DEHN FUNCTIONS AND HÖLDER EXTENSIONS IN ASYMPTOTIC CONES

ALEXANDER LYTCHAK, STEFAN WENGER, AND ROBERT YOUNG

ABSTRACT. The Dehn function measures the area of minimal discs that fill closed curves in a space; it is an important invariant in analysis, geometry, and geometric group theory. There are several equivalent ways to define the Dehn function, varying according to the type of disc used. In this paper, we introduce a new definition of the Dehn function and use it to prove several theorems. First, we generalize the quasi-isometry invariance of the Dehn function to a broad class of spaces. Second, we prove Hölder extension properties for spaces with quadratic Dehn function and their asymptotic cones. Finally, we show that ultralimits and asymptotic cones of spaces with quadratic Dehn function also have quadratic Dehn function. The proofs of our results rely on recent existence and regularity results for area-minimizing Sobolev mappings in metric spaces.

1. INTRODUCTION AND STATEMENT OF MAIN RESULTS

Isoperimetric inequalities and other filling functions describe the area of minimal surfaces bounded by curves or surfaces. Such minimal surfaces are important in geometry and analysis, and filling functions are particularly important in geometric group theory, where the Dehn function is a fundamental invariant of a group. The Dehn function, which measures the maximal area of a minimal surface bounded by curves of at most a given length, is connected to the difficulty of solving the word problem, and its asymptotics help describe the large-scale geometry of a group.

In this paper, we study spaces with a local or global quadratic isoperimetric inequality. The class of spaces that admit a global quadratic isoperimetric inequality includes not only spaces that satisfy a nonpositive curvature condition (such as δ–hyperbolicity or the CAT(0) condition), but also spaces with a mix of positive and negative curvature, such as the higher-dimensional Heisenberg groups and many solvable Lie groups.

Roughly speaking, spaces with a quadratic isoperimetric inequality tend to have many two-dimensional discs. Many techniques for proving that a space has a quadratic isoperimetric inequality show that in fact the space is Lipschitz 1–connected, i.e., that curves of length L are the boundaries of discs with Lipschitz constant of order L. It is an open question whether a quadratic isoperimetric inequality is equivalent to Lipschitz 1–connectedness. One way to approach this question is to study ultralimits and asymptotic cones of spaces with quadratic isoperimetric

inequalities. While Lipschitz 1–connectedness passes to asymptotic cones and ultralimits, it is an open question whether an asymptotic cone or ultralimit of a space with quadratic isoperimetric inequality also has a quadratic isoperimetric inequality.

In [31] it was shown that every asymptotic cone of a finitely presented group with a quadratic Dehn function is simply connected. In [37] it was shown that a homological quadratic isoperimetric inequality (using metric currents) is stable under taking ultralimits and asymptotic cones. This was used in [37] to produce the first examples of nilpotent groups whose Dehn function does not grow exactly like a polynomial function. A weaker version of the stability was also used in [36] to exhibit the smallest isoperimetric constant in a quadratic isoperimetric inequality at large scales which implies that the underlying space is Gromov hyperbolic.

The aim of this article is to prove several facts about spaces with a quadratic isoperimetric inequality. First, we will generalize the fact that the Dehn function of a manifold or simplicial complex is a quasi-isometry invariant to broader classes of spaces (Theorems 1.2 and 4.5). Second, we will show that quadratic Dehn functions are nearly stable under ultralimits (Theorem 1.6). Third, we will prove Hölder extension properties for spaces with quadratic Dehn functions (Theorems 1.4 and 6.6).

For the convenience of the geometrically minded reader we will first formulate our results using Lipschitz maps, but the proofs rely on a generalization of the Dehn function based on Sobolev maps with values in metric spaces. In manifolds, simplicial complexes, or other spaces that are Lipschitz 1–connected up to some scale, this Dehn function is equal to the usual Dehn function based on Lipschitz maps (Proposition 3.1), but this definition is better suited to some analytic arguments. For example, we will show that an ultralimit of spaces with quadratically bounded (Sobolev) Dehn functions has a quadratically bounded Sobolev Dehn function (Theorem 1.8).

1.1. The Lipschitz Dehn function and quasi-isometry invariance.

Let (X, d) be a complete metric space, and let D denote the open unit disc in $\mathbb{R}^2$ and $\overline{D}$ its closure. The (parameterized Hausdorff) area of a Lipschitz map $u \colon \overline{D} \to X$ is defined by

$$(1) \qquad \mathrm{Area}(u) = \int_X \#\{z \mid u(z) = x\} \, d\mathcal{H}^2(x),$$

where $\mathcal{H}^2$ denotes the 2–dimensional Hausdorff measure on X. In particular, if u is injective then $\mathrm{Area}(u) = \mathcal{H}^2(u(D))$. If X is a Riemannian manifold or simplicial complex with piecewise Riemannian metric then our definition of area coincides with the usual one which one obtains by integrating the Jacobian of the derivative of u, see Section 2. The Lipschitz filling area of a Lipschitz curve $c \colon S^1 \to X$ is given by

$$\mathrm{Fill\,Area}^{\mathrm{Lip}}(c) := \inf \left\{ \mathrm{Area}(v) \ \middle| \ v \colon \overline{D} \to X \text{ is Lipschitz}, \ v|_{S^1} = c \right\}.$$

Finally, the Lipschitz Dehn (or isoperimetric) function δ_X^{Lip} of X is defined by

$$\delta_X^{\mathrm{Lip}}(r) = \sup \left\{ \mathrm{Fill\,Area}^{\mathrm{Lip}}(c) \ \middle| \ c \colon S^1 \to X \text{ Lipschitz}, \ \ell(c) \leq r \right\}$$

for every $r > 0$, where $\ell(c)$ denotes the length of c. Thus, δ_X^{Lip} measures how the filling area of a curve depends on its length. It is important to notice that $\delta_X^{\mathrm{Lip}}(r)$

provides an upper bound on the area of suitable fillings of a given curve of length at most r but does not give control on the Lipschitz constants of such fillings.

There are several other notions of isoperimetric functions commonly used in the setting of large scale geometry and geometric group theory. One of them is the coarse isoperimetric function $\mathrm{Ar}_{X,\varepsilon}(r)$ introduced by Gromov [17]. Roughly speaking, this function measures how difficult it is to partition a given closed Lipschitz curve into closed curves of length at most ε. For the precise definition of $\mathrm{Ar}_{X,\varepsilon}$ we refer to Section 2 below, see also e.g. [13] or [9].

Our first result below shows that under mild conditions on the metric space X the functions $\delta_X^{\mathrm{Lip}}(r)$ and $\mathrm{Ar}_{X,\varepsilon}(r)$ have the same asymptotic growth as $r \to \infty$. In order to state our theorem, we recall the following definition. A metric space X is called Lipschitz 1–connected up to some scale if there exist $\lambda_0 > 0$ and $L \geq 1$ such that every λ–Lipschitz curve $c \colon S^1 \to X$ with $\lambda < \lambda_0$ extends to an $L\lambda$–Lipschitz map on $\overline{D}$. If the above holds with $\lambda_0 = \infty$ then X is called Lipschitz 1–connected. Spaces that are Lipschitz 1–connected up to some scale include Riemannian manifolds, finite dimensional simplicial complexes, and Alexandrov spaces when equipped with a cocompact group action by isometries, as well as CAT(κ)-spaces.

Theorem 1.1. *Let X be a locally compact, geodesic metric space. If X is Lipschitz 1–connected up to some scale and satisfies $\delta_X^{\mathrm{Lip}}(r) < \infty$ for all $r > 0$ then $\delta_X^{\mathrm{Lip}} \simeq \mathrm{Ar}_{X,\varepsilon}$ for all $\varepsilon > 0$.*

For the definition of the equivalence relation $\simeq$ for functions, see Section 2.

The assumptions on X and δ_X^{Lip} made in the theorem are not restrictive. Indeed, every locally compact, geodesic metric space Z which satisfies $\mathrm{Ar}_{Z,\varepsilon_0}(r) < \infty$ for some $\varepsilon_0 > 0$ and all $r > 0$ is quasi-isometric to a space X satisfying the hypotheses of Theorem 1.1. In fact, X can be obtained by suitably thickening up Z and Z then even embeds isometrically into X, see Section 3. In particular, for such X we have $\delta_X^{\mathrm{Lip}} \simeq \mathrm{Ar}_{X,\varepsilon} \simeq \mathrm{Ar}_{Z,\varepsilon}$ for every $\varepsilon \geq \varepsilon_0$.

It is well-known and easy to prove that the coarse isoperimetric function is a quasi-isometry invariant. Thus, Theorem 1.1 implies the quasi-isometry invariance of the Lipschitz Dehn function as well.

Theorem 1.2. *Let X and Y be locally compact, geodesic metric spaces such that $\delta_X^{\mathrm{Lip}}(r) < \infty$ and $\delta_Y^{\mathrm{Lip}}(r) < \infty$ for all $r > 0$. If X and Y are quasi-isometric and Lipschitz 1–connected up to some scale then $\delta_X^{\mathrm{Lip}} \simeq \delta_Y^{\mathrm{Lip}}$.*

Theorems 1.1 and 1.2 generalize results in [2], [15, 10.3.3], [8], and [12], which proved similar results for Riemannian manifolds and simplicial complexes. Our proof is most similar to that of Bridson in [8], which proves that a geometric version of the Dehn function is equivalent to a coarse version by using a characterization of minimal surfaces in a manifold.

1.2. **Spaces with quadratic Lipschitz Dehn function.**
We now turn to metric spaces whose Lipschitz Dehn function has at most quadratic growth. A complete metric space X is said to admit a quadratic isoperimetric inequality with respect to Lipschitz maps if there exists $C \geq 0$ such that $\delta_X^{\mathrm{Lip}}(r) \leq C \cdot r^2$ for all $r \geq 0$. Any complete metric space X that is Lipschitz 1–connected admits a quadratic isoperimetric inequality. These include Banach spaces, complete CAT(0) spaces, and, more generally, spaces with a convex geodesic bicombing. The higher Heisenberg

groups (with a left-invariant Riemannian or Carnot-Carathéodory distance) are also Lipschitz 1–connected.

More generally, every complete length space X that satisfies $\mathrm{Ar}_{X,\varepsilon}(r) \leq r^2$ for some $\varepsilon > 0$ is quasi-isometric to a complete length space Y which admits a quadratic isoperimetric inequality with respect to Lipschitz maps, see Proposition 3.7. If X is locally compact then Y can be chosen locally compact as well. See Section 2 for the definition of the relation $\preceq$. Consequently, if X is the universal cover of a closed Riemannian manifold or compact simplicial complex M, then X admits a quadratic isoperimetric inequality with respect to Lipschitz maps if and only if $\pi_1(M)$ has a quadratic Dehn function.

We first study Hölder extension properties of spaces with a quadratic isoperimetric inequality. Let Z and X be metric spaces. A map $\varphi\colon Z \to X$ is called (ν, α)–Hölder continuous, where $\nu \geq 0$ and $0 < \alpha \leq 1$, if

$$d_X(\varphi(z), \varphi(z')) \leq \nu \cdot d_Z(z, z')^{\alpha}$$

for all $z, z' \in Z$. When $\alpha = 1$, this is equivalent to Lipschitz continuity.

Definition 1.3. *Let $0 < \alpha \leq 1$. A pair (Z, X) of metric spaces Z and X is said to have the α–Hölder extension property if there exists $L \geq 1$ such that every (ν, α)– Hölder map $\varphi\colon A \to X$ with $A \subset Z$ and $\nu > 0$ admits an $(L\nu, \alpha)$–Hölder extension $\bar{\varphi}\colon Z \to X$.*

If X is a complete metric space and the pair $(\mathbb{R}^2, X)$ has the 1–Hölder (thus Lipschitz) extension property then X admits a quadratic isoperimetric inequality with respect to Lipschitz maps. The following theorem gives an "almost" converse.

Theorem 1.4. *Let X be a locally compact, geodesic metric space admitting a quadratic isoperimetric inequality with respect to Lipschitz maps. Then the pair $(\mathbb{R}^2, X)$ has the α–Hölder extension property for every $\alpha \in (0, 1)$.*

For every $0 < \alpha \leq 1$, the α–Hölder extension property for $(\mathbb{R}^2, X)$ is stable under taking ultralimits (Corollary 2.5). We thus obtain a strengthening of Papasoglu's result [31] that every asymptotic cone of a finitely presented group with quadratic Dehn function is simply connected.

Corollary 1.5. *Let X be a locally compact, geodesic metric space admitting a quadratic isoperimetric inequality with respect to Lipschitz maps. Then X is simply connected. Moreover, every asymptotic cone X_ω of X is simply connected and the pair $(\mathbb{R}^2, X_\omega)$ has the α–Hölder extension property for every $\alpha \in (0, 1)$.*

More generally, these two results will hold for metric spaces admitting a quadratic isoperimetric inequality with respect to Sobolev maps; see Section 1.3.

For sufficiently small $\alpha \in (0, 1)$, only depending on the isoperimetric constant, we produce α–Hölder extensions with additional properties, see Theorem 6.6.

Our next result yields almost-stability of a quadratic isoperimetric inequality under taking asymptotic cones.

Theorem 1.6. *Let X be a locally compact, geodesic metric space such that $\delta_X^{\mathrm{Lip}}(r) \leq Cr^2$ for all $r > 0$. Let X_ω be an asymptotic cone of X and let $\varepsilon > 0$. Then X_ω is $(1, \varepsilon)$–quasi-isometric to a complete length space Y such that $\delta_Y^{\mathrm{Lip}}(r) \leq C'r^2$ for all $r > 0$, where C' is a constant depending only on C.*

See Section 2 for the definition of an $(1, \varepsilon)$–quasi-isometry. It is an open question whether X_ω itself admits a quadratic isoperimetric inequality with respect to Lipschitz maps, but we will see that it admits a slightly weaker version of the quadratic isoperimetric inequality based on Sobolev maps (see Theorem 1.8).

As an application of Theorem 1.6 we obtain the following special case of the main result of [36]. Unlike the proof in [36] our arguments do not rely on the theory of currents in metric spaces.

Theorem 1.7. *Let X be a locally compact, geodesic metric space and $\varepsilon, r_0 > 0$. If every Lipschitz curve $c\colon S^1 \to X$ with $\ell(c) \geq r_0$ extends to a Lipschitz map $v\colon \overline{D} \to X$ with*

$$\text{Area}(v) \leq \frac{1 - \varepsilon}{4\pi} \cdot \ell(c)^2$$

then X is Gromov hyperbolic.

Theorem 1.6 can also be used to recover results in [37] (whose proofs relied on the theory of metric currents) concerning super-quadratic lower bounds for the growth of the Dehn function of certain Carnot groups, see Theorem 5.5 below. This result plays a role in the construction of nilpotent groups whose Dehn function do not grow exactly polynomially, see [37].

1.3. **Sobolev filling functions.** The proofs of the results above are based on the theory of Sobolev mappings from a Euclidean domain to a complete metric space X and our recent results on the existence and regularity of area-minimizing discs in proper metric spaces [27], [29], [26]. These results let us connect Gromov's coarse isoperimetric function $\text{Ar}_{X,\varepsilon}(r)$ to a filling function based on Sobolev maps.

There exist various equivalent definitions of Sobolev mappings with values in metric spaces, see [23], [32], and Section 2 below and the references therein. For $p > 1$ the space of p–Sobolev maps from D to X will be denoted by $W^{1,p}(D, X)$. Elements of $W^{1,p}(D, X)$ are equivalence classes of maps from D to X, so they are defined only up to sets of measure zero. Every Lipschitz map is p–Sobolev for every $p > 1$, but if $p \leq 2$ then a p–Sobolev map need not even have a continuous representative.

Nevertheless, if $u \in W^{1,2}(D, X)$, then one can define a parameterized Hausdorff area of u, denoted by $\text{Area}(u)$, see Section 2. This coincides with (1) when u is Lipschitz. Moreover, if $u \in W^{1,p}(D, X)$, then there is a map $\text{tr}(u) \in L^p(S^1, X)$, called the trace of u, such that if u has a continuous extension $\hat{u}$ to the closed unit disc $\overline{D}$, then $\text{tr}(u) = \hat{u}|_{S^1}$.

We can thus define a variant of the Lipschitz Dehn function $\delta_X^{\text{Lip}}(r)$ by replacing the Lipschitz filling area $\text{Fill Area}^{\text{Lip}}(c)$ of a Lipschitz curve c by its Sobolev variant

$$\text{Fill Area}(c) := \inf \left\{ \text{Area}(v) \mid v \in W^{1,2}(D, X), \text{tr}(v) = c \right\}.$$

Using this definition of filling area we obtain a Sobolev variant of the Dehn function which we denote by $\delta_X(r)$ and call the Dehn function of X, see Section 3.

A complete metric space X is said to admit a quadratic isoperimetric inequality (with respect to Sobolev maps) if there exists C such that $\delta_X(r) \leq C \cdot r^2$ for all $r > 0$. We will usually omit the phrase "with respect to Sobolev maps" in the sequel if there is no danger of ambiguity. A key property of Sobolev maps is that this notion is stable under taking ultralimits and asymptotic cones.

Theorem 1.8. *Let $C > 0$, let (X_n) be a sequence of locally compact, geodesic metric spaces, and let X be an ultralimit of (X_n). If $\delta_{X_n}(r) \leq Cr^2$ for all $r > 0$, then $\delta_X(r) \leq Cr^2$ for all $r > 0$.*

Corollary 1.9. *Let X be a locally compact, geodesic metric space. If X admits a quadratic isoperimetric inequality with constant C then so does every asymptotic cone of X.*

An analogous result for homological quadratic isoperimetric inequalities was proved in [37] using the theory of integral currents in metric spaces [4]. A similar stability property is also known for the coarse Dehn function. Indeed, if X is the Cayley graph of a finitely presented group and $\mathrm{Ar}_{X,\varepsilon}(r) \leq r^2$ then it follows from Papasoglu's result [31] that every asymptotic cone X_ω of X satisfies $\mathrm{Ar}_{X_\omega,\varepsilon}(r) \leq r^2$. Apart from the more general context, our result provides a bound on the Dehn function of X_ω on all (also small) scales and yields the sharp isoperimetric constant. We mention that the isoperimetric fillings of curves in the ultralimit which we construct in the theorem above can moreover be taken to be globally Hölder continuous and having Lusin's property (N), that is, they send sets of Lebesgue measure zero to sets of Hausdorff 2–measure zero.

Though $\delta_X(r)$ and $\delta_X^{\mathrm{Lip}}(r)$ are equal in many spaces, the general relationship between the two functions is unclear. We clearly have $\delta_X(r) \leq \delta_X^{\mathrm{Lip}}(r)$ for every complete metric space X, but since Sobolev maps need not have continuous representatives it is a priori not clear that a space admitting a quadratic isoperimetric inequality is even simply connected. However, we will show that if $\delta_X(r) \leq Cr^2$ for all sufficiently small r, then $\mathrm{Ar}_{X,\varepsilon} \simeq \delta_X$ for any $\varepsilon > 0$ (Theorem 4.4). Similarly, if X is Lipschitz 1–connected up to some scale, then $\delta_X(r) = \delta_X^{\mathrm{Lip}}(r)$ (Proposition 3.1). These equivalences will imply Theorems 1.1, 1.2, and 1.6.

A particular class of spaces to which the results above apply is the class of complete geodesic metric spaces X which are Ahlfors 2–regular, linearly locally contractible, and homeomorphic to $\mathbb{R}^2$ or S^2. Such spaces are of importance in various contexts, see e.g. [7]. It follows from the results in [28] that every such space X admits a quadratic isoperimetric inequality (with respect to Sobolev maps). By Theorem 7.1, every such X is α–Hölder 1–connected for every $\alpha \in (0, 1)$, and thus the pair $(\mathbb{R}^2, X)$ has the α–Hölder extension property. It is not known whether every such space is Lipschitz 1–connected.

1.4. **Outline.** In Section 2, we will recall some definitions and facts about Sobolev maps, isoperimetric inequalities, and ultralimits. In Sections 3 and 4, we will prove the equivalence of the Dehn function, the Lipschitz Dehn function, and the coarse isoperimetric function, showing Theorems 1.1 and 1.2. Then, in Section 5, we show that the Dehn function is stable under ultralimits and asymptotic cones, proving Theorems 1.6–1.8 and Corollary 1.9. Finally, in Sections 6–8, we prove extension results for Sobolev and Hölder maps, including Theorem 1.4 and Corollary 1.5.

2. Preliminaries

2.1. **Basic notation and definitions.** The Euclidean norm of a vector $v \in \mathbb{R}^n$ will be denoted by $|v|$. The unit circle in $\mathbb{R}^2$ with respect to the Euclidean norm is

denoted by

$$S^1 := \{z \in \mathbb{R}^2 \mid |z| = 1\}$$

and will be endowed with the Euclidean metric unless otherwise stated. The open unit disc in $\mathbb{R}^2$ is denoted by

$$D := \{z \in \mathbb{R}^2 \mid |z| < 1\},$$

its closure by $\overline{D}$.

Let (X, d) be a metric space. A curve in X is a continuous map $c\colon I \to X$, where I is an interval or S^1. If I is an interval then the length of c is defined by

$$\ell(c) := \sup \left\{ \sum_{i=0}^{k-1} d(c(t_i), c(t_{i+1})) \;\middle|\; t_i \in I \text{ and } t_0 < t_1 < \cdots < t_k \right\}$$

and an analogous definition applies in the case $I = S^1$. The space X is proper if every closed ball of finite radius in X is compact. For $\lambda \geq 1$ the space X is λ–quasi-convex if any two points $x, y \in X$ can be joined by a curve of length at most $\lambda \cdot d(x, y)$. If $\lambda = 1$ then X is called geodesic. A metric space which is λ–quasi-convex for every $\lambda > 1$ is called a length space. A complete, quasi-convex metric space is proper if and only if it is locally compact.

For $s \geq 0$ the Hausdorff s–measure on a given metric space is denoted by $\mathcal{H}^s$. We choose the normalization constant in such a way that on Euclidean $\mathbb{R}^n$ the Hausdorff n–measure coincides with the Lebesgue measure. A map from a subset of $\mathbb{R}^n$ to a metric space is said to satisfy Lusin's property (N) if it sends sets of Lebesgue measure zero to sets of Hausdorff n–measure zero.

The Lipschitz constant of a map $\varphi\colon X \to Y$ between metric space X and Y is denoted by $\mathrm{Lip}(\varphi)$.

2.2. **Metric space valued Sobolev maps.** In this subsection we briefly review the main definitions and results concerning Sobolev maps from a Euclidean domain into a metric space used throughout the present paper. There exist several equivalent definitions of Sobolev maps from Euclidean domains with values in a metric space, see e.g. [3], [23], [32], [33], [34], [19], [18], [5]. Here, we recall the definition of [32] using compositions with real-valued Lipschitz functions. We will restrict ourselves to Sobolev maps defined on the open unit disc D of $\mathbb{R}^2$.

Let (X, d) be a complete metric space and $p > 1$. We denote by $L^p(D, X)$ the set of measurable and essentially separably valued maps $u\colon D \to X$ such that for some and thus every $x \in X$ the function

$$u_x(z) := d(x, u(z))$$

belongs to $L^p(D)$, the classical space of p–integrable functions on D.

Definition 2.1. *A map $u \in L^p(D, X)$ belongs to the Sobolev space $W^{1,p}(D, X)$ if there exists $h \in L^p(D)$ such that for every $x \in X$ the function u_x belongs to the classical Sobolev space $W^{1,p}(D)$ and has weak gradient bounded by $|\nabla u_x| \leq h$ almost everywhere.*

The Reshetnyak p–energy $E_+^p(u)$ of a map $u \in W^{1,p}(D, X)$ is defined by

$$E_+^p(u) := \inf \left\{ \|h\|_{L^p(D)}^p \;\middle|\; h \text{ as in the definition above} \right\}.$$

If $u \in W^{1,p}(D, X)$ then there exists a representative $\bar{u}$ of u such that for almost every $v \in S^1$ the curve $t \mapsto \bar{u}(tv)$ with $t \in [1/2, 1)$ is absolutely continuous. The trace of u is defined by

$$\operatorname{tr}(u)(v) := \lim_{t \nearrow 1} \bar{u}(tv)$$

for almost every $v \in S^1$. It can be shown that $\operatorname{tr}(u) \in L^p(S^1, X)$, see [23]. Clearly, if u has a continuous extension $\hat{u}$ to $\overline{D}$ then $\operatorname{tr}(u)$ is simply the restriction of $\hat{u}$ to S^1.

As was shown in [21] and [27], every Sobolev map $u \in W^{1,p}(D, X)$ has an approximate metric derivative at almost every point $z \in D$ in the following sense. There exists a unique seminorm on $\mathbb{R}^2$, denoted $\operatorname{ap\,md} u_z$, such that

$$\operatorname{ap} \lim_{z' \to z} \frac{d(u(z'), u(z)) - \operatorname{ap\,md} u_z(z' - z)}{|z' - z|} = 0.$$

Here, $\operatorname{ap} \lim$ denotes the approximate limit, see [16]. If u is Lipschitz then the approximate limit can be replaced by an honest limit. It can be shown, see [27], that

$$E_+^p(u) = \int_D \mathcal{I}_+^p(\operatorname{ap\,md} u_z)\, dz,$$

where for a seminorm s on $\mathbb{R}^2$ we have set $\mathcal{I}_+^p(s) := \max\{s(v)^p \mid |v| = 1\}$.

The (parameterized Hausdorff) area of a map $u \in W^{1,2}(D, X)$ is defined by

$$\operatorname{Area}(u) := \int_D \mathbf{J}(\operatorname{ap\,md} u_z)\, dz,$$

where the Jacobian $\mathbf{J}(\|\cdot\|)$ of a norm $\|\cdot\|$ on $\mathbb{R}^2$ is set to be the Hausdorff 2–measure in $(\mathbb{R}^2, \|\cdot\|)$ of the Euclidean unit square and $\mathbf{J}(s) := 0$ for a degenerate seminorm s on $\mathbb{R}^2$. If $u \in W^{1,2}(D, X)$ satisfies Lusin's property (N) then

$$\operatorname{Area}(u) = \int_X \#\{z \mid u(z) = x\}\, d\mathcal{H}^2(x)$$

by the area formula [22]. In particular, if u is injective then $\operatorname{Area}(u) = \mathcal{H}^2(u(D))$. The area and energy are related by $\operatorname{Area}(u) \le E_+^2(u)$ for every $u \in W^{1,2}(D, X)$.

The following well-known properties of Sobolev maps with super-critical Sobolev exponent will be used later.

Proposition 2.2. *Let $u \in W^{1,p}(D, X)$ with $p > 2$. Then u has a unique representative $\bar{u} \colon \overline{D} \to X$ with the following properties:*

   (i) *$\bar{u}$ is (L, α)–Hölder continuous on $\overline{D}$ with $\alpha = 1 - \frac{2}{p}$ and $L \le M \left[E_+^p(u) \right]^{\frac{1}{p}}$ for some M depending only on p.*
   (ii) *$\bar{u}$ has Lusin's property (N).*

Proof. The existence of a continuous representative $\bar{u}$ which satisfies property (i) follows from Morrey's inequality, see e.g. [27, Proposition 3.3]. For the proof of statement (ii) we refer for example to Proposition 2.4 in [6]. $\square$

2.3. **The coarse isoperimetric function.** We recall the definition of coarse isoperimetric function of a metric space introduced by Gromov [17]. Our definition is a variant of that in [9, III.H.2.1]. In what follows, a triangulation of $\overline{D}$ is a homeomorphism from $\overline{D}$ to a combinatorial 2–complex τ in which every 2–cell is a 3–gon. We endow $\overline{D}$ with the induced cell structure from τ. For $i = 0, 1, 2$ the i–skeleton of τ will be denoted $\tau^{(i)}$ and will be viewed as a subset of $\overline{D}$. The 2–cells of τ will also be called triangles in τ.

Let X be a length space and $c\colon S^1 \to X$ a Lipschitz curve. Let $\varepsilon > 0$. An ε–filling of c is a pair (P, τ) consisting of a triangulation τ of $\overline{D}$ and a continuous map $P\colon \tau^{(1)} \to X$ such that $P|_{S^1} = c$ and such that $\ell(P|_{\partial F}) \le \varepsilon$ for every triangle F in τ. The ε–area of c is defined by

$$\mathrm{Ar}_\varepsilon(c) := \min\left\{|\tau| \ \middle|\ (P, \tau) \text{ an } \varepsilon\text{–filling of } c\right\},$$

where $|\tau|$ denotes the number of triangles in τ. If no ε–filling of c exists we set $\mathrm{Ar}_\varepsilon(c) := \infty$. The ε–coarse isoperimetric function of X is defined by

$$\mathrm{Ar}_{X,\varepsilon}(r) := \sup\{\mathrm{Ar}_\varepsilon(c) \mid c\colon S^1 \to X \text{ Lipschitz}, \ \ell(c) \le r\}$$

for all $r > 0$. If $\varepsilon_0 > 0$ is such that $\mathrm{Ar}_{X,\varepsilon_0}(r) < \infty$ for all $r > 0$ then for any $\varepsilon, \varepsilon' \ge \varepsilon_0$ the functions $\mathrm{Ar}_{X,\varepsilon}$ and $\mathrm{Ar}_{X,\varepsilon'}$ have the same asymptotic growth, that is, $\mathrm{Ar}_{X,\varepsilon} \simeq \mathrm{Ar}_{X,\varepsilon'}$. Here, for functions $f, g\colon [0, \infty] \to [0, \infty)$ one writes $f \preceq g$ if there exists $C > 0$ such that

$$f(r) \le Cg(Cr + C) + Cr + C$$

for all $r \ge 0$, and one writes $f \simeq g$ if $f \preceq g$ and $g \preceq f$.

If X is the Cayley graph of a finitely presented group Γ then for every sufficiently large $\varepsilon > 0$ we have $\mathrm{Ar}_{X,\varepsilon} \simeq \delta_\Gamma$, where δ_Γ is the Dehn function of Γ, see [9, III.H.2.5].

Let $\lambda \ge 1$ and $\varepsilon \ge 0$. A map $\varphi\colon X \to Y$ between metric spaces (X, d_X) and (Y, d_Y) is a (λ, ε)–quasi-isometry if

$$\lambda^{-1} d_X(x, x') - \varepsilon \le d_Y(\varphi(x), \varphi(x')) \le \lambda d_X(x, x') + \varepsilon$$

for all $x, x' \in X$ and for every $y \in Y$ there exists $x \in X$ with $d_Y(\varphi(x), y) \le \varepsilon$. If such a map φ exists then X and Y are called (λ, ε)–quasi-isometric. The coarse isoperimetric function is a quasi-isometry invariant, see e.g. [9, III.H.2.5]. More precisely, we have the following:

Proposition 2.3. *Let X and Y be quasi-isometric length spaces. If there exists $\varepsilon_0 > 0$ such that $\mathrm{Ar}_{X,\varepsilon_0}(r) < \infty$ and $\mathrm{Ar}_{Y,\varepsilon_0}(r) < \infty$ for all $r > 0$ then $\mathrm{Ar}_{X,\varepsilon} \simeq \mathrm{Ar}_{Y,\varepsilon}$ for all $\varepsilon \ge \varepsilon_0$.*

2.4. **Ultralimits and asymptotic cones of metric spaces.** Let ω be a non-principal ultrafilter on $\mathbb{N}$, that is, ω is a finitely additive measure on $\mathbb{N}$ such that every subset $A \subset \mathbb{N}$ is ω–measurable with $\omega(A) \in \{0, 1\}$ and such that $\omega(\mathbb{N}) = 1$ and $\omega(A) = 0$ whenever A is finite.

Let (Z, d) be a compact metric space. For every sequence $(z_n) \subset Z$ there exists a unique point $z \in Z$ such that

$$\omega(\{n \in \mathbb{N} \mid d(z_n, z) > \varepsilon\}) = 0$$

for every $\varepsilon > 0$. We will denote this point z by $\lim_\omega z_n$.

Let (X_n, d_n, p_n) be a sequence of pointed metric spaces. A sequence of points $x_n \in X_n$ is called bounded if

$$\sup_{n \in \mathbb{N}} d_n(x_n, p_n) < \infty.$$

One defines a pseudo-metric on the set $\tilde{X}$ of bounded sequences by

$$\tilde{d}_\omega((x_n), (x'_n)) := \lim_\omega d_n(x_n, x'_n).$$

The ω–ultralimit of the sequence (X_n, d_n, p_n) is defined to be the metric space obtained from $\tilde{X}$ by identifying points in $\tilde{X}$ of zero $\tilde{d}_\omega$–distance. We denote this space

by (X_ω, d_ω). An element of X_ω will be denoted by $[(x_n)]$, where (x_n) is an element of $\check{X}$. A basepoint in X_ω is given by $p_\omega := [(p_n)]$. Ultralimits of sequences of pointed metric spaces are always complete, see e.g. [9, I.5.53].

We have the following relationship between pointed Gromov-Hausdorff limits and ultralimits, see [9, Exercise I.5.52]. If (X_n, d_n) is proper for every n and converges in the pointed Gromov-Hausdorff sense to some metric space (X_∞, d_∞) then for every non-principal ultrafilter ω the ultralimit of (X_n, d_n, p_n) with respect to ω is isometric to (X_∞, d_∞).

Let Z be a metric space and $\alpha \in (0, 1]$. Let $\varphi_n : Z \to X_n$ be (ν_n, α)–Hölder maps, $n \in \mathbb{N}$, where $\nu_n \geq 0$ is uniformly bounded in n. If the sequence (φ_n) is bounded in the sense that

$$\sup_{n \in \mathbb{N}} d_n(\varphi_n(z), p_n) < \infty$$

for some and thus every $z \in Z$ then the assignment $z \mapsto [(\varphi_n(z))]$ defines an (ν, α)–Hölder map from Z to X_ω with $\nu = \lim_\omega \nu_n$. We denote this map by $(\varphi_n)_\omega$ or $\lim_\omega \varphi_n$.

It follows that if X_n is λ_n–quasi-convex for every n and λ_n is uniformly bounded then X_ω is λ–quasi-convex with $\lambda = \lim_\omega \lambda_n$. In particular, ultralimits of sequences of length spaces are geodesic.

Let (X, d) be a metric space, $(p_n) \subset X$ a sequence of basepoints and (r_n) a sequence of positive real numbers satisfying $\lim_{n \to \infty} r_n = 0$. Fix a non-principal ultrafilter ω on $\mathbb{N}$. The asymptotic cone of X with respect to (p_n), (r_n), and ω is the ultralimit of the sequence $(X, r_n d, p_n)$ with respect to ω. It will be denoted by $(X, r_n, p_n)_\omega$ or simply by X_ω if there is no danger of ambiguity.

2.5. **Lipschitz and Hölder maps to ultralimits.** Recall the definition of the α–Hölder extension property for a pair (Z, X) of metric spaces Z and X from the introduction. When $\alpha = 1$ the α–Hölder extension property is known as the Lipschitz extension property and has been well-studied, see e.g. [10] and references therein. If a metric space X is such that the pair $(\mathbb{R}, X)$ has the Lipschitz extension property with constant $L = \lambda$ then X is λ–quasi-convex. The converse is true if X is complete. In the absence of completeness one still obtains Lipschitz extensions of maps defined on closed subsets of $\mathbb{R}$.

Proposition 2.4. *Let Z be a separable metric space, $A \subset Z$ and $0 < \alpha \leq 1$. Let X_ω be an ultralimit of a pointed sequence of metric spaces X_n such that (Z, X_n) has the α–Hölder extension property with a fixed constant L for all n. If $\varphi \colon A \to X_\omega$ is (ν, α)–Hölder and $\varepsilon > 0$ then there exists a bounded sequence of $((1 + \varepsilon)L\nu, \alpha)$–Hölder maps $\varphi_n \colon Z \to X_n$ such that the restriction of $\lim_\omega \varphi_n$ to A coincides with φ.*

Proof. We denote the metric on Z by d and the metric on X_n by d_n. Let $A \subset Z$ be a non-empty subset, $\varphi \colon A \to X_\omega$ an (ν, α)–Hölder map for some $\nu > 0$, and let $\varepsilon > 0$. Let $\{z_k \mid k \in \mathbb{N}\} \subset A$ be a countable dense set. For each $k \in \mathbb{N}$ choose a bounded sequence of points $x_{k,n} \in X_n$ such that $\varphi(z_k) = [(x_{k,n})]$. Set

$$\delta_j := \min\{d(z_k, z_m) \mid 1 \leq k < m \leq j\}$$

for every $j \geq 2$. Define inductively a decreasing sequence $\mathbb{N} = N_1 \supset N_2 \supset \ldots$ of subsets N_j such that for each $j \geq 2$ we have $\omega(N_j) = 1$ and

$$|d_\omega(\varphi(z_k), \varphi(z_m)) - d_n(x_{k,n}, x_{m,n})| \leq \varepsilon \nu \delta_j^\alpha$$

for all $1 \le k, m \le j$ and all $n \in N_j$. For every $j \in \mathbb{N}$ define $M_j := N_j \setminus N_{j+1}$ and set $M_\infty := \cap_{i \in \mathbb{N}} N_i$. Note that the M_j are pairwise disjoint and satisfy

$$\mathbb{N} = M_\infty \cup \bigcup_{j \in \mathbb{N}} M_j.$$

Let $n \in \mathbb{N}$. If $n \in M_\infty$ then set $j(n) := n$. If $n \notin M_\infty$ then let $j(n)$ be the unique number for which $n \in M_{j(n)}$. Define $\varphi_n \colon \{z_1, \ldots, z_{j(n)}\} \to X_n$ by $\varphi_n(z_k) := x_{k,n}$ and note that

$$d_n(\varphi_n(z_k), \varphi_n(z_m)) \le d_\omega(\varphi(z_k), \varphi(z_m)) + \varepsilon v \delta_{j(n)}^\alpha \le (1 + \varepsilon) v d(z_k, z_m)^\alpha$$

for all $1 \le k < m \le j(n)$. This shows that the map φ_n is $((1+\varepsilon)v, \alpha)$–Hölder. By the α–Hölder extension property there thus exists an $((1 + \varepsilon)Lv, \alpha)$–Hölder extension of φ_n to the whole Z, again denoted by φ_n. The sequence (φ_n) is bounded since $\varphi_n(z_1) = x_{1,n}$ for all $n \in \mathbb{N}$ and hence the ultralimit map $\varphi_\omega := \lim_\omega \varphi_n$ exists and is $((1 + \varepsilon)Lv, \alpha)$–Hölder.

It remains to show that $\varphi_\omega = \varphi$ on A. Let $k \in \mathbb{N}$ and observe that $j(n) \ge k$ for every $n \in N_k$ and hence $\varphi_n(z_k) = x_{k,n}$ for such n. Since $\omega(N_k) = 1$ it follows that $\varphi_\omega(z_k) = \varphi(z_k)$. Since this is true for every $k \in \mathbb{N}$ we conclude that $\varphi_\omega = \varphi$ on all of A. $\qquad\qquad\square$

The following consequences of Proposition 2.4 will be useful in the sequel.

Corollary 2.5. *Let Z be a separable metric space and $0 < \alpha \le 1$. If X_ω is an ultralimit of a pointed sequence of metric spaces X_n such that (Z, X_n) has the α–Hölder extension property with a fixed constant L for all n then (Z, X_ω) has the α–Hölder extension property with constant L' for every $L' > L$.*

Since the α–Hölder extension property is preserved under rescaling the metric we obtain furthermore that if Z is separable and X such that the pair (Z, X) has the α–Hölder extension property then so does the pair (Z, X_ω) for every asymptotic cone X_ω of X.

In view of the remark preceding Proposition 2.4 and the fact that the proof of that proposition only needed extensions of maps defined on finite sets the proof also yields the following useful fact.

Corollary 2.6. *Let X_ω be an ultralimit of some sequence of metric spaces X_n each of which is λ–quasi-convex with the same $\lambda \ge 1$. Let $c \colon [0, 1] \to X_\omega$ be v–Lipschitz. Then for every $\varepsilon > 0$ there exists a bounded sequence of curves $c_n \colon [0, 1] \to X_n$ with $c = \lim_\omega c_n$ and such that c_n is $(1 + \varepsilon)v\lambda$–Lipschitz and satisfies*

$$\ell(c_n) \le (1 + \varepsilon)\lambda \cdot \ell(c)$$

for every $n \in \mathbb{N}$.

The interval $[0, 1]$ can be replaced by any interval $[a, b]$ or by S^1.

3. Lipschitz and Sobolev filling functions

Let X be a complete metric space and recall the definition of the Lipschitz Dehn function δ_X^{Lip} given in the introduction. In this section, we compare δ_X^{Lip} to the (Sobolev) Dehn function δ_X defined as follows. The (Sobolev) filling area of a Lipschitz curve $c \colon S^1 \to X$ is given by

$$\mathrm{Fill\,Area}(c) := \inf \left\{ \mathrm{Area}(v) \mid v \in W^{1,2}(D, X), \mathrm{tr}(v) = c \right\}.$$

The (Sobolev) Dehn function of X is defined by

$$\delta_X(r) = \sup \left\{ \text{Fill Area}(c) \mid c \colon S^1 \to X \text{ Lipschitz}, \ell(c) \leq r \right\}$$

for $r > 0$. We will usually omit the word "Sobolev" in the sequel. The following proposition shows that for many spaces δ_X^{Lip} and δ_X are equal.

Proposition 3.1. *If X is a complete length space that is Lipschitz 1–connected up to some scale, then $\delta_X(r) = \delta_X^{\text{Lip}}(r)$ for all $r > 0$.*

Proof. It suffices to show that Fill Area(c) = Fill Area$^{\text{Lip}}(c)$ for every Lipschitz curve $c \colon S^1 \to X$. Since every Lipschitz map defined on D is Sobolev we have Fill Area$(c) \leq$ Fill Area$^{\text{Lip}}(c)$. We will prove the reverse inequality by approximating a given Sobolev disc by a Lipschitz disc with almost the same area. For this, let $c \colon S^1 \to X$ be a Lipschitz curve and let $u \in W^{1,2}(D, X)$ be such that $\text{tr}(u) = c$. Let $B := B(0, 2)$ be the open disc of radius 2 and let $\hat{u} \colon \overline{B} \to X$ be the extension of u to the closed disc $\overline{B}$ such that:

$$\hat{u}(z) = \begin{cases} u(z) & z \in D \\ c\left(\frac{z}{\|z\|}\right) & z \notin D. \end{cases}$$

Then $\hat{u} \in W^{1,2}(B, X)$ and Area$(\hat{u})$ = Area(u). We next use the well-known fact (see for example [18, Theorem 8.2.1]) that Sobolev maps are Lipschitz on big subsets. For the convenience of the reader we provide a proof of this. Since $\hat{u} \in W^{1,2}(B, X)$ there exist $g \in L^2(B)$ and a negligible set $N \subset B$ such that

$$d(\hat{u}(z), \hat{u}(z')) \leq |z - z'| \cdot (g(z) + g(z'))$$

for all $z, z' \in B \setminus N$, see for example [27, Proposition 3.2]. Clearly, $\hat{u}$ is $\frac{t}{2}$–Lipschitz on the set

$$A_t := \left\{ z \in B \setminus N \mid g(z) \leq \frac{t}{4} \right\}$$

whenever $t > 0$. Chebyshev's inequality and the absolute continuity of integrals imply that if $\varepsilon \in (0, 1)$ then the set $B \setminus A_t$ has Lebesgue measure at most $\frac{\varepsilon^2}{t^2}$ for all t sufficiently large. In particular, A_t is $\frac{\varepsilon}{t}$–dense in B. Since $\hat{u}$ is Lipschitz on the annulus $\overline{B} \setminus B(0, 1)$ it follows that $\hat{u}$ is t–Lipschitz on the set $A := A_t \cup (\overline{B} \setminus B(0, 3/2))$ whenever $t > 0$ is sufficiently large.

Now, X is L–Lipschitz 1–connected up to some scale $\lambda_0 > 0$ for some $L \geq 1$ and $\lambda_0 > 0$. Therefore the map $\hat{u}|_A$ extends to an (MLt)–Lipschitz map v defined on

$$U = \left\{ z \in \overline{B} \mid \text{dist}(z, A) < \frac{M\lambda_0}{t} \right\},$$

where M is a universal constant, see for example the appendix below. Since A is $\frac{\varepsilon}{t}$–dense in $\overline{B}$ we see that $U = \overline{B}$ whenever $\varepsilon > 0$ is chosen small enough (depending only on M and λ_0). For such ε the map $v \colon \overline{B} \to X$ is thus a Lipschitz filling of c and satisfies

$$\text{Area}(v) \leq \text{Area}(\hat{u}|_A) + \text{Area}(v|_{B \setminus A}) \leq \text{Area}(u) + \frac{\varepsilon^2}{t^2} \cdot (MLt)^2.$$

It follows that for any $\varepsilon > 0$,

$$\text{Fill Area}^{\text{Lip}}(c) \leq \text{Fill Area}(c) + \varepsilon^2 (ML)^2,$$

so Fill Area$^{\text{Lip}}(c) \leq$ Fill Area(c) as desired. $\qquad\square$

We now compare δ_X and δ_X^{Lip} to the coarse isoperimetric function $\mathrm{Ar}_{X,\varepsilon}$. A sort of converse to the following statement will follow from Corollary 4.3.

Proposition 3.2. *Let X be a complete length space and $\varepsilon > 0$. If $L := \delta_X(\varepsilon) < \infty$ then*

$$\delta_X(r) \leq L \cdot \mathrm{Ar}_{X,\varepsilon}(r)$$

for every $r > \varepsilon$.

The same statement holds with δ_X replaced by δ_X^{Lip}. In the statement of the proposition we use the convention that $0 \cdot \infty = \infty$.

Proof. Let $r > \varepsilon$ and let $c \colon S^1 \to X$ be a Lipschitz curve with $\ell(c) \leq r$. We may assume that $\mathrm{Ar}_{X,\varepsilon}(r) < \infty$. By definition, there exists an ε–filling (P, τ) of c with $|\tau| \leq \mathrm{Ar}_{X,\varepsilon}(r)$. Endow τ with the length metric such that each edge in $\tau^{(1)}$ has length 1 and each triangle in $\tau^{(2)}$ is isometric to a Euclidean triangle. Then the space τ with this metric is biLipschitz homeomorphic to $\overline{D}$. Thus, we may assume that each triangle in τ, viewed as a subset of $\overline{D}$, is biLipschitz homeomorphic to a Euclidean triangle. Let $\nu > 0$ and let F be a triangle in τ. Since $\ell(P|_{\partial F}) \leq \varepsilon$ there exists a Sobolev map $u_F \in W^{1,2}(F, X)$ with $\mathrm{tr}(u_F) = P|_{\partial F}$ and such that $\mathrm{Area}(u_F) \leq \delta_X(\varepsilon) + \nu$. The gluing of all the maps u_F as F ranges over all triangles in τ yields a Sobolev map $u \in W^{1,2}(D, X)$ with $\mathrm{tr}(u) = P|_{S^1} = c$ and such that

$$\mathrm{Area}(u) \leq |\tau| \cdot (\delta_X(\varepsilon) + \nu).$$

See [23, Theorem 1.12.3] for the fact that the gluing of Sobolev maps is again Sobolev. Since $\nu > 0$ was arbitrary this completes the proof. $\square$

Every length space is quasi-isometric to a length space that is Lipschitz 1–connected up to some scale. In fact, a stronger property holds. Let X and Y be metric spaces and $\varepsilon > 0$. We say that Y is an ε–*thickening* of X if there exists an isometric embedding $\iota \colon X \to Y$ such that the Hausdorff distance between $\iota(X)$ and Y is at most ε. The embedding ι is then an $(1, \varepsilon)$–quasi-isometry.

Lemma 3.3. *Let X be a length space. There is a universal constant M such that for every $\varepsilon > 0$ there exists a complete length space X_ε which is an ε–thickening of X and has the following property. Let $\lambda > 0$ and let $c_0 \colon S^1 \to X_\varepsilon$ be λ–Lipschitz. If $\lambda \leq \frac{\varepsilon}{M}$, then c_0 is $M\lambda$–Lipschitz homotopic to a constant curve. If $\lambda \geq \frac{\varepsilon}{M}$ and $\varepsilon' > 0$, then c_0 is Lipschitz homotopic to a curve $c_1 \colon S^1 \to X$ with $\ell(c_1) \leq \ell(c) + \varepsilon'$ via a homotopy of area at most $M\varepsilon\lambda$. Furthermore, if X is locally compact then X_ε is locally compact.*

In particular, X_ε is Lipschitz 1–connected up to some scale.

Proof. The construction of the space X_ε is as in the proof of [36, Proposition 3.2]. The properties stated in the lemma above all follow from the proof of [36, Proposition 3.2], except the very last statement. Since the construction in the proof uses balls in an infinite dimensional Banach space the resulting space X_ε is never locally compact. In order to remedy this one replaces the balls X_z in $L^\infty(B_z)$ of the spaces B_z appearing in the proof by the injective hulls X_z of B_z, see below. If X is locally compact, then the B_z and the X_z are also locally compact. It follows that the resulting thickening X_ε is locally compact. $\square$

Injective hulls were constructed by Isbell in [20]. A metric space E is *injective* if for every subspace Y of a metric space Z, any 1-Lipschitz map $Y \to E$ extends to a 1–Lipschitz map $Z \to E$. An *injective hull E* of a metric space X is an injective metric space equipped with an isometric embedding $e \colon X \to E$ that is minimal in an appropriate sense among injective spaces containing X. We summarize, without proof, the properties of injective hulls that we used above and will use in Section 5. See [20] for the proofs of these properties.

Theorem 3.4. *Every metric space X has a unique injective hull $E(X)$ up to isometry. This hull is contractible via a Lipschitz map. If X is compact, then $E(X)$ is compact, and if X is a Banach space, then $E(X)$ is a Banach space.*

The Dehn function of the thickening X_ε constructed above is not much bigger than that of X.

Proposition 3.5. *Let X be a complete length space. Then for every $\varepsilon > 0$ there exists an ε–thickening Y of X that is a complete length space, Lipschitz 1–connected up to some scale, and satisfies*

$$\delta_Y(r) \le \varepsilon r + \delta_X(r + \varepsilon)$$

for all $r > 0$. If X is locally compact, then Y can be chosen to be locally compact as well.

The proof of the proposition will moreover show that if X admits a quadratic isoperimetric inequality with constant C then Y admits a quadratic isoperimetric inequality with a constant depending only on C.

The following construction will be needed here and also later. We set $I := [0, 1]$.

Lemma 3.6. *There exists $M > 0$ such that if $c, c' \colon I \to X$ are two Lipschitz parameterizations of the same curve, then there is a Lipschitz homotopy $h \colon I \times [0, 1] \to X$ such that $h(s, 0) = c(s)$, $h(s, 1) = c'(s)$, and $\mathrm{Lip}(h) \le M \cdot \max\{\mathrm{Lip}(c), \mathrm{Lip}(c')\}$. Moreover, $\mathrm{Area}(h) = 0$.*

The interval I can be replaced by S^1.

Proof. It suffices to consider the case that c' is parameterized proportional to arclength. Then $c'(s) = c(\varrho(s))$ for some Lipschitz function ϱ. Then the map

$$h(s, t) = c((1 - t)s + t\varrho(s))$$

is a Lipschitz homotopy from c to c'. The bound on the Lipschitz constant of h follows by calculation. Since the image of h is one-dimensional it follows that $\mathrm{Area}(h) = 0$. $\qquad\square$

Proof of Proposition 3.5. Let M be as in Lemma 3.3. Set $\bar\varepsilon = \frac{\varepsilon}{2M}$ and let $Y = X_{\bar\varepsilon}$ be as in Lemma 3.3. Let $c \colon S^1 \to Y$ be a Lipschitz curve. Let $\bar c$ be the constant speed parameterization of c so that $\mathrm{Fill\,Area}(c) = \mathrm{Fill\,Area}(\bar c)$ by Lemma 3.6. Note that $\bar c$ is λ–Lipschitz with $\lambda = \frac{\ell(c)}{4}$. If $\lambda \le \frac{\bar\varepsilon}{M}$, then $\bar c$ is $M\lambda$–Lipschitz homotopic to a constant curve. It follows that

$$\mathrm{Fill\,Area}(\bar c) \le 2\pi M^2 \lambda^2 \le 2\pi M\bar\varepsilon \cdot \frac{\ell(c)}{4} \le \varepsilon\ell(c).$$

If $\lambda \ge \frac{\bar\varepsilon}{M}$, then $\bar c$ is Lipschitz homotopic to a Lipschitz curve $c_1 \colon S^1 \to X$ via a homotopy of area at most $M\bar\varepsilon\lambda$, and $\ell(c_1) \le \ell(c) + \varepsilon$. We can thus fill c by a disc of area

$$\mathrm{Fill\,Area}(c) \le M\bar\varepsilon\lambda + \mathrm{Fill\,Area}(c_1) \le \varepsilon \cdot \ell(c) + \delta_X(\ell(c) + \varepsilon),$$

as desired. □

Similar thickenings allow us to work with spaces arising from geometric group theory such as Cayley graphs.

Proposition 3.7. *Let X be a length space such that* $\mathrm{Ar}_{X,\varepsilon}(r) < \infty$ *for some* $\varepsilon > 0$ *and all* $r > 0$*. Then there exists a thickening Y of X that is a complete length space, Lipschitz 1–connected up to some scale, and satisfies* $\delta_Y^{\mathrm{Lip}} \leq \mathrm{Ar}_{X,\varepsilon}$*. If X is locally compact then Y can be chosen to be locally compact as well.*

In particular, if X is such that $\mathrm{Ar}_{X,\varepsilon}(r) \leq r^2$ then there exists C such that $\delta_Y^{\mathrm{Lip}}(r) \leq Cr^2$ for all $r > 0$. Similarly, if X satisfies a quadratic isoperimetric inequality (with Lipschitz or Sobolev maps) only for curves of length $\geq r_0$ then there exists a thickening Y of X which is a complete length space, is Lipschitz 1–connected up to some scale, and satisfies $\delta_Y^{\mathrm{Lip}}(r) \leq Cr^2$ for some C and all $r > 0$.

Proof. This is a straight-forward consequence of Proposition 3.2 and Lemma 3.3. □

4. The coarse isoperimetric function

We give bounds on the coarse isoperimetric function using the constructions in [29] and [27]. We then prove the equivalence of the coarse isoperimetric function and the Dehn function and establish the quasi-isometry invariance of the Dehn function. This will imply Theorems 1.1 and 1.2.

Let Z be a metric space homeomorphic to $\overline{D}$. We denote by ∂Z the boundary circle of Z, that is, the image of S^1 under the homeomorphism from $\overline{D}$ to Z. Recall the definition of a triangulation of $\overline{D}$ given in Section 2.3. Triangulations of Z are defined in analogy.

The first result is a variant of [29, Theorem 8.11].

Theorem 4.1. *Let Z be a geodesic metric space homeomorphic to* $\overline{D}$ *such that* $\ell(\partial Z) < \infty$ *and* $\mathcal{H}^2(Z) < \infty$*. Suppose there exist* $C, l_0 > 0$ *such that every Jordan domain* $\Omega \subset Z$ *with* $\ell(\partial\Omega) < l_0$ *satisfies*

(2) $$\mathcal{H}^2(\Omega) \leq C \cdot \ell(\partial\Omega)^2.$$

Then for every $n \in \mathbb{N}$ *with* $n > \frac{8\ell(\partial Z)}{l_0}$ *there exists a triangulation of Z into at most*

$$K \cdot n + K \cdot n^2 \cdot \frac{\mathcal{H}^2(Z)}{\ell(\partial Z)^2}$$

triangles of diameter at most $\frac{\ell(\partial Z)}{n}$ *each, where K depends only on C. Moreover, every edge contained in* ∂Z *has length at most* $\frac{\ell(\partial Z)}{n}$*.*

Proof. We may assume that $C \geq 1$. By the proof of [29, Theorem 8.11] there exists a finite, connected graph $\partial Z \subset \hat{G} \subset Z$ such that $Z \setminus \hat{G}$ has at most

$$K \cdot n + K \cdot n^2 \cdot \frac{\mathcal{H}^2(Z)}{\ell(\partial Z)^2}$$

components, each of which is a topological disc of diameter at most $\frac{\ell(\partial Z)}{n}$. Here, K only depends on the isoperimetric constant C. Note that [29, Theorem 8.11] is formulated for a special metric space Z but the proof only relies on the inequality (2), see [29, Remark 8.4]. We may assume that every vertex of $\hat{G}$ has degree at

least 3. We will call the components of $Z \setminus \hat{G}$ the faces of the graph $\hat{G}$. By the above, every face F of $\hat{G}$ has diameter at most $\frac{\ell(\partial Z)}{n}$. Denote by v, e, and f the number of vertices, edges, and faces of $\hat{G}$, respectively. Notice that $e \geq \frac{3v}{2}$ and hence, by Euler's formula, $e = v + f - 1 < \frac{2e}{3} + f$. This implies $e < 3f$.

We now construct a subdivision G from $\hat{G}$ as follows. Firstly, add n new vertices on ∂Z in such a way as to divide ∂Z into n segments of equal length $\frac{\ell(\partial Z)}{n}$. Moreover, for each face of $\hat{G}$ whose boundary consists of a single edge, add an extra vertex in the interior of the edge. Denote this new graph by $\hat{G}$ again. The number e' of edges in this new graph is bounded by $e' \leq e + n + f$. Now, fix a point in the interior of each face F of $\hat{G}$ and connect it by an injective curve in F to each vertex of F such that the curves do not intersect except possibly at endpoints. Thus, if F has m vertices then F is replaced by m new faces. This yields a new planar graph $\partial Z \subset G \subset Z$ all of whose faces have boundary consisting of three edges and all of whose faces have diameter at most $\frac{\ell(\partial Z)}{n}$. The number L of faces of G is bounded by the sum of degrees of vertices of $\hat{G}$, which equals $2e'$. Consequently,

$$L \leq 2e' \leq 2e + 2n + 2f < 8f + 2n \leq K' \cdot n + K' \cdot n^2 \cdot \frac{\mathcal{H}^2(Z)}{\ell(\partial Z)^2}$$

for some constant K' only depending on C. The triangulation of Z associated with G has the desired properties. This completes the proof. $\square$

Theorem 4.2. *Let X be a locally compact, geodesic metric space such that, for some $C, r_0 > 0$, we have $\delta_X(r) \leq Cr^2$ for all $0 < r < r_0$. Let $c\colon S^1 \to X$ be a Lipschitz curve and $u \in W^{1,2}(D, X)$ such that $\mathrm{tr}(u) = c$. Then for every $n \in \mathbb{N}$ with $n > \frac{8\ell(c)}{r_0}$ there exists a triangulation τ of $\overline{D}$ with at most*

$$K \cdot n + K \cdot n^2 \cdot \frac{\mathrm{Area}(u)}{\ell(c)^2}$$

triangles and a continuous map $P\colon \tau^{(1)} \to X$ such that $P|_{S^1} = c$ and $\ell(P|_{\partial F}) \leq \frac{4\ell(c)}{n}$ for every triangle F in τ. Here, K only depends on C.

Note that if X is Lipschitz 1–connected up to some scale, then there exist $C, r_0 > 0$ such that $\delta_X(r) \leq Cr^2$ for all $0 < r < r_0$.

Proof. Consider the space $Y := X \times \mathbb{R}^2$, which satisfies $\delta_Y(r) \leq C'r^2$ for all $0 < r < r_0$, where C' only depends on C, see [26, Lemma 3.2]. Let $\varepsilon > 0$ be sufficiently small. Then the biLipschitz curve in Y given by $\hat{c}(z) := (c(z), \varepsilon z)$ satisfies

$$\ell(c) < \ell(\hat{c}) < \min \left\{ \frac{nr_0}{8}, \frac{4\ell(c)}{3} \right\}$$

and there exists $\hat{u} \in W^{1,2}(D, Y)$ with trace $\hat{c}$ and such that

$$\mathrm{Area}(\hat{u}) \leq \mathrm{Area}(u) + \frac{\ell(c)^2}{n},$$

compare with the proof of [26, Theorem 3.4]. It follows from the results in [29, Section 1] that there exist a geodesic metric space Z homeomorphic to $\overline{D}$, and a 1–Lipschitz map $\overline{u}\colon Z \to Y$ with the following properties. Firstly, the restriction of $\overline{u}$ to the boundary circle ∂Z of Z is an arc-length preserving homeomorphism onto

the image Γ of $\hat{c}$, in particular, $\ell(\partial Z) = \ell(\hat{c})$. Secondly, every Jordan domain $\Omega \subset Z$ with $\ell(\partial\Omega) < r_0$ satisfies

$$\mathcal{H}^2(\Omega) \leq C' \cdot \ell(\partial\Omega)^2.$$

Thirdly, the Hausdorff measure of Z is bounded by

$$\mathcal{H}^2(Z) \leq \text{Area}(\hat{u}) \leq \text{Area}(u) + \frac{\ell(c)^2}{n}.$$

Let τ be a triangulation of Z as in Theorem 4.1. Then the number N of triangles in τ is bounded by

$$N \leq K \cdot n + K \cdot n^2 \cdot \frac{\mathcal{H}^2(Z)}{\ell(\partial Z)^2} \leq 2K \cdot n + K \cdot n^2 \cdot \frac{\text{Area}(u)}{\ell(c)^2}$$

for some constant K only depending on C. Moreover, the diameter of each triangle in τ is bounded from above by $\frac{4\ell(c)}{3n}$. Finally, the length of each edge contained in ∂Z is also at most $\frac{4\ell(c)}{3n}$.

Let now $\varphi \colon S^1 \to \partial Z$ be the homeomorphism given by $\varphi := (\overline{u}|_{\partial Z})^{-1} \circ \hat{c}$. By the Schoenflies theorem, φ extends to a homeomorphism from $\overline{D}$ to Z, which we denote again by φ. Let $\hat{\tau}$ be the triangulation of $\overline{D}$ given by $\varphi^{-1}(\tau)$. We define a map $P \colon \hat{\tau}^{(1)} \to X$ as follows. Denote by $\pi \colon Y \to X$ the natural 1–Lipschitz projection. For $z \in S^1$ or $z \in \hat{\tau}^{(0)}$ let $P(z) := \pi \circ \overline{u} \circ \varphi(z)$. On each edge $[v, v']$ in $\hat{\tau}^{(1)}$ not contained in S^1 define P to be a geodesic from $P(v)$ to $P(v')$. It follows from the construction that for each triangle F in $\hat{\tau}$ we have

$$\ell(P|_{\partial F}) \leq 3 \cdot \frac{4\ell(c)}{3n} = \frac{4\ell(c)}{n}$$

and that P agrees with c on S^1. $\qquad\square$

As a consequence of Theorem 4.2 we obtain a bound on the coarse isoperimetric function as follows.

Corollary 4.3. *Let X be a locally compact, geodesic metric space such that, for some $C, r_0 > 0$, we have $\delta_X(r) \leq Cr^2$ for all $0 < r < r_0$. Then there exists K depending only on C and r_0 such that*

$$\text{Ar}_{X,\varepsilon}(r) \leq 1 + K \cdot r + K \cdot \delta_X(r)$$

for every $r > 0$ and every $\varepsilon \geq \frac{r_0}{2}$.

Proof. Let $\varepsilon \geq \frac{r_0}{2}$ and let $c \colon S^1 \to X$ be a Lipschitz curve. Set $L := \frac{8\ell(c)}{r_0}$, and let $n \in \mathbb{N}$ be the smallest integer with $n > L$. If $L < 1$ then $\ell(c) < \frac{r_0}{8} < \varepsilon$ and hence $\text{Ar}_{X,\varepsilon}(\ell(c)) = 1$. We may therefore assume that $L \geq 1$ so that $n \leq \frac{16\ell(c)}{r_0}$. Since

$$\frac{4\ell(c)}{n} < \frac{4\ell(c)}{L} = \frac{r_0}{2} \leq \varepsilon$$

it follows from Theorem 4.2 that for every $\nu > 0$

$$\text{Ar}_{X,\varepsilon}(\ell(c)) \leq K \cdot n + K \cdot n^2 \cdot \frac{\delta_X(\ell(c)) + \nu}{\ell(c)^2}$$

$$\leq \frac{K'}{r_0} \cdot \ell(c) + \frac{K'}{r_0^2} \cdot (\delta_X(\ell(c)) + \nu)$$

for some constants K and K' depending only on C. Since $\nu > 0$ was arbitrary this completes the proof. $\qquad\square$

We now obtain the following analog of Theorem 1.1.

Theorem 4.4. *Let X be a locally compact, geodesic metric space satisfying $\delta_X(r) < \infty$ for all $r > 0$. If there exist $C, r_0 > 0$ such that $\delta_X(r) \leq Cr^2$ for all $r \in (0, r_0)$ then $\mathrm{Ar}_{X,\varepsilon} \simeq \delta_X$ for every $\varepsilon > 0$.*

This follows from Corollary 4.3 and Proposition 3.2. Theorem 1.1 is a consequence of the theorem above together with Proposition 3.1. Finally, the following result together with Proposition 3.1 implies Theorem 1.2.

Theorem 4.5. *Let X and Y be locally compact, geodesic metric spaces such that $\delta_X(r) < \infty$ and $\delta_Y(r) < \infty$ for all $r > 0$ and such that there exist $C, r_0 > 0$ with*

$$\delta_X(r) \leq Cr^2 \quad and \quad \delta_Y(r) \leq Cr^2$$

for all $0 < r < r_0$. If X and Y are quasi-isometric then $\delta_X \simeq \delta_Y$.

Proof. This is a consequence of Theorem 4.4 together with Proposition 2.3. Finiteness of the functions $\mathrm{Ar}_{X,\varepsilon}$ and $\mathrm{Ar}_{Y,\varepsilon}$ follows from Corollary 4.3. □

5. Stability of the quadratic isoperimetric inequality

The main result of this section is the following theorem, which readily implies Theorem 1.8 in the introduction.

Theorem 5.1. *Let X_ω be an ultralimit of a sequence of proper, λ–quasi-convex metric spaces X_n such that, for some $C, r_0 > 0$, we have $\delta_{X_n}(r) \leq Cr^2$ for all $0 < r < r_0$ and all $n \in \mathbb{N}$. Then for every Lipschitz curve $c: S^1 \to X_\omega$ with $\ell(c) < \lambda^{-1} \cdot r_0$ and every $\varepsilon > 0$ there exists $u \in W^{1,p}(D, X_\omega)$ with $\mathrm{tr}(u) = c$ and satisfying*

$$(3) \qquad \mathrm{Area}(u) \leq (C + \varepsilon)\lambda^2 \cdot \ell(c)^2 \quad and \quad \left[E_+^p(u)\right]^{\frac{1}{p}} \leq C'\lambda \cdot \mathrm{Lip}(c),$$

where $p > 2$ and C' depend only on C and ε. In particular, X_ω satisfies $\delta_{X_\omega}(r) \leq \lambda^2 Cr^2$ for all $0 < r < \frac{r_0}{\lambda}$.

The main ingredient in the proof of the theorem is the following result obtained in [26, Theorem 3.4].

Theorem 5.2. *Let X be a proper metric space such that, for some $C, r_0 > 0$, we have $\delta_X(r) \leq Cr^2$ for all $0 < r < r_0$. Let $\varepsilon > 0$. Then every Lipschitz curve $c: S^1 \to X$ with $\ell(c) < r_0$ is the trace of some $u \in W^{1,p}(D, X)$ with*

$$\mathrm{Area}(u) \leq \mathrm{Fill\ Area}(c) + \varepsilon \cdot \ell(c)^2 \leq (C + \varepsilon) \cdot \ell(c)^2$$

and $\left[E_+^p(u)\right]^{\frac{1}{p}} \leq C' \cdot \mathrm{Lip}(c)$, where $p > 2$ and C' only depend on C, ε.

We can now give the proof of Theorem 5.1.

Proof. Suppose X_ω is the ultralimit (with respect to some non-principal ultrafilter ω) of the sequence of pointed metric spaces (X_n, d_n, p_n).

Let $c: S^1 \to X_\omega$ be a ν–Lipschitz curve of length $\ell(c) < \lambda^{-1} \cdot r_0$. Fix $\varepsilon > 0$ and let $\delta \in (0, 1)$ be such that $(1 + \delta)\lambda\ell(c) < r_0$ and $(1 + \delta)^2(C + \varepsilon/2) \leq C + \varepsilon$. By Corollary 2.6 there exists a bounded sequence of $(1 + \delta)\lambda\nu$–Lipschitz curves $c_n: S^1 \to X_n$ such that $\lim_\omega c_n = c$ and such that

$$\ell(c_n) \leq (1 + \delta)\lambda \cdot \ell(c) < r_0$$

for every n. By Theorem 5.2 there exist $p > 2$ and C' depending only on C and ε such that the following holds. For each $n \in \mathbb{N}$ there exists $u_n \in W^{1,p}(D, X_n)$ with $\mathrm{tr}(u_n) = c_n$ and such that

$$(4) \qquad \mathrm{Area}(u_n) \le \left(C + \frac{\varepsilon}{2} \right) \cdot \ell(c_n)^2 \le (C + \varepsilon)\lambda^2 \cdot \ell(c)^2$$

and

$$\left[E_+^p(u_n) \right]^{\frac{1}{p}} \le 2^{-1} C' \, \mathrm{Lip}(c_n) \le C' \lambda \nu.$$

In particular, there exists a representative of u_n, denoted by the same symbol, which is (L, α)–Hölder on $\overline{D}$, where $\alpha = 1 - \frac{2}{p}$ and where L depends on C', λ, ν and p, see Proposition 2.2.

Now note that (u_n) is a bounded sequence since each u_n is (L, α)–Hölder. Let $u \colon \overline{D} \to X_\omega$ be the ultralimit of (u_n). Then u is (L, α)–Hölder on $\overline{D}$ and satisfies $u|_{S^1} = c$. In order to complete the proof it thus suffices to show that $u \in W^{1,p}(D, X_\omega)$ and that the inequalities in (3) hold. Observe first that the sequence of subsets $Y_n := u_n(\overline{D})$, when endowed with the metric d_n, is a uniformly compact sequence of metric spaces in the sense of Gromov. Indeed, since u_n is (L, α)–Hölder it follows that the diameter of each Y_n is bounded by $2^\alpha L$ and given $\varepsilon > 0$ the image $u_n(F)$ of a finite $(\varepsilon L^{-1})^{\frac{1}{\alpha}}$–dense subset $F \subset \overline{D}$ is ε–dense in Y_n. By Gromov's compactness theorem for metric spaces, there exist a compact metric space (Y, d_Y) and isometric embeddings $\varphi_n \colon Y_n \hookrightarrow Y$ for every $n \in \mathbb{N}$. Let $v \colon \overline{D} \to Y$ be the ultralimit of the sequence of maps $v_n := \varphi_n \circ u_n$. We claim that a subsequence (v_{n_j}) of (v_n) converges uniformly to v and thus also in $L^p(D, Y)$. Indeed, let $B \subset \overline{D}$ be a countable dense set. Then there exists a subsequence (n_j) such that $v_{n_j}(z) \to v(z)$ as $j \to \infty$ for every $z \in B$. Since v_{n_j} is (L, α)–Hölder for every j it follows that v_{n_j} converges uniformly to v, which proves the claim. Thus, [23, Theorem 1.6.1] implies that $v \in W^{1,p}(D, Y)$. Moreover, v satisfies the inequalities in (3) by the weak lower semi-continuity of the E_+^p–energy and of area, see [27, Corollaries 5.7 and 5.8]. Finally, set $A := v(\overline{D})$. Since

$$d_Y(v(z), v(z')) = d_\omega(u(z), u(z'))$$

for all $z, z' \in \overline{D}$ there exists an isometric embedding $\iota \colon A \hookrightarrow X_\omega$ such that $u = \iota \circ v$. Since ι is isometric it follows that $u \in W^{1,p}(D, X_\omega)$ and that (3) holds. This completes the proof. $\qquad \square$

Proposition 3.7 and Theorem 5.1 imply that every asymptotic cone of a metric space with a quadratic isoperimetric inequality for sufficiently long curves admits a quadratic isoperimetric inequality with some constant which may be large. In the following corollary we show how to obtain a better bound for the isoperimetric constant of the asymptotic cones. This will be used in the proof of Theorem 5.4 below.

Corollary 5.3. *Let X be a locally compact, geodesic metric space. Suppose there exist $\beta, r_0 > 0$ such that every Lipschitz curve $c \colon S^1 \to X$ with $\ell(c) \ge r_0$ is the trace of a map $u \in W^{1,2}(D, X)$ with $\mathrm{Area}(u) \le \beta \cdot \ell(c)^2$. Then every asymptotic cone of X admits a quadratic isoperimetric inequality with constant β.*

Proof. By Proposition 3.7 and the remark following it, the space X isometrically embeds into a locally compact, geodesic metric space Y which is at finite Hausdorff distance from X and which admits a quadratic isoperimetric inequality with some

constant C. Note first that if $\hat{c}\colon S^1 \to X$ is a Lipschitz curve of length $\ell(\hat{c}) \geq r_0$ then, by assumption,

$$\text{Fill Area}_Y(\hat{c}) \leq \text{Fill Area}_X(\hat{c}) \leq \beta \cdot \ell(\hat{c})^2.$$

Here, $\text{Fill Area}_Y(\hat{c})$ and $\text{Fill Area}_X(\hat{c})$ denote the filling areas in Y and X, respectively. This fact together with Theorem 5.2 lets us improve inequality (4) in the proof of Theorem 5.1 for n large enough. Namely, we can find fillings u_n in Y which satisfy the properties in the proof of Theorem 5.1 and moreover

$$\text{Area}(u_n) \leq \text{Fill Area}_Y(c_n) + \frac{\varepsilon}{2} \cdot \ell(c_n)^2 \leq \left(\beta + \frac{\varepsilon}{2}\right) \cdot \ell(c_n)^2$$

for all n sufficiently large.   □

In the rest of this section we give some consequences of the isoperimetric stability proved above. We first provide the proof of Theorem 1.7. We in fact prove the following stronger version.

Theorem 5.4. *Let X be a locally compact, geodesic metric space and $\varepsilon, r_0 > 0$. If every Lipschitz curve $c\colon S^1 \to X$ with $\ell(c) \geq r_0$ is the trace of some $v \in W^{1,2}(D, X)$ with*

$$\text{Area}(v) \leq \frac{1 - \varepsilon}{4\pi} \cdot \ell(c)^2$$

then X is Gromov hyperbolic.

Proof. By [13, Proposition 3.1.1] it is enough to prove that every asymptotic cone of X is a metric tree. Let therefore $Y = (Y, d_Y)$ be an asymptotic cone of X. Since X is geodesic it follows that Y is geodesic. Note also that Y admits a quadratic isoperimetric inequality with some constant $C < \frac{1}{4\pi}$ by Corollary 5.3.

We first claim that every $u \in W^{1,2}(D, Y)$ has $\text{Area}(u) = 0$. We argue by contradiction and assume there exists u which has strictly positive area. It follows from [22] or [27, Proposition 4.3(ii)] that for some $p \in Y$ and some sequence $r_n \to \infty$ the ultralimit $Y_\omega := (Y, r_n d_Y, p)_\omega$ contains an isometric copy of a two-dimensional normed plane V. Let Z be the injective hull of V. Then Z is a Banach space and there exists a 1–Lipschitz map from Y_ω to Z which restricts to the identity on V. It follows that curves in V can be filled at least as efficiently in Z as in Y_ω. Since the isoperimetric constant of every normed plane is at least $\frac{1}{4\pi}$, see [36], and since affine discs in normed spaces are area minimizers by [11] it follows that the isoperimetric constant of Z and thus also of Y_ω must be at least $\frac{1}{4\pi}$. However, Y_ω is an asymptotic cone of X by [14] and thus admits a quadratic isoperimetric inequality with some constant $C < \frac{1}{4\pi}$ by Corollary 5.3, a contradiction. This proves the claim.

We finally show that Y is a metric tree. For this it suffices to show that Y does not contain any rectifiable Jordan curve. We argue by contradiction and assume that there exists a rectifiable Jordan curve Γ in Y. Let $c\colon S^1 \to \Gamma$ be a Lipschitz parameterization and let $u \in W^{1,2}(D, Y)$ be such that $\text{tr}(u) = c$. Then $\text{Area}(u) = 0$ by the claim above. Now, let Z be the injective hull of Γ. Since Γ is compact it follows that Z is compact. Moreover, Z admits a quadratic isoperimetric inequality. Denote by $\Lambda(\Gamma, Z)$ the set of all maps in $W^{1,2}(D, Z)$ whose trace is a weakly monotone parameterization of Γ. By [27, Theorem 1.1] and [27, Lemma 7.2], there exists $v \in \Lambda(\Gamma, Z) \cap C^0(\overline{D}, Z)$ which satisfies

$$\text{Area}(v) = \inf\{\text{Area}(w) \mid w \in \Lambda(\Gamma, Z)\}$$

and $E_+^2(v) \leq 2 \cdot \text{Area}(v)$. Let $\pi\colon Y \to Z$ be a 1–Lipschitz map which restricts to the identity on Γ. Then $w := \pi \circ u$ is in $\Lambda(\Gamma, Z)$ and $\text{Area}(w) = 0$. Since v is an area minimizer it follows that also $\text{Area}(v) = 0$ and hence that $E_+^2(v) = 0$ and hence that v is constant. This contradicts the fact that $\text{tr}(v)$ weakly monotonically parameterizes the Jordan curve Γ. □

As mentioned in the introduction, one can also prove the following theorem from [37] using the results above. For the definitions involving Carnot groups we refer to [37] .

Theorem 5.5. *Let G be a Carnot group of step 2 with grading $\mathfrak{g} = V_1 \oplus V_2$ of its Lie algebra. Suppose $u \in V_2$ is such that $u \neq [v, w]$ for all $v, w \in V_1$. Let H be the Carnot group of step 2 whose Lie algebra is $\mathfrak{h} = V_1 \oplus V_2/\langle u \rangle$ and endow H with a left-invariant Riemannian metric. Then H does not admit a quadratic isoperimetric inequality.*

In [37], this is proved by showing that if H admits a quadratic isoperimetric inequality, then minimal surfaces in H converge to integral currents in its asymptotic cone. By Pansu's Rademacher theorem for Carnot groups and Stokes' theorem for currents, there are curves in the asymptotic cone that do not bound integral currents, leading to a contradiction. A version of Stokes' theorem also holds for Sobolev maps (with Lipschitz trace), so one can use Corollary 5.3 to show that minimal discs in H converge to Sobolev discs and arrive at the same contradiction.

6. Hölder-Sobolev extensions

In this section we prove the first part of Corollary 1.5 as well as Theorem 6.6, which was alluded to in the introduction and which yields Hölder extensions with additional properties. We introduce the notion of Hölder 1–connectedness of a space and show that for quasi-convex metric spaces Hölder 1–connectedness implies simple connectedness and is equivalent to the Hölder extension property in case the ambient space is complete. Theorem 6.6 and the first part of Corollary 1.5 will then become consequences of the results proved in Section 5.

Definition 6.1. *Given $0 < \alpha \leq 1$, a metric space X is said to be α–Hölder 1–connected if there exists L such that every v–Lipschitz curve $c\colon S^1 \to X$ with $v > 0$ has an (Lv, α)–Hölder extension to $\overline{D}$.*

Quasi-convex metric spaces which are α–Hölder 1–connected with $\alpha = 1$ are called Lipschitz 1–connected in [25].

Proposition 6.2. *Let X be a quasi-convex metric space. If X is α–Hölder 1–connected for some $\alpha > 0$ then X is simply connected.*

Proof. Let $\lambda \geq 1$ be such that X is λ–quasi-convex. We denote the metric on X by d. Let $c\colon [0, 1] \to X$ be a continuous curve with $c(0) = c(1)$. Denote by Q the closed unit square in $\mathbb{R}^2$ and define $\varphi\colon \partial Q \to X$ by $\varphi(s, 0) := c(s)$ and $\varphi(s, t) := c(0)$ if $s = 0$ or $s = 1$ or $t = 1$. It suffices to show that φ extends continuously to Q. We construct such an extension as follows. For every $k \geq 0$ set

$$b_k := \max \left\{ d(c(a_k^m), c(a_k^{m+1})) \mid m = 0, 1, \ldots, 2^k - 1 \right\},$$

where $a_k^m := 2^{-k} m$ for $0 \leq m \leq 2^k$. On each segment $[a_k^m, a_k^{m+1}] \times \{2^{-k}\}$ with $k \geq 1$ and $0 \leq m \leq 2^k - 1$ define φ to be a Lipschitz curve from $c(a_k^m)$ to $c(a_k^{m+1})$

with Lipschitz constant at most $\lambda 2^k b_k$. This defines φ on the horizontal boundary segments of each of the rectangles

$$R_k^m := [a_k^m, a_k^{m+1}] \times [2^{-(k+1)}, 2^{-k}]$$

for $k \geq 0$ and $m = 0, 1, \ldots, 2^k - 1$. Extend φ to ∂R_k^m such that φ is a constant map on each vertical boundary segment. Then $\varphi|_{\partial R_k^m}$ is Lipschitz with constant at most $2^k b_k'$, where $b_k' := 4\lambda(2b_{k+1} + b_k)$. Since R_k^m is isometric to a 2^{-k}–scaled copy of R_0^0, which is in turn biLipschitz equivalent to $\overline{D}$, it follows from the α–Hölder 1–connectedness that $\varphi|_{\partial R_k^m}$ extends to a $(2^{k\alpha} L' b_k', \alpha)$–Hölder map on R_k^m, again denoted by φ. Here, L' is a constant only depending on the constant appearing in the definition of α–Hölder 1–connectedness. This defines φ on all of Q. Note that

$$d(\varphi(z), \varphi(a_k^m)) \leq 2^\alpha L' b_k'$$

for every $z \in R_k^m$ and all k and m. Since c is continuous and thus $b_k' \to 0$ as $k \to \infty$ it follows from the inequality above that φ is continuous at every point $(s, 0)$ with $s \in [0, 1]$. Since φ is continuous on $Q \setminus ([0, 1] \times \{0\})$ by construction the proof is complete.   □

In the setting of quasi-convex metric spaces, α–Hölder 1–connectedness is stable under taking ultralimits. We will not need this fact and thus leave the proof to the reader.

We turn to the relationship between a quadratic isoperimetric inequality and Hölder extendability of curves. As a direct consequence of Theorems 5.1 and 5.2 and Propositions 2.2 and 6.2 we obtain:

Corollary 6.3. *Let X be a locally compact, geodesic metric space admitting a quadratic isoperimetric inequality, and let X_ω be an asymptotic cone of X. Then X and X_ω are α_0–Hölder 1–connected for some $\alpha_0 \in (0, 1)$ depending only on the isoperimetric constant of X. In particular, X and X_ω are simply connected.*

The following result is the link between Hölder 1–connectedness and the Hölder extension property defined in the introduction.

Theorem 6.4. *Let X be a complete quasi-convex metric space and $0 < \alpha \leq 1$. Then X is α–Hölder 1–connected if and only if the pair $(\mathbb{R}^2, X)$ has the α–Hölder extension property.*

The equivalence is quantitative in the following sense. If one of the two statements holds with some constant $L \geq 1$ then the other statement holds with a constant $L' \leq M\lambda L$, where λ is the quasi-convexity constant of X and M is a universal constant. The theorem is well-known in the case $\alpha = 1$, see [1]. We refer to the Appendix below for the proof, which is an adaptation of the arguments in [1]. As a consequence of the theorem above we obtain:

Corollary 6.5. *Let X be a complete quasi-convex metric space. If the pair $(\mathbb{R}^2, X)$ has the α–Hölder extension property for some $0 < \alpha \leq 1$ then it also has the β–Hölder extension property for every $\beta \in (0, \alpha)$.*

Proof. This follows from Theorem 6.4 together with the fact that if a space is α–Hölder 1–connected then it is also β–Hölder 1–connected for all $\beta \in (0, \alpha)$.   □

We conclude this section with the following result.

Theorem 6.6. *Let X be a locally compact, geodesic metric space admitting a quadratic isoperimetric inequality or an ultralimit of a sequence of such spaces (with uniformly bounded isoperimetric constants). Then there exist $\alpha_0 > 0$ and $L \geq 1$ such that every (ν, α)–Hölder map $\varphi \colon A \to X$ with $\alpha \leq \alpha_0$ and $A \subset \mathbb{R}^2$ extends to a map $\bar{\varphi} \colon \mathbb{R}^2 \to X$ with the following properties:*

(i) *$\bar{\varphi}$ is $(L\nu, \alpha)$–Hölder continuous on $\mathbb{R}^2$.*
(ii) *$\bar{\varphi}$ sends compact subsets of $\Omega := \mathbb{R}^2 \setminus \overline{A}$ to sets of finite $\mathcal{H}^2$–measure.*
(iii) *$\bar{\varphi}$ sends subsets of Ω of Lebesgue measure zero to sets of $\mathcal{H}^2$–measure zero.*

The numbers α_0 and L only depend on the isoperimetric constant. If X has isoperimetric constant $C = \frac{1}{4\pi}$ then the conclusion also holds for $\alpha = L = 1$ by [30] and [24]. Note that properties (ii) and (iii) are in general not shared by $(1 - \varepsilon)$–Hölder maps, no matter how small $\varepsilon > 0$. We mention that Corollary 1.5 for $0 < \alpha \leq \alpha_0$ is also a consequence of Theorem 6.6.

Proof. We begin with the following observation. Let $c \colon S^1 \to X$ be a ν–Lipschitz curve. By Theorems 5.1 and 5.2 there exists $u \in W^{1,p}(D, X)$ with $\mathrm{tr}(u) = c$ and such that $[E_+^p(u)]^{\frac{1}{p}} \leq C\nu$ for some C and $p > 2$ only depending on the isoperimetric constant of X. Set $\alpha_0 := 1 - \frac{2}{p}$. By Proposition 2.2 there exists a representative $\varphi \colon \overline{D} \to X$ of u which is $(M\nu, \alpha_0)$–Hölder continuous on all of $\overline{D}$ with M only depending on the isoperimetric constant of X. Moreover, φ satisfies Lusin's property (N). Therefore, the area formula [22] yields

$$\mathcal{H}^2(\varphi(\overline{D})) \leq \int_X \#\{z \mid \varphi(z) = x\} \, d\mathcal{H}^2(x) = \int_D \mathbf{J}(\mathrm{ap} \, \mathrm{md} \, \varphi_z) \, dz = \mathrm{Area}(\varphi) < \infty.$$

Finally, φ is $(2M\nu, \alpha)$–Hölder continuous for every $0 < \alpha \leq \alpha_0$.

Now, let α_0 and M be as above and fix $0 < \alpha \leq \alpha_0$. Let $A \subset \mathbb{R}^2$ be non-empty and $\varphi \colon A \to X$ a (ν, α)–Hölder map. Since X is complete we may assume that A is closed. Let $\varphi \colon \mathbb{R}^2 \to X$ be the map constructed in the proof of Theorem 6.4, where $\varphi|_Q$ is a map given as in the observation above for every cube Q in the Whitney cube decomposition of $\mathbb{R}^2 \setminus A$. For the notion of Whitney cube decomposition see the proof of Theorem 6.4. Then φ is $(L\nu, \alpha)$–Hölder continuous for some L depending only on M by (the proof of) Theorem 6.4. Moreover, φ satisfies properties (ii) and (iii) in the statement of the theorem because each $\varphi|_Q$ satisfies these properties for every Q by the observation above. Notice here that every compact subset of $\mathbb{R}^2 \setminus A$ only intersects finitely many cubes from the Whitney cube decomposition. This completes the proof of the theorem. □

7. Hölder exponent arbitrarily close to one

The aim of this section is to prove the following result which, together with the results of Sections 6 and 4, yields Theorem 1.4 and Corollary 1.5 and their generalizations.

Theorem 7.1. *Let X be a locally compact, geodesic metric space. Suppose that there are $C > 0$ such that $\delta_X(r) \leq Cr^2$ for all $r > 0$. Then, for any $\alpha \in (0, 1)$, X is α–Hölder 1–connected, with multiplicative constant depending only on C and α.*

Theorem 6.4 then implies that the pair $(\mathbb{R}^2, X)$ has the α–Hölder extension property, and Corollary 6.3 implies that X is simply connected. Because the multiplicative constant depends only on C and α, Corollary 2.5 implies the following corollary.

Corollary 7.2. *Let X be a locally compact, geodesic metric space admitting a quadratic isoperimetric inequality. Then every asymptotic cone X_ω of X is simply connected and the pair $(\mathbb{R}^2, X_\omega)$ has the α–Hölder extension property for every $\alpha \in (0, 1)$.*

We prove Theorem 7.1 by using the triangulations produced by Theorem 4.2 to repeatedly subdivide a curve into smaller curves. If X is a length space, we say that X is (n, k)–triangulable if for all $r > 0$, we have $\mathrm{Ar}_{X, r/k}(r) \leq n$. If X admits a quadratic isoperimetric inequality, then Theorem 4.2 implies that there is some K such that X is (Kn^2, n)–triangulable for all $n \geq 1$.

Lemma 7.3. *Let $B = [0, 1]^2$ be the unit square. Let $n > 0$, $k > 1$. Divide B into an $n \times n$ grid of squares of side length n^{-1}. Let $B_1, \ldots, B_{n^2}$ be squares of side length $(2n)^{-1}$, one centered in each of the grid squares, and let E be the closure of $B \setminus \bigcup_i B_i$.*

There is an $L > 0$ depending on n such that if X is a space that is (n^2, k)–triangulable and $c\colon \partial B \to X$ is a constant-speed Lipschitz curve, then there is a map $f\colon E \to X$ such that:

(i) *f agrees with c on ∂B.*
(ii) *For each i, the restriction $f|_{\partial B_i}$ is a constant-speed curve and*

$$\ell(f|_{\partial B_i}) \leq \frac{\ell(c)}{k}.$$

(iii) *$\mathrm{Lip}(f) \leq L \,\mathrm{Lip}(c)$.*

Proof. Let $c\colon \partial B \to X$ be a constant-speed Lipschitz curve. Since X is (n^2, k)–triangulable, there is a triangulation τ of the closed disc $\overline{D}$ with at most n^2 triangles, and a continuous map $P\colon \tau^{(1)} \to X$ such that $P|_{S^1} = c$ and $\ell(P|_{\partial F}) \leq \frac{\ell(c)}{k}$ for every triangle F in τ. We equip τ with a metric so that each face is isometric to a equilateral triangle with unit sides and assume that P parameterizes each edge with constant speed, so that $\mathrm{Lip}(P) \leq \ell(c)$. Denote the faces of τ by $F_1, \ldots, F_j$. We will construct f by finding a map $v\colon E \to \tau^{(1)}$ that sends ∂B to $\partial \tau$ and sends ∂B_i to ∂F_i for all $i \leq j$.

For all i, let x_i be the center of the square B_i. For $1 \leq i \leq j$, let y_i be the center of the face F_i, and for $i > j$, let y_i be a point in the interior of τ, chosen so that the y_i are all distinct. Let $q\colon B \to B$ be a Lipschitz map that fixes the first j grid squares of B and collapses ∂B_i to x_i for $i > j$. Let $g\colon B \to \tau$ be a Lipschitz homeomorphism such that for all i, $g(x_i) = y_i$ and $g(\partial B_i)$ is a small circle around y_i. Then $g \circ q(E)$ avoids the center of each face of τ.

Let

$$r\colon \tau \setminus \{y_1, \ldots, y_j\} \to \tau^{(1)}$$

be the retraction such that if $x \in F_i$, then $r(x)$ is the intersection of the ray from y_i to x with the boundary ∂F_i. This map is locally Lipschitz, so $v = r \circ g \circ q$ is a Lipschitz map from E to $\tau^{(1)}$ such that $v(\partial B_i) = \partial F_i$ for all $i \leq j$ and $v(\partial B_i)$ is a point for all $i > j$.

The Lipschitz constant of v depends on τ, but there are only finitely many possible triangulations τ with at most n^2 faces. Each of these gives rise to a map v_τ, so if $L_0 = L_0(n) = \max_\tau \text{Lip}(v_\tau)$, then $\text{Lip}(v) \leq L_0$. Let $f_0 = P \circ v$. Then $\text{Lip}(f_0) \leq L_0 \text{Lip}(P) \leq L_0 \ell(c)$.

This map sends ∂B to c and ∂B_i to either a curve of length at most $\frac{\ell(c)}{k}$ or a point, but the parameterizations of the boundary components may not have constant speed. Let E' be the space E with an annulus of length $\frac{1}{n}$ glued to each boundary component ∂B_i and an annulus of length 1 glued to ∂B. This is biLipschitz equivalent to E, say by an equivalence $h\colon E' \to E$. Define $f\colon E \to X$ so that $f|_{\partial B} = c$, $f|_{\partial B_i}$ is a constant-speed parameterization of $f_0|_{\partial B_i}$, and $f(h(x)) = f_0(x)$ for all $x \in E$. This defines f on all of E except for an annulus around each boundary component, and we use Lemma 3.6 to define f on these annuli. Then f satisfies the desired conditions on each boundary component and there is a $L_1 > 0$ such that $\text{Lip}(f) \leq L_1 \text{Lip}(f_0) \leq L_1 L_0 \ell(c)$, as desired. $\qquad\square$

Now, we can prove the desired Hölder extension property.

Proposition 7.4. *If X is (n^2, k)–triangulable for some $1 < k < 2n$, then for any Lipschitz curve $c\colon S^1 \to X$, there is an extension $\beta\colon \overline{D} \to X$ such that β is $\left(C_H \text{Lip}(c), \frac{\log k}{\log 2n}\right)$–Hölder continuous, where C_H depends only on n and k.*

Proof. We identify S^1 with the boundary ∂B, where $B = [0,1]^2$. Let E^0 be B with n^2 holes of side length $(2n)^{-1}$ removed, as in Lemma 7.3. We denote these holes by $B_1^1, \ldots B_{n^2}^1$.

Let $c\colon \partial B \to X$ be a Lipschitz curve. By Lemma 3.6, we may assume that c is parameterized with constant speed. Let $\beta^0\colon E^0 \to X$ be a map satisfying Lemma 7.3. The curves $c_i = \beta^0|_{\partial B_i^1}$ are Lipschitz curves with $\ell(c_i) \leq k^{-1}\ell(c)$. Since c_i is a constant-speed curve, we have $\text{Lip}(c_i) \leq 2nk^{-1}\ell(c)$. For $i = 1, \ldots, n^2$, let E_i^1 be a copy of E^0, scaled to fit inside B_i^1. We can apply Lemma 7.3 to each curve c_i to obtain maps $\beta_i^1\colon E_i^1 \to X$ that agree with β^0 on ∂B_i^1 and satisfy

$$\text{Lip}(\beta_i^1) \leq L\,\text{Lip}(c_i) \leq 2nLk^{-1}\ell(c).$$

Let $E^1 = E^0 \cup \bigcup_i E_i^1$ and let $\beta^1\colon E^1 \to X$ be the map that extends β^0 and all of the β_i^1's.

Now, E^1 has n^4 smaller square holes, which we denote $B_1^2, \ldots B_{n^4}^2$. We can repeat the process to produce E_i^2, β_i^2, etc. In fact, for each $j > 0$ and $i = 1, \ldots, n^{2j}$, there are B_i^j, E_i^j, and $\beta_i^j\colon E_i^j \to X$ with the following properties:

   (i) The set B_i^j is one of the holes in $E_{\lceil i/n^2 \rceil}^{j-1}$; it has side length $(2n)^{-j}$.

   (ii) E_i^j is a copy of E^0, scaled to fit in B_i^j.

   (iii) $\beta_i^j\colon E_i^j \to X$ is a map such that β_i^j agrees with $\beta_{\lceil i/n^2 \rceil}^{j-1}$ on ∂B_i^j.

   (iv) $\ell(\beta_i^j|_{\partial B_i^j}) \leq k^{-j}\ell(c)$.

   (v) $\text{Lip}(\beta_i^j) \leq (2n)^j Lk^{-j}\ell(c)$.

Let $E^m = \bigcup_{j=1}^m \bigcup_{i=1}^{n^{2j}} E_i^j$ and let $E^\infty = \bigcup_{j=1}^\infty E^j$. Then $B \setminus E^\infty$ is a Cantor set of measure zero. Let $\beta^j\colon E^j \to X$ be the common extension of the β_i^m for $m \leq j$. By (v), $\text{Lip}(\beta^j) \leq (2n)^j Lk^{-j}\ell(c)$. Let $\beta^\infty\colon E^\infty \to X$ be the common extension of the maps β^j. Then β^∞ is an extension of c, and we claim that β^∞ is Hölder continuous.

Suppose that $x \in E^\infty$. For each j, let $x_j \in E^j$ be a point such that $|x - x_j| = \mathrm{dist}(x, E^j)$. The holes in E^j have side length $(2n)^{-j-1}$, so $|x - x_j| \leq (2n)^{-j-1}$. Then

$$
\begin{aligned}
d(\beta^\infty(x), \beta^\infty(x_j)) &\leq \sum_{i=j}^{\infty} \mathrm{Lip}(\beta^{i+1}) \cdot |x_i - x_{i+1}| \\
&\leq \sum_{i=j}^{\infty} (2n)^{i+1} L k^{-i-1} \ell(c) \cdot [(2n)^{-i-1} + (2n)^{-i-2}] \\
&\leq \frac{1 + (2n)^{-1}}{1 - k^{-1}} \cdot L k^{-(j+1)} \ell(c) \\
&\leq L' k^{-j} \ell(c),
\end{aligned}
$$

where L' is a constant depending on n and k.

Suppose that $u, v \in E^\infty$, and let $m \geq 0$ be such that

$$
\sqrt{2}(2n)^{-m-1} \leq |u - v| \leq \sqrt{2}(2n)^{-m}.
$$

Let $u_m, v_m \in E^m$ be the nearest-point projections of u and v. Then

$$
|u_m - v_m| \leq \sqrt{2}(2n)^{-m} + 2(2n)^{-m-1} \leq 3(2n)^{-m}
$$

and

$$
\begin{aligned}
d(\beta^\infty(u), \beta^\infty(v)) &\leq 2L' \ell(c) k^{-m} + d(\beta^\infty(u_m), \beta^\infty(v_m)) \\
&\leq 2L' \ell(c) k^{-m} + 3 \mathrm{Lip}(\beta^m)(2n)^{-m} \\
&\leq 2L' \ell(c) k^{-m} + 3L\ell(c) k^{-m} \\
&\leq (2L' + 3L) \cdot \ell(c) \cdot k^{-m} \\
&\leq C_H \ell(c) \cdot |u - v|^{\frac{\log k}{\log 2n}}
\end{aligned}
$$

where C_H depends on n and k. Therefore, β^∞ is $\frac{\log k}{\log 2n}$–Hölder continuous, and so there exists a unique extension $\beta \colon B \to X$ of β^∞ which is $\frac{\log k}{\log 2n}$–Hölder continuous. $\qquad \square$

Proof of Theorem 7.1. Let $\alpha \in (0, 1)$. By Theorem 4.2, there is a K depending only on C such that X is (Kn^2, n)–triangulable for all $n \geq 1$. We have

$$
\lim_{n \to \infty} \frac{\log n}{\log(2\sqrt{Kn^2})} = 1,
$$

so there is an n such that $\frac{\log n}{\log(2\sqrt{Kn^2})} > \alpha$. By Proposition 7.4, X is $\frac{\log n}{\log(2\sqrt{Kn^2})}$–Hölder 1–connected with a constant depending only on K and n. $\qquad \square$

8. Appendix: Proof of Theorem 6.4

In this section, we prove Theorem 6.4. The proof is a variation of the proof of Theorem (1.2) in [1], which is the special case of our theorem for $\alpha = 1$. The construction of our Hölder continuous extension is exactly as in [1], while the proof of the Hölder continuity is slightly more technical than the proof of the Lipschitz continuity in [1].

The proof of the theorem will moreover show the following. If X only satisfies the α–Hölder 1–connectedness condition for Lipschitz curves with Lipschitz constant at most some number ν_0 then (ν, α)–Hölder maps $\varphi \colon A \to X$ with $A \subset \mathbb{R}^2$ and

$v > 0$ arbitrary can be extended to (MLv, α)–Hölder maps defined on the neighborhood

$$\left\{ z \in \mathbb{R}^2 \; \middle| \; \text{dist}(z, A) < \left(\frac{Mv_0}{\lambda v} \right)^{\frac{1}{\alpha}} \right\}$$

of A, where λ is the quasi-convexity constant and M is a universal constant.

Proof of Theorem 6.4. Suppose first that the pair $(\mathbb{R}^2, X)$ has the α–Hölder extension property with some constant L. Let $c \colon S^1 \to X$ be a v–Lipschitz curve. Then c is $(2v, \alpha)$–Hölder and thus, by the Hölder extension property, there exists an $(2vL, \alpha)$–Hölder extension of c defined on all of $\mathbb{R}^2$. It follows that X is α–Hölder 1–connected with constant $L' = 2L$.

Now, suppose that X is α–Hölder 1–connected with some constant $L \geq 1$. In what follows, M will denote a suitable universal constant, which might change with each occurrence in the text. Denote by λ the quasi-convexity constant of X. Let $A \subset \mathbb{R}^2$ and let $\varphi \colon A \to X$ be (v, α)–Hölder continuous for some $v > 0$. Since X is complete we may assume that A is closed. Let $\mathcal{Q}$ be a Whitney cube decomposition of the complement A^c of A, see Theorem VI.1.1 in [35]. Thus, $\mathcal{Q}$ is a collection of pairwise almost disjoint closed squares $Q \subset A^c$ of the form

$$Q = [p, p + 2^k] \times [q, q + 2^k]$$

with $p, q \in 2^k \mathbb{Z}$ and $k \in \mathbb{Z}$ such that $A^c = \cup_{Q \in \mathcal{Q}} Q$ and

(5) $$4^{-1} \text{diam}(Q) \leq \text{dist}(A, Q) \leq 4 \text{diam}(Q)$$

for every $Q \in \mathcal{Q}$. We first observe that if $Q, Q' \in \mathcal{Q}$ have non-trivial intersection then $\text{diam}(Q) \leq 20 \text{diam}(Q')$ by (5). Moreover, either $Q \cap Q'$ consists of a single point or it contains an entire edge of Q or Q'. Denote by K_0 the set of vertices and by K_1 the union of edges of squares in $\mathcal{Q}$. An edge of a square in $\mathcal{Q}$ will be called minimal edge in K_1 if it does not contain any shorter edges of squares in $\mathcal{Q}$. Note that for every $Q \in \mathcal{Q}$ and every minimal edge I in K_1 with $I \subset \partial Q$ we have

(6) $$\text{diam}(I) \geq (\sqrt{2} \cdot 20)^{-1} \text{diam}(Q)$$

by the above.

We extend φ first to points in K_0 as follows. For each $x \in K_0$ choose $r_0(x) \in A$ such that

$$|x - r_0(x)| = \text{dist}(x, A)$$

and set $\varphi(x) := \varphi \circ r_0(x)$. We next extend $\varphi|_{K_0}$ to K_1 using the quasi-convexity of X. For this, let $I \subset K_1$ be a minimal edge in the sense above and let a and b denote the endpoints of I. It follows from (5) that

$$|r_0(a) - r_0(b)| \leq M|a - b|$$

and hence

$$d(\varphi(a), \varphi(b)) \leq v|r_0(a) - r_0(b)|^{\alpha} \leq vM|a - b|^{\alpha}.$$

Thus, φ extends to some $(M\lambda v \, \text{diam}(I)^{\alpha-1})$–Lipschitz map on I by the λ–quasi-convexity of X. Since K_1 is the union of minimal edges in K_1 this defines φ on all of K_1. We finally extend $\varphi|_{K_1}$ to A^c using the α–Hölder 1–connectedness. For this, let $Q \in \mathcal{Q}$. It follows with (6) that $\varphi|_{\partial Q}$ is $(M\lambda v \, \text{diam}(Q)^{\alpha-1})$–Lipschitz. Hence $\varphi|_{\partial Q}$ extends to some $(ML\lambda v, \alpha)$–Hölder map defined on Q. Since Q was arbitrary this defines φ on all of A^c and hence on all of $\mathbb{R}^2$.

It remains to show that φ is $(ML\lambda\nu, \alpha)$–Hölder continuous on all of $\mathbb{R}^2$. Let $x, y \in \mathbb{R}^2$. We first assume that $x \in A$ and $y \in A^c$. Let $Q \in Q$ be such that $y \in Q$ and let y' be a vertex of Q. Then

$$|y - y'| \le \operatorname{diam}(Q) \le 4 \operatorname{dist}(Q, A) \le 4|x - y|$$

and hence

$$
\begin{aligned}
|r_0(y') - x| &\le |r_0(y') - y'| + |y' - y| + |y - x| \\
&\le \operatorname{dist}(y', A) + 5|y - x| \\
&\le \operatorname{dist}(Q, A) + \operatorname{diam}(Q) + 5|y - x| \\
&\le 5 \operatorname{dist}(Q, A) + 5|y - x| \\
&\le 10|y - x|.
\end{aligned}
$$

From this we infer that

$$
\begin{aligned}
d(\varphi(x), \varphi(y)) &\le d(\varphi(x), \varphi(y')) + d(\varphi(y'), \varphi(y)) \\
&\le \nu|x - r_0(y')|^\alpha + ML\lambda\nu|y' - y|^\alpha \\
&\le ML\lambda\nu|x - y|^\alpha.
\end{aligned}
$$

We now assume that $x, y \in A^c$. Let $Q, Q' \in Q$ be such that $x \in Q$ and $y \in Q'$ and let x' and y' be vertices of Q and Q', respectively. We distinguish two cases and first consider the case that

$$|x - y| \ge \frac{1}{30} \max\{\operatorname{dist}(Q, A), \operatorname{dist}(Q', A)\}.$$

From this and (5) we infer that $|x - x'| \le 120|x - y|$ and $|y - y'| \le 120|x - y|$ and moreover

$$|r_0(x') - r_0(y')| \le M|x - y|.$$

It follows that

$$
\begin{aligned}
d(\varphi(x), \varphi(y)) &\le d(\varphi(x), \varphi(x')) + d(\varphi(x'), \varphi(y')) + d(\varphi(y'), \varphi(y)) \\
&\le ML\lambda\nu(|x - x'|^\alpha + |y - y'|^\alpha) + \nu|r_0(x') - r_0(y')|^\alpha \\
&\le ML\lambda\nu|x - y|^\alpha,
\end{aligned}
$$

which proves the first case. We now consider the case that

$$|x - y| < \frac{1}{30} \max\{\operatorname{dist}(Q, A), \operatorname{dist}(Q', A)\}.$$

We may assume without loss of generality that $x \ne y$ and $\operatorname{dist}(Q', A) \le \operatorname{dist}(Q, A)$. Denote by Z the square of side length $2|x - y|$ centered at x. We claim that at most 4 squares from Q intersect Z. Indeed, if $Q'' \in Q$ intersects Z in some point z then

$$\operatorname{diam}(Q'') \ge \frac{1}{4} \operatorname{dist}(Q'', A) \ge \frac{1}{4}(\operatorname{dist}(z, A) - \operatorname{diam}(Q''))$$

and thus $\operatorname{diam}(Q'') \ge \frac{1}{5} \operatorname{dist}(z, A)$. Since

$$\operatorname{dist}(z, A) \ge \operatorname{dist}(x, A) - |x - z| \ge \operatorname{dist}(Q, A) - |x - z| > (30 - \sqrt{2}) \cdot |x - y|$$

it follows that $\operatorname{diam}(Q'') \ge 5^{-1}(30 - \sqrt{2}) \cdot |x - y|$ and hence the side length of Q'' is strictly larger than the side length of Z. This implies that Z intersects at most 4 squares from Q, as claimed. It follows that the straight segment from x to y intersects at most 4 squares from Q. Since φ is $(ML\lambda\nu, \alpha)$–Hölder on each such square it follows that

$$d(\varphi(x), \varphi(y)) \le 4ML\lambda\nu|x - y|^\alpha.$$

This completes the proof. □

References

[1] Frederick Justin Almgren, Jr. The homotopy groups of the integral cycle groups. *Topology*, 1:257–299, 1962.

[2] Juan M. Alonso. Inégalités isopérimétriques et quasi-isométries. *C. R. Acad. Sci. Paris Sér. I Math.*, 311(12):761–764, 1990.

[3] Luigi Ambrosio. Metric space valued functions of bounded variation. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 17(3):439–478, 1990.

[4] Luigi Ambrosio and Bernd Kirchheim. Currents in metric spaces. *Acta Math.*, 185(1):1–80, 2000.

[5] Luigi Ambrosio and Paolo Tilli. *Topics on analysis in metric spaces*, volume 25 of *Oxford Lecture Series in Mathematics and its Applications*. Oxford University Press, Oxford, 2004.

[6] Zoltán M. Balogh, Roberto Monti, and Jeremy T. Tyson. Frequency of Sobolev and quasiconformal dimension distortion. *J. Math. Pures Appl. (9)*, 99(2):125–149, 2013.

[7] Mario Bonk and Bruce Kleiner. Quasisymmetric parametrizations of two-dimensional metric spheres. *Invent. Math.*, 150(1):127–183, 2002.

[8] Martin R. Bridson. The geometry of the word problem. In *Invitations to geometry and topology*, volume 7 of *Oxf. Grad. Texts Math.*, pages 29–91. Oxford Univ. Press, Oxford, 2002.

[9] Martin R. Bridson and André Haefliger. *Metric spaces of non-positive curvature*, volume 319 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1999.

[10] Alexander Brudnyi and Yuri Brudnyi. *Methods of geometric analysis in extension and trace problems. Volume 1*, volume 102 of *Monographs in Mathematics*. Birkhäuser/Springer Basel AG, Basel, 2012.

[11] Dmitri Burago and Sergei Ivanov. Minimality of planes in normed spaces. *Geom. Funct. Anal.*, 22(3):627–638, 2012.

[12] José Burillo and Jennifer Taback. Equivalence of geometric and combinatorial Dehn functions. *New York J. Math.*, 8:169–179 (electronic), 2002.

[13] Cornelia Druţu. Quasi-isometry invariants and asymptotic cones. *Internat. J. Algebra Comput.*, 12(1-2):99–135, 2002. International Conference on Geometric and Combinatorial Methods in Group Theory and Semigroup Theory (Lincoln, NE, 2000).

[14] Cornelia Druţu and Mark Sapir. Tree-graded spaces and asymptotic cones of groups. *Topology*, 44(5):959–1058, 2005. With an appendix by Denis Osin and Sapir.

[15] D. B. A. Epstein, J. W. Cannon, D. F. Holt, S. V. F. Levy, M. S. Paterson, and W. P. Thurston. *Word processing in groups*. Jones and Bartlett Publishers, Boston, MA, 1992.

[16] Lawrence C. Evans and Ronald F. Gariepy. *Measure theory and fine properties of functions*. Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1992.

[17] M. Gromov. Asymptotic invariants of infinite groups. In *Geometric group theory, Vol. 2 (Sussex, 1991)*, volume 182 of *London Math. Soc. Lecture Note Ser.*, pages 1–295. Cambridge Univ. Press, Cambridge, 1993.

[18] Juha Heinonen, Pekka Koskela, Nageswari Shanmugalingam, and Jeremy Tyson. *Sobolev spaces on metric measure spaces*, volume 27 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2015.

[19] Juha Heinonen, Pekka Koskela, Nageswari Shanmugalingam, and Jeremy T. Tyson. Sobolev classes of Banach space-valued functions and quasiconformal mappings. *J. Anal. Math.*, 85:87–139, 2001.

[20] J. R. Isbell. Six theorems about injective metric spaces. *Comment. Math. Helv.*, 39:65–76, 1964.

[21] M. B. Karmanova. Area and co-area formulas for mappings of the Sobolev classes with values in a metric space. *Sibirsk. Mat. Zh.*, 48(4):778–788, 2007.

[22] Bernd Kirchheim. Rectifiable metric spaces: local structure and regularity of the Hausdorff measure. *Proc. Amer. Math. Soc.*, 121(1):113–123, 1994.

[23] Nicholas J. Korevaar and Richard M. Schoen. Sobolev spaces and harmonic maps for metric space targets. *Comm. Anal. Geom.*, 1(3-4):561–659, 1993.

[24] U. Lang and V. Schroeder. Kirszbraun's theorem and metric spaces of bounded curvature. *Geom. Funct. Anal.*, 7(3):535–560, 1997.

[25] Urs Lang and Thilo Schlichenmaier. Nagata dimension, quasisymmetric embeddings, and Lipschitz extensions. *Int. Math. Res. Not.*, (58):3625–3655, 2005.

[26] Alexander Lytchak and Stefan Wenger. Regularity of harmonic discs in spaces with quadratic isoperimetric inequality. *Calc. Var. Partial Differential Equations*, 55(4):Paper No. 98, 19, 2016.

[27] Alexander Lytchak and Stefan Wenger. Area minimizing discs in metric spaces. *Arch. Ration. Mech. Anal.*, 223(3):1123–1182, 2017.

[28] Alexander Lytchak and Stefan Wenger. Canonical parametrizations of metric discs. *preprint*, 2017.

[29] Alexander Lytchak and Stefan Wenger. Intrinsic structure of minimal discs in metric spaces. *Geom. Topol.*, 22(1):591–644, 2018.

[30] Alexander Lytchak and Stefan Wenger. Isoperimetric characterization of upper curvature bounds. *Acta Math., to appear*, 2018.

[31] P. Papasoglu. On the asymptotic cone of groups satisfying a quadratic isoperimetric inequality. *J. Differential Geom.*, 44(4):789–806, 1996.

[32] Yu. G. Reshetnyak. Sobolev classes of functions with values in a metric space. *Sibirsk. Mat. Zh.*, 38(3):657–675, iii–iv, 1997.

[33] Yu. G. Reshetnyak. Sobolev classes of functions with values in a metric space. II. *Sibirsk. Mat. Zh.*, 45(4):855–870, 2004.

[34] Yu. G. Reshetnyak. On the theory of Sobolev classes of functions with values in a metric space. *Sibirsk. Mat. Zh.*, 47(1):146–168, 2006.

[35] Elias M. Stein. *Singular integrals and differentiability properties of functions.* Princeton Mathematical Series, No. 30. Princeton University Press, Princeton, N.J., 1970.

[36] Stefan Wenger. Gromov hyperbolic spaces and the sharp isoperimetric constant. *Invent. Math.*, 171(1):227–255, 2008.

[37] Stefan Wenger. Nilpotent groups without exactly polynomial Dehn function. *J. Topol.*, 4(1):141–160, 2011.

Mathematisches Institut, Universität Köln, Weyertal 86 – 90, 50931 Köln, Germany
E-mail address: alytchak@math.uni-koeln.de

Department of Mathematics, University of Fribourg, Chemin du Musée 23, 1700 Fribourg, Switzerland
E-mail address: stefan.wenger@unifr.ch

Courant Institute of Mathematical Sciences, New York University, 251 Mercer St., New York, NY 10012, USA
E-mail address: ryoung@cims.nyu.edu