (F, d_o) : X \to \mathbb{R}^k$ be a distance map and $\delta \leq \frac{1}{64 \cdot k}$. Suppose that $\hat{F}$ is an extended (k, δ)-strainer map at a point x with extended strainer radius $\hat{\sigma}_x$. Denote by W a ball $B_r(x)$ with radius $r \leq \hat{\sigma}_x$. Then there exists a deformation retraction of W onto the halfspace $\hat{\Pi}_x^+ \cap W$.*



Proof. The proof is an adaption of the proof of [LN19, Theorem 9.1]. For convenience of the reader, we stick to the notation of [LN19, Theorem 9.1]. Hence, (p_i) denotes the strainer defining F and (q_i) is a k-tuple in $X \setminus W$ such that x lies on the geodesic $p_i q_i$ and the tuples (p_i) and (q_i) are opposite $(k, 2\delta)$-strainers in W.

Set $s_0 := d(o,x)$ and define the function $M : W \to \mathbb{R}$ by

$$M(z) := \max_{1\leq i\leq k} |d(p_i,z) - d(p_i,x)|.$$

Denote by $\Phi : W \times [0,1] \to W$ the homotopy which retracts W onto $\Pi_x \cap W$, provided by [LN19, Theorem 9.1]. Recall that the length of the path $\gamma_z(t) := \Phi(z,t)$ is at most $8k \cdot M(z)$. Moreover, $\gamma_z(t)$ is an infinite piecewise geodesic all of whose segments are directed towards one of the points p_i or q_i. By the first variation formula, the value of d_o changes along $\gamma_z(t)$ with velocity at most 4δ. Hence, for all $t \in [0,1]$,

$$|d_o(z) - d_o(\gamma_z(t))| \leq 4\delta \cdot 8k \cdot M(z) \leq \frac{1}{2} \cdot M(z)\ . \tag{7.1}$$

Denote by $\varphi : W \times [0,1] \to W$ the flow which deformation retracts W onto $\bar{B}_{s_0}(o) \cap W$. More precisely, φ moves a point $z \in W \setminus \bar{B}_{s_0}(o)$ towards o at unit speed until it reaches $S_{s_0}(o)$ and then stops. Note that φ does indeed preserve W, by the CAT(0) property of X. We define a concatenated homotopy Ψ by setting $\Psi(z,t) = \Phi(z,2t)$ for $t \leq \frac{1}{2}$ and $\Psi(z,t) = \varphi(\Phi(z,1), 2t-1)$ for $t \geq \frac{1}{2}$.

By definition, $d(o, \Psi(z,1)) \leq s_0$, for all $z \in W$ and Ψ fixes $\hat{\Pi}_x^+ \cap W$.

The length of the Ψ-flow line of a general point $z \in W$ is bounded above by $8k \cdot M(z) + 2r$. However, if $d(o,z) \leq s_0$ holds, then the length of the Ψ-flow line starting at z is at most $(1+4\delta)\cdot 8k \cdot M(z)$ by (7.1).

Along the homotopy φ the value of M changes at most with velocity 4δ, due to the first variation formula. Hence, for any z with $d(o,z) \leq s_0$, we deduce, using $M(\Phi(z,1)) = 0$ and (7.1):

$$M(\Psi(z,1)) \leq 2\delta \cdot M(z)\ .$$

To obtain the required deformation retraction, we take a limit of iterated concatenations of Ψ. More precisely, for $m \geq 1$, we define homotopies $\Psi_m : W \times [0,1] \to W$ as follows. The homotopy Ψ_m is the identity on the interval $[1-2^{-m}, 1]$ and it equals a rescaling of Ψ on any of the intervals $[1-2^{-l}, 1-2^{-l-1}]$, for $l = 0, ..., m-1$.

The above inequalities imply $M(\Psi_m(z,1)) \leq (2\delta)^m \cdot M(z)$, by induction. Moreover, the flow line of Ψ_m starting at $z \in W$ has length uniformly bounded above by $2r + 24k \cdot M(z)$. Therefore, (Ψ_m) converges uniformly to a homotopy $\Psi_\infty : W \times [0,1] \to W$ as required. □

Recall that a closed subset Π of a topological space Y is called *homotopy negligible in* Y if for each open set U of Y the inclusion $U \setminus \Pi \to U$ is a homotopy equivalence. If Y is an ANR, this condition is satisfied, if any point $z \in \Pi$ has a neighborhood basis $\mathcal{U}_z$ of contractible neighborhoods U_z with contractible complements $U_z \setminus \Pi$, [EK69, Theorem 1]. In our setting, we have:

Corollary 7.2. *Let* $\hat{F} = (F, d_o) : U \to \mathbb{R}^k$ *be an extended* (k,δ)*-strainer map on an open set* $U \subset X$ *with* $\delta \leq \frac{1}{64\cdot k}$*. Then, for every*

$x \in U$, the halfspace $\hat\Pi^+_x$ is an ANR and the fiber $\hat\Pi_x$ is homotopy negligible in $\hat\Pi^+_x$.

Proof. By [LN19, Theorem 9.1], for all $y \in \hat\Pi^+_x \setminus \hat\Pi_x$ the set $\hat\Pi^+_x \cap B_r(y)$ is contractible as a retract of $B_r(y)$, as long as the radius r is less than the straining radius σ_y and the difference of levels $d_o(x) - d_o(y)$. Similarly, by Proposition 7.1, for all $z \in \hat\Pi_x$, the set $\hat\Pi^+_x \cap B_r(z)$ is contractible as a retract of $B_r(z)$ for some radius $r < \hat\sigma_z$. Hence, $\hat\Pi^+_x$ is an ANR. Now for $z \in \hat\Pi_x$ set $W = B_{\hat\sigma_z}(z)$ as above. It remains to show that $(\hat\Pi^+_x \setminus \hat\Pi_x) \cap W$ is contractible. Since it is an ANR as an open subset of $\hat\Pi^+_x$, it suffices to verify that all of its homotopy groups vanish, [Hu65, Corollary VII.8.5]. This will follow, once we have shown that for any compact subset $K \subset (\hat\Pi^+_x \setminus \hat\Pi_x) \cap W$, the inclusion map $K \hookrightarrow (\hat\Pi^+_x \setminus \hat\Pi_x) \cap W$ is nullhomotopic. By continuity of the straining radius $\hat\sigma_x$, for a given such set K, we find a point $w \in (\hat\Pi^+_x \setminus \hat\Pi_x) \cap W$ and $s \leq \hat\sigma_w$ with $K \subset B_s(w) \subset W$. By Proposition 7.1, $\hat\Pi^+_w \cap B_s(w) \subset (\hat\Pi^+_x \setminus \hat\Pi_x) \cap W$ is contractible and the proof is complete. □

Proof of Proposition 5.4. We have already seen in Corollary 7.2 that the halfspace $\hat\Pi^+_x$ is an ANR.

Assume now that U is a homology n-manifold. Since $\delta < \frac{1}{20 \cdot k}$, Theorem 4.1 implies that the fibers of F are homology $(n-k)$-manifolds. The complement of $\hat\Pi_x$ in $\hat\Pi^+_x$ is open in Π_x and therefore a homology $(n-k)$-manifold. By Corollary 7.2, $\hat\Pi_x$ is homotopy negligible in $\hat\Pi^+_x$. In particular, every singleton $\{y\} \subset \hat\Pi_x$ is homotopy negligible in $\hat\Pi^+_x$ [Tor78, Corollary 2.6]. We conclude that the local homology groups $H_*(\hat\Pi^+_x, \hat\Pi^+_x \setminus \{y\})$ vanish at all points $y \in \hat\Pi_x$. By [Mit90], $\hat\Pi_x$ is the boundary of a homology manifold and therefore is itself a homology manifold without boundary. □

7.3. **Contractibility of hemispheres.** At last, we provide

Proof of Proposition 5.5. Suppose for contradiction that there is a sequence $\delta_l \to 0$ and distance maps $\hat F_l = (F_l, d_{q^l}) : X \to \mathbb{R}^{k+1}$ with $F_l = (d_{p^l_1}, \dots, d_{p^l_k})$ which are extended (k, δ_l)-strainer maps at points $x_l \in V$ where the statement fails. Thus we find arbitrary small 'hemispheres' around x_l which are either not contractible or not locally contractible. More precisely, we find sequences $\epsilon_l \to 0$, $s_l \to 0$ and a sequence of points $y_l \in X$ with $d(x_l, y_l) < s_l \cdot \epsilon_l$ and the following additional properties.

(1) (*Gromov–Hausdorff close to tangent space*)

$$|\bar B_{\epsilon_l \cdot l}(x_l), \bar B_{\epsilon_l \cdot l}(o_{x_l})|_{GH} < \frac{\epsilon_l}{l};$$

(2) (*improved strainer*) the map (d_{x_l}, F_l) is a $(k+1, 4 \cdot \delta_l)$-strainer map on $\dot B_{\epsilon_l \cdot l}(x_l)$;

(3) (*improved extended strainer*) the map $(d_{x_l}, \hat{F}_l)$ is an extended $(k+1, 4\cdot\delta_l)$-strainer map on an open neighborhood V_l of $\hat{\Pi}_{x_l} \cap \dot{B}_{\epsilon_l \cdot l}(x_l)$;
(4) (*large levels*)
$$\min\{d_{p_1^l}(x_l), \ldots, d_{p_k^l}(x_l), d_{q^l}(x_l)\} \geq l \cdot \epsilon_l;$$
(5) (*fiber lies in extended domain*)
$$S_{\epsilon_l}(x_l) \cap \hat{\Pi}_{y_l} \subset V_l;$$
(6) (*non-contractible*) $S_{\epsilon_l}(x_l) \cap \hat{\Pi}^+_{y_l}$ is either not contractible or not locally contractible.

The first item can be arranged because in our setting tangent spaces are Gromov–Hausdorff limits of rescaled balls around a particular point [LN19, Corollary 5.7]. The second and third item follow from [LN19, Proposition 9.4] and Lemma 5.3, respectively, by choosing ϵ_l small enough. Similarly, the forth item can be achieved by choosing ϵ_l small enough. Finally, the fifth item can then be guaranteed by choosing s_l small enough.

We define the shifted strainer maps $\Phi_l = (d_{p_1^l} - d_{p_1^l}(x_l), \ldots, d_{p_k^l} - d_{p_k^l}(x_l))$, as well as the shifted distance function $b_l = d_{q^l} - d_{q^l}(x_l)$. In particular, $\Phi_l(x_l) = 0$. Now we rescale space and functions by $\frac{1}{\epsilon_l}$. Since V is relatively compact, up to passing to subsequences, we can take a pointed Gromov–Hausdorff limit $(X_\infty, x_\infty) = \lim\limits_{l\to\infty}(\frac{1}{\epsilon_l}\cdot X, x_l)$ [LN19, Proposition 5.10]. We also pass to corresponding limits of functions: $\Phi_\infty = \lim\limits_{l\to\infty} \frac{1}{\epsilon_l}\cdot\Phi_l$ and $b_\infty = \lim\limits_{l\to\infty}\frac{1}{\epsilon_l}\cdot b_l$. Item four above ensures that all coordiantes of Φ_∞, as well as the function b_∞, are Busemann functions on X_∞ [KL97, Lemma 2.3].

By condition (1) above, (X_∞, x_∞) is isometric to a pointed Gromov–Hausdorff limit of the sequence of tangent spaces $(T_{x_l}X, o_{x_l})$. In particular, (X_∞, x_∞) is isometric to a Euclidean cone with tip x_∞. Therefore, $S_1(x_\infty)$ is a CAT(1) space [Ber83]. Moreover, the spaces of directions $\Sigma_{x_l}X$ converge to $S_1(x_\infty)$ [LN19, Theorem 13.1]. By assumption, the negative gradients of the components of Φ_l provide a δ_l-spherical k-tuple of directions at x_l. [LN19, Proposition 6.6] implies that $S_1(x_\infty)$ splits isometrically as a spherical join $S_1(x_\infty) \cong \mathbb{S}^{k-1} * \Sigma'$; equivalently, X_∞ splits isometrically as a direct product $X_\infty \cong \mathbb{R}^k \times X'_\infty$. Moreover, the negative gradients of the components of Φ_∞ form a spherical k-tuple inside the $\mathbb{S}^{k-1}$-factor of $S_1(x_\infty)$.

From the L-openness of $(d_{x_l}, \hat{F}_l)$ (Lemma 5.1), we conclude the Gromov–Hausdorff convergence $\lim\limits_{l\to\infty}\frac{1}{\epsilon_l}\cdot(S_{\epsilon_l}(x_l)\cap\hat{\Pi}^+_{y_l}) = S_1(x_\infty)\cap\hat{\Pi}^+_{x_\infty}$.

Sublemma. The sequence $\frac{1}{\epsilon_l}\cdot(S_{\epsilon_l}(x_l)\cap\hat{\Pi}^+_{y_l})$ is uniformly locally contractible.

By compactness of $S_1(x_\infty)$, we find $r > 0$ such that the straining radius of Φ_∞ satisfies $\sigma_z > 2r$ at all $z \in S_1(x_\infty) \cap \Pi_{x_\infty}$ and the extended straining radius of (Φ_∞, b_∞) satisfies $\hat{\sigma}_z > 2r$ at all $z \in S_1(x_\infty) \cap \hat{\Pi}_{x_\infty}$. Then, for l large enough, the straining radius of F_l and the extended straining radius of (F_l, b_l) is larger than $r \cdot \epsilon_l$ on $S_{\epsilon_l}(x_l) \cap \Pi_{y_l}$ and $S_{\epsilon_l}(x_l) \cap \hat{\Pi}_{y_l}$, respectively.

Let $z_l \in S_{\epsilon_l}(x_l) \cap \hat{\Pi}^+_{y_l}$ be a point. If the distance from z_l to the fiber $S_{\epsilon_l}(x_l) \cap \hat{\Pi}_{y_l}$ is at least $\frac{r}{2} \cdot \epsilon_l$, then $S_{\epsilon_l}(x_l) \cap \hat{\Pi}^+_{y_l} \cap B_{\frac{r}{2}\cdot\epsilon_l}(z_l)$ is contractible by [LN19, Theorem 9.1]. On the other hand, if the distance from z_l to the fiber $S_{\epsilon_l}(x_l) \cap \hat{\Pi}_{y_l}$ is smaller than $\frac{r}{2} \cdot \epsilon_l$, then $S_{\epsilon_l}(x_l) \cap \hat{\Pi}^+_{y_l} \cap B_{\frac{r}{2}\cdot\epsilon_l}(z_l)$ is contained in an ball $B_{r\cdot\epsilon_l}(w_l)$ with $w_l \in S_{\epsilon_l}(x_l) \cap \hat{\Pi}_{y_l}$. By Proposition 7.1, the set $S_{\epsilon_l}(x_l) \cap \hat{\Pi}^+_{y_l} \cap B_{r\cdot\epsilon_l}(w_l)$ is contractible. It follows that any $\frac{r}{2}$-ball in $\frac{1}{\epsilon_l} \cdot (S_{\epsilon_l}(x_l) \cap \hat{\Pi}^+_{y_l})$ is contractible inside its concentric $2r$-ball.

As one consequence, the same local contractibility holds for the hemispheres $S_1(x_\infty) \cap \hat{\Pi}^+_{x_\infty}$ [Pet97, Theorem 9]. Moreover, $S_1(x_\infty) \cap \hat{\Pi}^+_{x_\infty}$ is homotopy equivalent to $S_{\epsilon_l}(x_l) \cap \hat{\Pi}^+_{y_l}$, for large enough l. Hence, to arrive at a contradiction, it remains to show that $S_1(x_\infty) \cap \hat{\Pi}^+_{x_\infty}$ is contractible.

Let $v \in S_1(x_\infty)$ denote the point corresponding to the negative gradient of b_∞. By semi-continuity of angles and the splitting $S_1(x_\infty) \cong \mathbb{S}^{k-1} * \Sigma'$, we see $v \in \Sigma'$ (cf. Section 5.2). In particular, $b_\infty = b'_\infty \circ \pi'$ where π' denotes the projection $X_\infty \cong \mathbb{R}^k \times X'_\infty \to X'_\infty$, and b'_∞ is a Busemann function on X'_∞. We infer $\hat{\Pi}^+_{x_\infty} \cong \{0\} \times \{b'_\infty \le 0\}$ since $\Phi_\infty(x_\infty) = 0$. Hence,

$$S_1(x_\infty) \cap \hat{\Pi}^+_{x_\infty} \cong \Sigma_{x_\infty} X'_\infty \cap \{b'_\infty \le 0\}.$$

But $\Sigma_{x_\infty} X'_\infty \cap \{b'_\infty \le 0\} = \bar{B}_{\frac{\pi}{2}}(v) \subset \Sigma_{x_\infty} X'_\infty$ and therefore $\Sigma_{x_\infty} X'_\infty \cap \{b'_\infty \le 0\}$ is contractible, since $\Sigma_{x_\infty} X'_\infty$ is CAT(1). Consequently, the hemisphere $S_1(x_\infty) \cap \hat{\Pi}^+_{x_\infty}$ is contractible. Contradiction. $\square$

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