ALMOST CONVEX SUBSETS

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ABSTRACT. We study almost convex subsets of spaces with one-sided curvature bounds. We derive some characterisations and properties of sets of positive reach in Riemannian manifolds.

1. INTRODUCTION

1.1. **The aim.** This paper is devoted to the study of the geometry of nice subsets of nice metric spaces. The nice metric spaces we have in mind are Riemannian manifolds or more generally spaces with one-sided curvature bound in the sense of Alexandrov. The nice subsets we are interested in are almost convex subsets in some precise (parameter depending) sense. For each positive number α we define the notion of α-convexity, such that the case $\alpha = \infty$ corresponds to convex subsets and the most interesting case $\alpha = 2$ describes subsets of positive reach in Riemannian manifolds, that play an important role in the integral geometry, compare for example [Hug98]. The class of subsets of positive reach is very big and contains for example smooth submanifolds with boundaries, for which the inner geometry was studied in a series of papers of Alexander, Berg and Bishop (see [ABB87] and [ABB93]). In spaces with one-sided curvature bound α-convex subsets for $\alpha = 2$ are closely related to semi-convex functions, that play a major role in the investigations of such spaces. In spaces with one-sided curvature bounds we create in this way surrogates for submanifolds and methods for studying them. Restricted to manifolds we obtain

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a geometric access to the geometry of non-smooth submanifolds and of sets of positive reach.

1.2. **The definitions.** The idea is very simple. In order to study the property of a subset Z of a metric space X we want first to neglect the ambient geometry of X and to work only with the induced metric d on Z. The following definition just says that Z is convex up to order α.

Definition 1.1. Let $C, \alpha \geq 0, \rho > 0$ be real numbers. Let (Z, d) be a metric space. Denote by d^Z the inner metric induced by d. We say that Z is (C, α, ρ)-geodesic if $d(x, y) \leq \rho$ implies $d^Z(x, y) \leq d(x, y)(1 + Cd^\alpha(x, y))$. We say that a subset Z of a metric space X is (C, α, ρ)-convex if Z is (C, α, ρ)-geodesic with respect to the induced metric.

Remark 1.1. In the definition the constant ρ plays the role of the minimal distance between connected components of a locally convex subset and can be neglected in local considerations. The constant α is the most important one and describes the degree of non-convexity. Finally the constant C plays the role of the bound of the second fundamental form for $\alpha = 2$ and can be interpreted similiary for general α.

Remark 1.2. The above definition is global, i.e. it requires the fixed choice of constants C and ρ for the whole space Z. It is easy to extend the definition and to allow C to grow and ρ to become small if one goes to infinity in Z. We leave this extension to the interested reader.

In general it is difficult to find geodesics in a space, it is often easier to find midpoints and to observe that this and the completeness garantuee the geodesicity of the space. On the other hand geometric assumptions on the ambient space X (such as curvature bounds) can be used to compare midpoints between two given points with approximate midpoints, and therefore to relate the geometry of almost convex subsets to the geometry of X. This motivates the following definition:

Definition 1.2. Let $C, \alpha \geq 0, \rho > 0$ be such that $C\rho^\alpha < 1$. A metric space X is called (C, α, ρ)-almost midpoint space if for all

$x_0, x_1 \in X$ with $s = d(x_0, x_1) < \rho$ there is a point $m \in X$ with $d(x_i, m) \le \frac{s}{2}(1 + Cs^\alpha)$.

One sees immediately that a (C, α, ρ)-geodesic space is also (C, α, ρ)-almost midpoint space. On the other hand we have:

Proposition 1.1. *Let $\alpha, C \ge 0, \rho > 0$ be real numbers with $C\rho^\alpha < 1$. If $\alpha > 0$, then there are constants $\bar{C}, \bar{\rho}$ depending only on C, ρ and α such that each complete (C, α, ρ)-almost midpoint space is $(\bar{C}, \alpha, \bar{\rho})$-geodesic.*

The result does not hold for $\alpha = 0$, see however Lemma 3.2.

1.3. **Almost convex subsets of manifolds.** Comparing midpoints to almost midpoints and using a metric interpretation of $\mathcal{C}^{1,\alpha}$ maps we obtain:

Theorem 1.2. *Let M be a smooth Riemannian manifold, $Z \subset M$ a compact subset, $0 < \alpha \le 1$. Then Z is $(C, 2\alpha, \rho)$-convex for some $C, \rho > 0$, iff for some $\bar{\rho}, \bar{C} > 0$ each two points in Z at distance $s < \bar{\rho}$ can be connected in Z by an arclength parametrized $\mathcal{C}^{1,\alpha}$ curve of length smaller $\bar{C}s$, whose $\mathcal{C}^{1,\alpha}$ norm is bounded by $\bar{C}$.*

This shows that the notion of 2α-convexity does not depend on the Riemannian metric but only on the $\mathcal{C}^{1,\alpha}$ atlas of M. Moreover we obtain that geodesics in a $\mathcal{C}^{1,\alpha}$ submanifold of a smooth Riemannian manifold are uniformly $\mathcal{C}^{1,\alpha}$. This result is a little bit astonishing as far as the geodesics of an abstract manifold with an α-Hölder continuous Riemannian tensor do not have the same regularity for $\alpha < 1$ ([LY04],T.1.1). This provides the non-embeddability result ([LY04],Cor.1.2) for such manifolds.

Theorem 1.3. *Let M be a smooth Riemannian manifold, Z a compact subset of X. Then the following are equivalent:*

(1) *Z has positive reach;*
(2) *Z is $(C, 2, \rho)$-convex for some $C, \rho > 0$;*
(3) *Z is a locally convex subset with respect to a Lipschitz continuous Riemannian metric.*

The point (3) in the theorem above shows that sets of positive reach are indeed closely related to convex subsets of Riemannian manifold (despite the fact that their local topology may be very complicated) not only in the sense of integral but also in the sense of metric geometry. It seems a difficult task to obtain a similiar characterization of (C,α)-convex subsets for $0<\alpha<2$.

As was already mentioned our approach allows us to study the inner geometry and topology of the sets of positive reach. As an example we prove:

Proposition 1.4. *Let M be a Riemannian manifold and $Z \subset M$ a subset of positive reach. The following are equivalent:*

(1) *Z is a $\mathcal{C}^{1,1}$ submanifold;*
(2) *Z is a topological manifold;*
(3) *(Z,d^Z) is geodesically complete;*
(4) *Each tangent space T_xZ is a Euclidean space.*

Remark 1.3. Federer has proved in [Fed59] that a set of positive reach that is a Lipschitz manifold must be a $\mathcal{C}^{1,1}$ submanifold.

1.4. **Almost convex subsets in spaces with one-sided curvature bounds.** The notion of (C,α)-convexity is closely related to semi-concave functions for $\alpha=2$. The following lemma is proved in [Lyt04c],Cor.1.9 (see Subsection 2.4 for the definition of semi-concave functions and regular sub-level sets).

Proposition 1.5. *Let X be a space with a one-sided curvature bound. If $Z=f^{-1}([t,\infty))$ is a regular sublevel set of a semi-concave function $f: X\to\mathbb{R}$, then Z is a $(C,2,\rho)$-almost midpoint space for some $C,\rho>0$ and therefore it is $(\bar{C},2,\bar{\rho})$-convex.*

Actually, the last result is not only true for spaces with one-sided curvature bounds but also for surfaces with an integral curvature bound or manifolds with a Hölder continuous Riemannian tensor ([Lyt04c]).

In the presence of an upper curvature bound this condition is also necessary. Moreover in this case one can relate the geometry of the subset to the geometry of a neighborhood:

Theorem 1.6. *Let X be a $CAT(\kappa)$ space, Z a closed $(C,2,\rho)$-convex subset. Then there are numbers $r,A,\lambda,i>0$ depending only on κ, C and ρ, such that the distance function d_Z to the subset Z is λ-convex in the tubular neighborhood $U=U_r(Z)$ of radius r around Z. For each point $x\in U_r$ there is a unique foot point $p\in Z$ next to x and the map $P^Z: U\to Z$ is Lipschitz. Moreover its local*

Lipschitz point at x is bounded by $1 + A \cdot d(x, Z)$. Finally Z does not contained a closed geodesic (with respect to the inner metric) of length smaller than i.

Remark 1.4. Some dimension dependant bounds on the (normal) injectivity radius of a smooth submanifold of a Riemannian manifold in terms of its second fundamental form (essentially equivalent to the constant C due to Theorem 1.2) was obtained in [Cor90] and [She95]. Recently an optimal bound of r and i in the theorem above was obtained in [AB04].

The restriction of each semi-concave function on X to Z is semi-concave with respect to the inner metric of Z. A closer look on this object reveals:

Corollary 1.7. *Let Z be a closed $(C, 2+\epsilon, \rho)$-convex subset of a $CAT(\kappa)$ space X, for some $\rho, \epsilon > 0$. Then Z is locally convex.*

Remark 1.5. The proof of this fact (Lemma 6.3) shows that there is no meaningful analogue of the notion of quasigeodesics in spaces with an upper curvature bound (compare [PP94],6.1).

Remark 1.6. In [Lyt04b] we prove that a subset Z as above is a $CAT(\tilde{\kappa})$ space with respect to the inner metric for some $\tilde{\kappa}(\kappa, C)$.

Remark 1.7. For Riemannian manifolds the connection between sets of positive reach and semi-concave functions was discovered in [Kle80] and [Ban82].

The situation in spaces with a lower curvature bound is (as usual) much more complicated. We cannot say much about the geometry of (C, α)-convex subsets. The only thing we show in this direction is that positive reach implies $(C, 2)$-convexity. Actually the positive reach condition does no make much sense in spaces with a lower curvature bound, since usually even points do not have positive reach. The following definition seems to be more appropriate:

Definition 1.3. Let X be space with a lower curvature bound, $r, \epsilon > 0$. We say that a closed subset Z of X has ϵ-almost reach r if for each point $x \in X$ and each p in Z with $d(x, Z) = d(x, p) < r$ there is a point $\bar{x} \in X$ with $d(p, \bar{x}) = d(Z, \bar{x}) = r$ and such that $\angle xp\bar{x} \leq \epsilon$.

We prove:

Proposition 1.8. *Let X be a proper space with lower curvature bound k. Let $r > 0, 0 < \epsilon < \frac{\pi}{2}$ be given. Then there is some $C = C(k, r, \epsilon)$ and $\rho = \rho(k, r, \epsilon)$, such that each closed subset Z in X of ϵ-almost reach r is $(C, 2, \rho)$-convex.*

Remark 1.8. For manifolds this result (together with Theorem 1.3) says that almost positive reach implies positive reach. We could not find a direct argument for this fact.

1.5. **The way.** After preliminaries we discuss in Section 3 basics about almost geodesicity and prove Proposition 1.1. In Section 4 we discuss implications of curvature bounds on almost midpoints. In Section 5 a proof of Proposition 1.8 is given. In Section 6 we study (C, α)-embedded subsets in $CAT(\kappa)$ spaces. Finally in Section 7 we study almost convex subsets in Riemannian manifolds and prove Theorem 1.2, Theorem 1.3 and Proposition 1.4.

1.6. **Aknowlegments.** I am grateful to Werner Ballmann for helpful comments. I would like to thank Joseph Fu for some explanations about sets of positive reach and for bringing the papers of Alexander, Berg and Bishop to my attention. I am grateful to Stephanie Alexander for her interest in my work.

2. Preliminaries and Notations

2.1. **Notations.** By $\mathbb{R}^n$ we denote the Euclidean n-dimensional space. By M^2_κ we denote the two-dimensional simply connected Riemannian manifold with constant curvature κ.

We shall denote by d the distance in metric spaces. For a subset A of a metric space X we denote by d_A the distance function to the set A. By $B_r(A)$ resp. $U_r(A)$ we denote the set of all points $x \in X$ with $d_A(x) = d(x, A) \leq r$ resp. $d(x, A) < r$. By tX we denote the space X with the metric scaled by the factor t.

For a curve γ in X we denote its length by $L(\gamma)$. A geodesic or more precisely an X-geodesic is an isometric embedding of an interval in X. We call X a geodesic space if each two points in X can be connected by a geodesic. A subset Z of a metric space X is called convex if Z is geodesic with respect to the induced metric. A metric space X is proper if its closed bounded subsets are compact. A midpoint between two points x_0, x_1 in a metric space X is a point m with $d(m, x_i) = \frac{1}{2}d(x_0, x_1)$ for $i = 0, 1$.

2.2. **Hölder differentials.** Let $0 < \alpha \leq 1$ be a real number. A map $f : U \to \mathbb{R}^m$ of an open subset $U \subset \mathbb{R}^n$ is called $\mathcal{C}^{1,\alpha}$ if it is $\mathcal{C}^1$ and the differential $Df : U \to \mathbb{L}(\mathbb{R}^n, \mathbb{R}^m)$ is locally α-Hölder, i.e. for each compact subset K of U there is a constant $C(K)$ such that for all $x, y \in K$ and all unit vectors $v \in \mathbb{R}^n$ the inequality $|D_x f(v) - D_y f(v)| < C|x - y|^\alpha$ holds.

The following lemma allows a metric approach to $\mathcal{C}^{1,\alpha}$ maps. We refer to [CH70],L.2.1 for an easy proof.

Lemma 2.1. *Let $f : U \to \mathbb{R}^m$ be a map. Then f is $\mathcal{C}^{1,\alpha}$ iff for each compact subset K of U, there is some $C > 0$ such that for each x in K and all small $h \in \mathbb{R}^n$ one has $|f(x+h) - 2f(x) + f(x-h)| \leq C|h|^{1+\alpha}$.*

For $\mathcal{C}^{1,\alpha}$ manifolds M and N and a family $\mathcal{F}$ of maps $f_j : U_j \to N$ defined on open subsets $U_j \subset N$ is called locally uniformly $\mathcal{C}^{1,\alpha}$ if it is locally uniformly Lipschitz and for relatively compact charts in M and N the constant C of Lemma 2.1 can be chosen uniformly for all $f_j \in \mathcal{F}$. We refer to [LY04] for more on this subject.

2.3. **Spaces with a curvature bound.** We say that a complete geodesic space X is a $CAT(\kappa)$ space resp. has curvature $\geq \kappa$, if all triangles in X are not thicker resp. not thinner than the corresponding triangles in M^2_κ. We refer to [BH99] and [BBI01] for the theory of such spaces.

2.4. **Semi-concave functions.** We refer to [PP94],[AB03] and [Lyt04c] for the theory of semi-concave functions. A function $f : \mathbb{R} \to \mathbb{R}$ is called λ-concave if $f(t) + \lambda t^2$ is a concave function. A locally Lipschitz map $f : X \to \mathbb{R}$ of a geodesic space X is called λ-concave if the restriction of f onto each geodesic is λ-concave. A map $f : X \to \mathbb{R}$ is called λ-convex if $-f$ is $(-\lambda)$-concave.

A 1-Lipschitz map $f : X \to \mathbb{R}$ is λ-convex if and only if, for all points $x_0, x_1 \in X$ and each midpoint m between x_0 and x_1 one has $f(x_0) + f(x_1) - 2f(m) \geq -2\lambda d(x_0, x_1)^2$.

A map $f : U \to \mathbb{R}$ defined on an open subset U of X is called semi-convex (concave), if each point $x \in U$ has a neighborhood V such that f is λ-convex on each geodesic in V for some $\lambda = \lambda(V)$.

Example 2.1. A map $f : \mathbb{R}^n \to \mathbb{R}$ is semi-concave and semi-convex iff f is of class $\mathcal{C}^{1,1}$.

The absolute gradient of function $f : U \to \mathbb{R}$ at the point $x \in U$ is given by $|\nabla_x f| = \max\{0, \limsup_{\bar{x} \to x} \frac{f(\bar{x}) - f(x)}{d(\bar{x}, x)}\}$. Let $f : U \to \mathbb{R}$ be a λ-concave function. We say that the sublevel set $U_t := f^{-1}([t, \infty))$ is (C, r)-regular if for each $z \in X$ with $f(z) < t$ and $d(z, U_t) \leq r$ one has $|\nabla_z f| \geq C$. The sublevel set U_t is called regular if it is (C, r)-regular for some $C, r > 0$.

2.5. **Riemannian manifolds.** We refer to [BN93] for the theory of spaces with two-sided curvature bounds. We will denote by $\mathcal{M}_{n,i,\kappa}$ the class of all complete manifolds with curvature bounded from below resp. from above by $-\kappa$ resp. by κ and injectivity radius bounded below by $\geq i$. If you do not know [BN93] just assume that all manifolds in question are smooth (or at least C^3 with a C^2 Riemannian tensor).

2.6. **Positive reach and unique footpoints.** Recall from [Fed59]:

Definition 2.1. A closed subset Z in a metric space X is said to have UFP (unique footpoint property) if for all $x \in X$ there is a unique point $z \in Z$ with $d(z, x) = d(Z, x)$.

Definition 2.2. We say that a closed subset Z in a proper geodesic space X has reach $\rho > 0$, if for each $x \in X \setminus Z$, each $p \in Z$ with $d(x, p) = d(x, Z) < \rho$ there is some $y \in X$ with $\rho = d(p, y) = d(p, x) + d(x, y) = d(y, Z)$.

Federer observed in [Fed59] that a subset Z of a smooth Riemannian manifold has reach $\geq r$ iff the subset Z has the unique foot point property in the tubular neighborhood $U_r(Z)$ around Z.

2.7. **Ultralimits.** When dealing with non-proper spaces we will use ultralimits of a sequence of pointed metric spaces (X_i, x_i) that we will denote by $\lim_\omega(X_i, x_i)$. We refer to [BH99] or [Lyt04a] for more on ultralimits. If you do not like this concept just assume that all spaces are proper and replace the ultalimit by a limit in the

Gromov-Hausdorff topology. An ultralimit is always a complete space and an ultralimit of inner metric spaces is a geodesic space.

3. Almost convexity

3.1. **Basics.** We start with:

Definition 3.1. For $0 \leq r < 1$ we call a point m an r-almost midpoint between points x_0 and x_1 in a metric space X, if $d(x_i, m) \leq \frac{1}{2} d(x_0, x_1)(1+r)$ holds.

For $r = 0$ we get usual midpoints. Observe that $d(x_i, m) < d(x_0, x_1)$. Under rescaling of the metric r-almost midpoints remain r-almost midpoints. Moreover r-almost midpoints converge to r-almost midpoints under an ultraconvergence of spaces.

Recall Definition 1.1 and Definition 1.2 from the introduction. We immediatly see:

Lemma 3.1. *If X is (C, α, ρ)-geodesic then the completion of X is also (C, α, ρ)-geodesic. For $t > 0$ the space tX is $(t^\alpha C, \alpha, t\rho)$-geodesic. If (X_i, x_i) is a sequence of (C_i, α, ρ_i)-geodesic spaces with $C_i \to C < \infty$ and $\rho_i \to \rho > 0$, then the ultralimit $(X, x) = \lim_\omega (X_i, x_i)$ is (C, α, ρ)-geodesic. A direct product of (C, α, ρ)-geodesic spaces is (C, α, ρ)-geodesic.*

Moreover in all the statements one can replace (C, α, ρ)-geodesic by (C, α, ρ)-almost midpoint.

Example 3.1. The set of all closed (C, α, ρ)-convex subsets of a given proper metric space Z is closed in the pointed Hausdorff topology.

Example 3.2. If X is (C, α, ρ)-geodesic for some $\alpha > 0$ then for each sequence $x_i \in X$ and each sequence $t_i \to 0$ the blow-up $\lim_\omega(\frac{1}{t_i} X, x_i)$ is a geodesic space.

Example 3.3. Let X be a (C, α, ρ)-geodesic space. Then for all $0 < \alpha' < \alpha$ and each $C' > 0$ there is some $\rho' = \rho'(C, \alpha, \rho, C', \alpha') > 0$ such that X is (C', α', ρ')-geodesic.

Example 3.4. Let X be a (C, α, ρ)-geodesic space. Each convex subset of X with respect to the inner metric d^X is a (C, α, ρ)-geodesic space with respect to the induced metric. In particular this holds for each d^X-geodesic.

Example 3.5. Let X be a (C, α, ρ)-geodesic space. Let Z be a (C', α', ρ')-convex subset of (X, d^X). If $\alpha, \alpha' > 0$ then an easy computation shows that Z is a (C'', β, ρ'')-convex subset of X for some constants $C'', \rho'' > 0$, where $\beta = \min\{\alpha, \alpha'\}$.

Example 3.6. Let X be a geodesic metric space, CX the Euclidean cone over X. Then X considered as the unit sphere in CX is a $(1, 2, \rho)$-embedded, for some sufficiently small ρ, as one sees by an easy computation in the Euclidean plane. The constant $C = 1$ is certainly not optimal (see [AB04]).

The next example shows that in Banach spaces a subset can be (C, α)-convex for each $\alpha \geq 0$ and some $C = C(\alpha) > 0$ without being locally convex (compare Corollary 1.7).

Example 3.7. Consider the vector space $\mathbb{R}^2$ with the norm given by $|(x_1, x_2)| = |x_1| + |x_2|$. Let $f : [0, 1] \to \mathbb{R}^+$ be a monoton $\mathcal{C}^\infty$ function with $f(0) = 0, f(1) = 1$ and such that all derivatives vanish at 0. The graph $\gamma(t) = (t, f(t))$ parametrized by the length is a geodesic. For $0 < t \leq 1$ set $\gamma(-t) = (-t, f(t))$. The curve $\gamma : [-1, 1] \to \mathbb{R}^2$ consists of two geodesics glued together at the origin. The trace Γ of γ is not convex, however for arbitrary big α one can find $C, \rho > 0$, such that Γ is (C, α, ρ)-convex.

3.2. **Connectivity.** We are going to study connectivity properties of (C, α, ρ)-almost midpoint spaces.

Lemma 3.2. *For $C < 1$, let X a be complete $(C, 0, \rho)$-almost midpoint space. Then arbitrary points x_0, x_1 with $s = d(x_0, x_1) < \rho$ in X can be connected by a continuous curve. In particular connected components of X have distance $\geq \rho$ from each other.*

Proof. Consider some C-almost midpoint m between x_0 and x_1 and denote it by $x_{\frac{1}{2}}$. Choose C-almost midpoints $x_{\frac{1}{4}}$ resp. $x_{\frac{3}{4}}$ between x_0 and $x_{\frac{1}{2}}$ resp. between $x_{\frac{1}{2}}$ and x_1. Continuing in this fashion we define for each dyadic number $0 < t = \frac{m}{2^n} < 1$ a point $x_t \in X$. By induction on n we see $d(x_{\frac{m}{2^n}}, x_{\frac{m+1}{2^n}}) < \frac{s(1+C)^n}{2^n}$.

Thus the map $f : D \to X$ of the dyadic numbers $D \subset [0, 1]$ given by $f(t) = x_t$ is uniformly continuous and since X is complete, it can be extended to a continuous map $f : [0, 1] \to X$ connecting x_0 and x_1. □

Remark 3.8. From the proof we actually can deduce that the curve $f : [0,1] \to X$ defined above is q-Hölder and satisfies $d(f(x), f(z)) \leq sd(x,z)^q$ with $q = \frac{\ln(2)-\ln(1+C)}{\ln(2)}$. It is easy to show that the existence of such curves between arbitrary points x_0, x_1 with $d(x_0, x_1) = s < \rho$ is also sufficient for being $(C, 0, \rho)$-geodesic with $C = 2^{1-q} - 1$.

3.3. **Connectivity, $\alpha > 0$.** The case $\alpha > 0$ is much more interesting. First an easy observation. Let $\epsilon > 0, a < 1$ be given. By taking logarithms of both sides we observe, that for each natural n the inequality $\prod_{i=1}^{n}(1 + \epsilon a^i) \leq e^{\frac{\epsilon}{1-a}}$ holds. This implies:

Lemma 3.3. *For each $1 > \bar{a} > 0$ there is some $\bar{\epsilon} > 0$, such that for all $0 < a < \bar{a}$ and all $\epsilon < \bar{\epsilon}$ we have $\prod_{i=1}^{n}(1 + \epsilon a^i) \leq 1 + \frac{2\epsilon}{1-a}$.*

Now we are going to show Proposition 1.1:

Proof of Proposition 1.1. Set $\bar{a} = (\frac{(1+\frac{1}{2})}{2})^{\alpha}$. Choose $\bar{\epsilon} = \bar{\epsilon}(\bar{a}) < \frac{1}{2}$ as in Lemma 3.3. Let now $s < \rho$ be such that $\epsilon = Cs^{\alpha} < \bar{\epsilon}$.

Let x_0, x_1 be arbitrary points in X with $d(x_0, x_1) = s$. For each dyadic number $t = \frac{m}{2^n}$ with odd m we define as in the proof of Lemma 3.2 a point x_t as an r-almost midpoint between $x_{\frac{m+1}{2^n}}$ and $x_{\frac{m-1}{2^n}}$ with $r = Cd(x_{\frac{m+1}{2^n}}, x_{\frac{m-1}{2^n}})^{\alpha}$. For $n \in \mathbb{N}$ denote by d_n the maximum of the distances $d(x_{\frac{m}{2^n}}, x_{\frac{m+1}{2^n}})$. The construction implies $d_n \leq \frac{d_{n-1}}{2}(1 + Cd_{n-1}^{\alpha})$.

As in Lemma 3.2 we know $d_n < \frac{s(1+\epsilon)^n}{2^n}$. Now we see $d_n \leq \frac{1}{2}d_{n-1}(1 + Cd_{n-1}^{\alpha}) \leq \frac{1}{2}d_{n-1}(1 + Cs^{\alpha}((\frac{(1+\epsilon)}{2})^{\alpha})^n)$. Set $a = (\frac{(1+\epsilon)}{2})^{\alpha}$. From the above inequality we get by induction: $d_n \leq \frac{s}{2^n}(1 + \epsilon a^n)(1 + \epsilon a^{n-1})...(1 + \epsilon a)$. By Lemma 3.3 we obtain $2^n d_n < s(1 + \frac{2\epsilon}{1-a}) = s(1 + \frac{2Cs^{\alpha}}{1-a})$. Hence setting $\bar{C} = \frac{2C}{1-a}$ we see, that the curve f is $s(1 + \bar{C}s^{\alpha})$-Lipschitz. □

Remark 3.9. The proof shows that $\bar{C}$ depends linearly on C.

Finally we discuss a useful extension of Proposition 1.1:

Lemma 3.4. *Let X be a complete metric space, $C, \rho > 0, \alpha > \beta > 0$. Assume that for all $x_0, x_1 \in X$ with $s = d(x_0, x_1) < \rho$ there is a Cs^{β}-almost midpoint $m \in X$ between x_0 and x_1 satisfying $d(x_0, m) + d(x_1, m) \leq s(1 + Cs^{\alpha})$. Then X is $(\bar{C}, \alpha, \bar{\rho})$-geodesic for some $\bar{C}, \bar{\rho} > 0$.*

Proof. The already common construction using almost midpoints gives us for points $x_0, x_1 \in X$ with $s = d(x_0, x_1) \leq \rho$ an L-Lipschitz curve $f : [0,1] \to X$ connecting x_0 and x_1, with $L < 2s$ and such that for each n and each odd $m < 2^n$ one has $d(f(\frac{m-1}{2^n}), f(\frac{m}{2^n})) + d(f(\frac{m}{2^n}), f(\frac{m+1}{2^n}))$
$\leq d(f(\frac{m-1}{2^n}), f(\frac{m+1}{2^n}))(1 + Cd((\frac{m-1}{2^n}), f(\frac{m}{2^n}))^\alpha)$.

To estimate the length of this curve we denote by t_n the partial sum $\Sigma_{m=1}^{2^n} d(f(\frac{m-1}{2^n}), f(\frac{m}{2^n}))$.

From above we get $t_{n+1} \leq t_n + C\Sigma_{m=1}^{2^n} d(f(\frac{m-1}{2^n}), f(\frac{m}{2^n}))^{1+\alpha}$. Now we can estimate the last sum by $Ct_n(\frac{L}{2^n})^\alpha = \tilde{C}t_n s^\alpha(\frac{1}{2^\alpha})^n$. Thus we derive $t_{n+1} \leq t_n(1 + \tilde{C}s^\alpha(\frac{1}{2^\alpha})^n)$. Applying Lemma 3.3 and arguing as in the proof of Proposition 1.1 we can estimate the length of the curve f by $s(1 + \bar{C}s^\alpha)$. $\square$

4. Almost midpoints under geometric assumptions

4.1. **Constant curvature.**

The subsequent estimations follow from the trigonometry in the space form M^2_κ.

Lemma 4.1. *For each sufficiently small $\epsilon > 0$, there is some $D = D(\epsilon) > 0$, such that for each triangle xyz in M_2^κ with sidelengthes $|xy| = a, |xz| = b, |yz| = c$ of perimeter $a + b + c < D$ the following estimations for the midpoint m of the side xy do hold:*

1. *If $\kappa = -1$ and $d(m, z) < \epsilon a$ then $b + c \leq a(1 + 4\epsilon^2)$;*
2. *If $\kappa = 1$ and $b, c \leq \frac{a}{2}(1 + \epsilon)$, then $d(z, m) \leq \frac{a}{2}\sqrt{\epsilon}$;*
3. *If $\kappa = 1$ and $\angle yxz \geq \frac{\pi}{2}$ then $c^2 \geq a^2 + b^2 - \epsilon b^2$.*
4. *If $\kappa = -1$ and $\angle yxz \leq \frac{\pi}{2}$ then $c^2 \leq a^2 + b^2 + \epsilon b^2$.*
5. *If $\kappa = -1$ and $\angle yxz \leq \frac{\pi}{2} - \rho$ for some $\rho > 0$ then there is a constant $K = K(\rho) > 0$, such that $c^2 \leq a^2 + b^2 - Kab$*

4.2. **Curvature bounds.**

Lemma 4.1 allows us now to compare almost midpoints with midpoints in space with curvature bounds. Namely rescaling and using comparison triangles one directly gets from Lemma 4.1(1)-(2) the next two corollaries:

Corollary 4.2. *For each $\kappa \in \mathbb{R}$ there is some $\rho > 0$ such that for points x_0, x_1 with $d(x_0, x_1) < \rho$ in a space with curvature $\geq \kappa$, each midpoint $\bar{m}$ between x_0 and x_1 and each point m satisfying*

$d(m, \bar{m}) \leq rd(x_0, x_1)$ *one has* $d(x_0, m) + d(x_1, m) - d(x_0, x_1) \leq 4r^2 d(x_0, x_1)$ *if* r *is small enough.*

Corollary 4.3. *For each* $\kappa \in \mathbb{R}$ *there is some* $\rho > 0$ *such that for points* x_0, x_1 *in a* $CAT(\kappa)$ *space* X *with* $d(x_0, x_1) < \rho$*, all small* r*, each* r*-almost midpoint* m *and the midpoint* $\bar{m}$ *between* x_0 *and* x_1 *we have* $d(m, \bar{m}) \leq \frac{\sqrt{r}}{2} d(x_0, x_1)$*.*

4.3. **Manifolds.** In spaces with a lower and an upper curvature bound Corollary 4.2 and Corollary 4.3 hold true. In particular it is the case for each manifold $M \in \mathcal{M}_{n,i,\kappa}$.

5. Lower curvature bound

We are going to prove Proposition 1.8 in this section. Let X be a space with curvature $\geq \kappa$. Let Z be a closed subset of X. Assume first that for some $C, \rho, \alpha > 0$ and all $x_0, x_1 \in Z$ with $s = d(x_0, x_1) \leq \rho$ there is a midpoint $\bar{m} \in X$ between x_0 and x_1 and a point $m \in Z$ with $d(m, \bar{m}) \leq Cs^{1+\alpha}$. Then from Corollary 4.2 and Lemma 3.4 we see that Z is a $(\bar{C}, 2\alpha, \bar{\rho})$-convex subset of X for some $\bar{C}, \bar{\rho} > 0$ depending only on ρ, C, κ and α.

Proof of Proposition 1.8. Rescaling X we may assume that the curvature bound is -1. Let z_0, z_1 be points in Z with $2s = d(z_0, z_1) < \epsilon(r) << r$. Let $\bar{m}$ a midpoint between z_0 and z_1. Let m be a point on Z with $d(m, \bar{m}) = d(Z, \bar{m})$. Choose a point $x \in X$ with $d(x, Z) = d(x, m) = r$ and $\angle x m \bar{m} \leq \rho$.

Set $a = d(m, \bar{m}), b = d(x, \bar{m})$. For $i = 0$ or for $i = 1$ we have $\angle x \bar{m} z_i \leq \frac{\pi}{2}$. Since $r = d(x, Z) \leq d(x, z_i)$ we get from Lemma 4.1(4): $r^2 \leq b^2 + s^2 + \epsilon s^2$. On the other hand Lemma 4.1(5) and the assumption $\angle x m \bar{m} \leq \rho$ give us $b^2 \leq r^2 + a^2 - Kar$, for some $K = K(\rho)$.

Altogether we get $a^2 - Kar + (1 + \epsilon)s^2 \geq 0$, hence $ar \leq \tilde{K} s^2$. Therefore $a \leq Cs^2$ with $C = \frac{\tilde{K}}{r}$. Thus $d(m, \bar{m}) \leq Cs^2$ and the argument preceeding the proof shows that the subset Z is $(\bar{C}, 2, \bar{\rho})$-convex in X. □

6. Upper curvature bound

6.1. **Arbitrary $\alpha > 0$.** Let X be a $CAT(\kappa)$ space. By rescaling we may and will assume $\kappa = 1$. Let Z be a $(C, 2\alpha, \rho)$-convex subset of X. We will assume from now on that all points in question have distances $<< \rho$ and forget about ρ. Consequently we will write $(C, 2\alpha)$-convex instead of $(C, 2\alpha, \rho)$-convex.

Let $\gamma : [0, a] \to Z$ be a (Z, d^Z)-geodesic. Observe that γ is also $(C, 2\alpha)$-convex in X. Set $x_t = \gamma(t)$. We have $|t - s| \geq d(x_t, x_s) \geq |t-s|(1 + C|t-s|^{2\alpha})$. From the comparison triangle of $x_0 x_t x_{2t}$ we derive for the angle at x_0: $\angle x_t x_0 x_{2t} \leq \bar{K} t^\alpha$ for a fixed $\bar{K}$. This implies, that γ has a unique initial direction $v \in C_{x_0}X$ and for some constant K and all small t the angle between v and the initial direction of the X-geodesic $x_0 x_t$ is at most $K t^\alpha$. This implies, that the angle between each two Z-geodesics γ_1, γ_2 starting at $z \in Z$ is well defined and coincides with the angle between these two curves in X.

We will show now that there are no small closed geodesics in Z:

Lemma 6.1. *Let $Z \subset X$ be as above. Let $\gamma : tS^1 \to (Z, d^Z)$ be an isometric embedding of a circle. Then $t \geq \epsilon(C) > 0$.*

Proof. Consider γ as a map $\gamma : [0, 2\pi t] \to Z$ with $\gamma(0) = \gamma(2\pi t) = x$. Let η be the X-geodesic between x and $\gamma(\pi t)$. Denote by $\eta^+ \in C_x X$ the starting direction of η and by $\gamma^+ \in C_x X$ resp. $\gamma^- \in C_x X$ the starting resp. the ending direction of γ. From above we get $\angle(\gamma^-, \gamma^+) = \pi$ and $\angle(\gamma^\pm, \eta^+) \leq K(\pi t)^\alpha$. Hence $\pi t \geq (\frac{K}{2})^{\frac{1}{\alpha}}$. ∎

Remark 6.1. Assume now in addition that X is proper and geodesically complete. Then $C_x X$ is naturally isometric to $T_x X = X_x^{(t_i)}$ for each sequence $t_i \to 0$. In this case $Z_x^{(t_i)}$ coincides with the closure of the cone CV in $T_x X$ over the set V of all starting directions of Z-geodesics. Hence the tangent space $T_x Z \subset T_x X$ is well defined. Remark that $T_x Z$ coincides with the tangent cone $T_x Z$ at x to Z when Z is considered with the inner metric. By Example 3.2 the tangent cone $T_x Z \subset T_x X$ is a convex subcone of $T_x X$. It is easy to deduce as in [Lyt04a],10.5 that a $(C, 2\alpha)$-embedded subset of a proper geodesically complete $CAT(\kappa)$ space X is geometric with respect to the inner metric, i.e. the first formula of variation holds in (Z, d^Z).

6.2. **The case $\alpha \geq 2$ and semi-concave functions.** Let X be a $CAT(\kappa)$ space, Z a $(C,2)$-convex subset. First we are going to investigate the behavior of semi-concave functions on X restricted to Z. Let namely $f : X \to \mathbb{R}$ be an L-Lipschitz and λ-concave function. Let γ be an arbitrary (sufficiently short) Z-geodesic. Then γ is $(C,2)$-convex too. Let t,s be in the interval of definition of γ and let $\bar{m}$ be the X-midpoint between $\gamma(t)$ and $\gamma(s)$. Then using Corollary 4.3 we see that for $m = \gamma(\frac{t+s}{2})$ one has $d(m,\bar{m}) \leq \frac{\sqrt{C}}{2} d(\gamma(s),\gamma(t))^2$. Hence $f(m) \geq f(\bar{m}) - Ld(m,\bar{m}) \geq \frac{f(\gamma(t))+f(\gamma(s))}{2} + \lambda(s-t)^2 - \frac{\sqrt{C}}{2} L(s-t)^2$.

We have proved:

Lemma 6.2. *Let X be a $CAT(\kappa)$ space, Z a $(C,2)$-embedded subset and assume that (Z,d^Z) is locally geodesic. Let $f : X \to \mathbb{R}$ be a semi-concave function. Then the restriction $\bar{f}$ of f to (Z,d^Z) is semi-concave too. More precisely if f is L-Lipschitz and λ-concave at x then $\bar{f}$ is $\lambda - \frac{\sqrt{C}}{2}L$-concave at x.*

Remark 6.2. The assumption that (Z,d^Z) is locally geodesic is satisfied if X is a proper space. In [Lyt04b] we show that it is always fulfilled.

Assume now $\alpha > 2$ and let Z be a (C,α)-convex subset of a $CAT(\kappa)$ space X. Replacing X by its ultraproduct X^ω and Z by $Z^\omega \subset X^\omega$ we may assume that (Z,d^Z) is a geodesic space (this step is superficial if X is proper). The subset Z is $(\epsilon,2,\rho)$-embedded for each $\epsilon > 0$ and sufficiently small $\rho = \rho(\epsilon)$ (due to Example 3.3). The last lemma shows that each L-Lipschitz semi-convex function f restricted to Z is still semi-convex. Moreover if f is λ-convex then the restriction of f to Z is again λ-convex. Therefore to prove Corollary 1.7 it is enough to verify the following

Lemma 6.3. *Let γ be a curve parametrized by the arclength in a $CAT(\kappa)$ space X. If the restriction of each λ-convex function $f : X \to \mathbb{R}$ onto γ is still λ-convex, then γ is a local geodesic.*

Proof. Assume again that $\kappa = 1$ and set $x = \gamma(a)$ and $y = \gamma(b)$ where $|b - a| \leq \epsilon << \frac{\pi}{2}$. Let l be the X-geodesic between x and y. We consider the distance function $f = d_l : X \to \mathbb{R}$ and the restriction $\bar{f} : [a, b] \to \mathbb{R}$ of d_l to γ. The distance function to the convex subset l in the $CAT(1)$ space X satisfies $d_l'' \geq -2d_l$ (in the weak sense, when restricted to a geodesic) at all points with distance $\leq \epsilon$ to l. Thus we get $\bar{f}'' \geq -2\bar{f}$. On the other hand we have $f(a) = f(b) = 0$. A standard comparison argument shows that $|b-a| \leq \epsilon$ implies that f must be constant 0. Thus l coincides with γ. $\square$

6.3. **Geometric properties.** We are going to prove Theorem 1.6 in this subsection.

Proof of Theorem 1.6. Thus let X be a $CAT(\kappa)$ space and Z a closed $(C, 2)$-convex subset. As usual we assume that $\kappa = 1$.

In the first step we are going to show that X locally has the unique foot point property. Let $r_0 << \rho$ be sufficiently small, $x \in X$ a point with $r_0 = d(x, Z)$. Let z_n be a sequence in Z with $d(x, z_n) \to r_0$. Assume that z_n is not a Cauchy sequence. Then we find subsequences p_n, q_n with $d(x, p_n) < r_0 + \frac{1}{n}$ and $d(x, q_n) < r_0 + \frac{1}{n}$ and $s_n = d(p_n, q_n) > \epsilon$ for all n. Let $\bar{m} \in X$ be the midpoint between p_n and q_n. From the comparison triangle we get by Lemma 4.1(3): $d(x, \bar{m})^2 \leq (r_0 + \frac{1}{n})^2 - (\frac{s_n}{2})^2 + \epsilon^2 s_n^2$.

Take now a point m in Z with $d(m, \bar{m}) \leq \bar{C} s_n^2$. We see $d(x, m) \leq d(x, \bar{m}) + d(m, \bar{m}) \leq r_0 + \frac{1}{n} - \frac{s_n^2}{4r_0} + \bar{C} s_n^2$. If r_0 has been chosen such that $\frac{1}{4r_0} > \bar{C}$ we get a contradiction to $d(x, Z) = r$.

Thus z_n must be a Cauchy sequence. This shows that the map $P^Z : B_{r_0}(Z) \to Z$ that sends a point $x \in B_{r_0}(Z)$ to the point p in Z with $d(x, p) = d(x, Z)$ is well-defined and continuous.

Remark 6.3. Let $Z \subset X$ be as above. Then the ultraproduct X^ω is again a $CAT(\kappa)$ space and $Z^\omega \subset X^\omega$ is $(C, 2)$-convex. Remark that the restriction of d_{Z^ω} to $X \subset X^\omega$ coincides with d_Z and the restriction of the projection $P^{Z^\omega} : B_{r_0}(Z^\omega) \subset X^\omega \to Z^\omega$ to $B_{r_0}(Z) \subset X$ coincides with P^Z.

Now we are going to show that the projection P^Z is in fact much better than just continuous. Using the above remark we may assume that (Z, d^Z) is geodesic. Take $r_0 << \rho$ as above. Let $x \in X$ be a point with $d(x, Z) = r << r_0$. Let $y \in X$ be another point with $a = d(x, y) << r$. Set $p = P^Z(x)$ and $q = P^Z(y)$. We have to estimate $s = d(p, q)$. Consider a Z-geodesic γ between p and q. Let $\bar{\gamma}$ be the X-geodesic between p and q and let η_1 resp. η_2 be the geodesics between p and x resp. between q and y.

By the very definition of the projection we get $\angle(\gamma^+, \eta_1^+) \geq \frac{\pi}{2}$ and $\angle(\gamma^-, \eta_2^+) \geq \frac{\pi}{2}$. From Subsection 6.1 we derive $\angle(\gamma^\pm, \bar{\gamma}^\pm) \leq Ks$ for some $K = K(C)$. Hence $\angle(\bar{\gamma}^+, \eta_1^+) \geq \frac{\pi}{2} - Ks$ and $\angle(\bar{\gamma}^-, \eta_2^+) \geq \frac{\pi}{2} - Ks$.

Now by comparing of the quadrangle $\bar{\gamma}, \eta_1, \eta_2$ with the corresponding quadrangle in the round sphere M_1^2 we obtain $d(x, y) \geq s - 4Ksr$ if r is small enough. Hence $a \geq s(1 - Ar)$ for some $A = A(C)$. This shows that the Lipschitz constant of P^Z at the point x is not bigger than $(1 + A \cdot d(x, Z))$.

Let $r > 0$ be such that P^Z is a well defined L-Lipschitz function in $B_r(Z)$, for some constant $L = L(C)$. We are going to show that the distance function d_Z is semi-convex in $U_r(Z)$.

Again we assume that Z is geodesic. The question is local. Consider points x, y, p, q and the curves γ and $\bar{\gamma}$ as in the last step. Let m_0 be the midpoint of the X-geodesic between x and y, $\bar{m}$ the midpoint of $\bar{\gamma}$ and m the midpoint of γ. We have $d(m_0, \bar{m}) \leq \frac{1}{2}(d(x, p) + d(y, q)) + 2d(x, y)^2$. Moreover we have $d(\bar{m}, m) \leq \bar{C}d(p, q)^2 \leq L\bar{C}d(x, y)^2$. This shows $d_Z(m_0) \leq d(m_0, m) \leq \frac{1}{2}(d_Z(x) + d_Z(y)) + (2 + L\bar{C})d(x, y)^2$. Thus the function d_Z is $(1 + 2\bar{C})$-convex in $U_r(Z)$.

Observe finally the semi-concave function $-d_Z$ has Z as a regular sublevel set. This recovers the statement of [Kle80], if X is a Riemannian manifold and finishes the proof of Theorem 1.6. $\square$

Remark 6.4. The same argument as in the first step of the above proof shows that for all sufficiently small $r = r(C, \kappa) > 0$ for each $x \in X$ and all $z_0, z_1 \in B_r(x)$ each $(C, 2)$-midpoint between z_0 and z_1 is contained in $B_r(x)$. Hence the intersection of each $(C, 2)$-convex subset Z with $B_r(x)$ is again $(C, 2)$-convex, moreover this intersection is a convex subset of (Z, d^Z).

7. Differentiable properties

7.1. **Basics.** From now on let n, i, k be fixed and let $\mathcal{M}_{n,i,\kappa}$ be the space of Riemannian manifolds as in Subsection 2.5. Let $M \in \mathcal{M}_{n,i,\kappa}$ be a fixed manifold. Let finally $0 < \alpha \leq 1$ be a fixed number.

From Corollary 4.2 and Corollary 4.3 we deduce that a subset $Z \subset M$ is $(C, 2\alpha, \rho)$-convex iff for some $\rho > 0$ and some $\bar{C} > 0$ all points $x_0, x_1 \in Z$ with $s = d(x_0, x_1) < \rho$ there is a point $m \in Z$ such that for the distance to the midpoint $\bar{m} \in M$ between x_0 and x_1 one has $d(m, \bar{m}) \leq \bar{C} s^{1+\alpha}$.

We are going to prove Theorem 1.2 for the manifold M.

Proof of Theorem 1.2. Consider two points x_0, x_1 in Z with $s = d(x_0, x_1) < \rho << i$. Let γ be the M-geodesic between x_0 and x_1 and denote by m its midpoint. Consider the open ball $U = U_{\frac{i}{3}}(m)$. With respect to the distance coordinates it can be conidered as an open ball in $\mathbb{R}^n$ with a Lipschitz continuos Riemannian metric. Observe that the Euclidean distance on this ball is L-Bilipschitz to the M-distance, where L is some fixed constant. (Choosing the ball small one can assume that the constant is very close to 1). Since the M-geodesics are uniformly $C^{1,1}$ we see (from Lemma 2.1) that $||m - \frac{x_0+x_1}{2}|| \leq C_0 s^2$ for some fixed constant C_0 (that depends only on n, κ, i).

Assume that x_0, x_1 are connected by curve $\gamma : [0, t] \to Z$ parametrized by the arclength with $t \leq \bar{C} s$ whose $C^{1,\alpha}$ norm is bounded by $\bar{C}$. From Lemma 2.1 we derive $||\gamma(\frac{t}{2}) - \frac{x_0+x_1}{2}|| \leq \bar{C} t^{1+\alpha}$. Hence we obtain $d(m, \gamma(\frac{t}{2})) \leq C s^{1+\alpha}$ for some fixed C. Hence if such a curve exists for each pair of points in Z we obtain that Z is $(C, 2\alpha)$-convex.

Assume now that Z is $(C, 2\alpha)$-convex. Choose sufficiently close points $x_0, x_1 \in Z$ and consider a Z-geodesic connecting them. This geodesic is also $(C, 2\alpha)$-convex. For the midpoint $\bar{m}$ of this geodesic we derive from Corollary 4.3 $d(\bar{m}, m) \leq C s^{1+\alpha}$. Again from Lemma 2.1 we derive that this geodesic must be $C^{1,\alpha}$ with some fixed bound on the $C^{1,\alpha}$ norm. $\square$

Let M be again a manifold in $\mathcal{M} = \mathcal{M}_{n,i,\kappa}$; $Z \subset M$ a closed subset. We are going to establish the equivalences of Theorem 1.3

Proof of Proposition 1.3. By [Fed59] positive reach is equivalent to the local uniqueness of foot points (it follows directly from the geodesic completness and the existence of a lower curvature bound). Thus due to Proposition 1.8 and Theorem 1.6 the conditions (1) and (2) are equivalent. Due to [LY04] geodesics with respect to a Lipschitz continuous Riemannian metric are uniformly $\mathcal{C}^{1,1}$. Hence by Theorem 1.2 (3) implies (1).

Assume (2). By Theorem 1.6 for some constant $A > 0$ and some small $r > 0$ the foot point projection $P^z : B_r(Z) \to Z$ is locally Lipschitz and the Lipschitz constant at $x \in B_r(Z)$ is bounded by $1 + A \cdot d(x, Z)$. Consider on M the Riemannian metric $\tilde{g}$ (conformally equivalent to the metric g) given by $\tilde{g}_x = (1 + 2A \cdot d(x, Z))g_x$. The metric is by definition Lipschitz continuous and coincides with g on Z.

By d we will denote the metric on M induced by g. Let x_0, x_1 be points in Z with $d(x_0, x_1) \leq \frac{r}{2}$. Let γ be a geodesic between these two points with respect to the Riemannian metric $\tilde{g}$. If we choose r small enough we may assume that γ is contained in $B_r(Z)$. Consider the curve $\tilde{\gamma} = P^Z(\gamma)$.

We see $|\tilde{\gamma}'(t)|_{\tilde{g}} = |\tilde{\gamma}'(t)|_g \leq |\gamma'(t)|_g(1 + A \cdot d(\gamma(t), Z))$. On the other hand one has $|\gamma'(t)|_g \leq \frac{1}{1+2A \cdot d(\gamma(t),Z)} |\gamma'(t)|_{\tilde{g}}$. Hence $|\tilde{\gamma}'(t)|_{\tilde{g}} \leq \frac{1+A \cdot d(\gamma(t),Z)}{1+2A \cdot d(\gamma(t),Z)} |\gamma'(t)|_{\tilde{g}}$. Since γ is a geodesic with respect to $\tilde{g}$, this inequality implies $d(\gamma(t), Z) = 0$ for all t. Thus $\gamma = \tilde{\gamma}$, therefore Z is locally convex with respect to $\tilde{g}$ and we have (3). □

Finally using the main result of [Lyt04b] we are going to prove Proposition 1.4 now:

Proof of Proposition 1.4. We may assume that Z is connected. The subset Z is $(C, 2)$-convex for some $C > 0$, hence due to [Lyt04b] we know that Z is a $CAT(\kappa)$ space for some κ depending only on the bounds of the curvature of M and on C.

(1) certainly implies (2). The implication (2) $\to$ (3) holds in arbitrary $CAT(\kappa)$ spaces, because in a non-geodesically complete

$CAT(\kappa)$ some point x has arbitrary small neighborhoods U, such that $U \setminus \{x\}$ is contractible (compare [BH99]).

In a proper geodesically complete $CAT(1)$ space each tangent cone is geodesically complete too. Since tangent cones T_xZ to a subset of positive reach Z are convex subsets in T_xM (compare [Fed59] or Remark 6.1) and since the only geodesically complete convex subsets of a Euclidean space are Euclidean spaces, we see that (3) implies (4).

Assume (4). Then Z is geodesically complete, since otherwise some link S_xZ (= the unit sphere in the tangent cone T_xZ) must be contractible, that is not the case for unit spheres in Euclidean spaces. Let D be the maximal dimension of tangent spaces T_xZ. The set U of all points $x \in Z$ where the tangent space T_xZ is a D-dimensional Euclidean space is open in Z and it is a $\mathcal{C}^{1,1}$ submanifold of M, due to [Fed59]. On the other hand in a proper geodesically complete $CAT(\kappa)$ space Z the links vary semi-continuously in the following sense: if x_i converge to x in Z, then there is a (natural) surjective 1-Lipschitz map $p : S_x \to \lim(S_{x_i})$. This shows that if S_{x_i} are D-dimensional Euclidean spheres, then S_x cannot be a Euclidean sphere of dimension smaller than D. Thus the subset U is also closed in Z. Since we assumed Z to be connected, we derive $U = Z$. □

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