Centers of convex subsets of buildings

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Abstract. We prove that two dimensional convex subsets of spherical buildings are either buildings or have a center.

1. Introduction

Recently, convex cores of isometric group actions on symmetric spaces were studied in [7]. The results of Kleiner and Leeb imply the following: Assume that the group Γ acts by isometries on a symmetric space H in a non-elementary way (i.e. without fixed point at infinity). Then the convex core of Γ is a direct product of symmetric and Gromov-hyperbolic spaces. In particular the boundary at infinity of the convex core must be a spherical building with respect to the Tits metric.

This gives rise to the following more abstract question. Let Γ be a group acting by isometries on a spherical building G. Let X be a convex (by which we actually mean π-convex, compare section 2.2) subset of G invariant under Γ. Is it true that Γ must have a fixed point if X is not a building? In the case of small dimensions (of the building or at least the subset) we answer this question in the affirmative.

Theorem 1.1. *Let G be a spherical building, $X \subset G$ a convex subset. If $\dim(X) \leq 2$ then either X is itself a building or the isometry group of X has a fixed point in X.*

In the proof we use some observations about diameter and radius of general $CAT(1)$ spaces (see section 2.1 for the definitions), which we consider to be of independent interest. In important special situations they were obtained in [4] by different methods.

The next result is shown in [8, Thm. B] for $\mathrm{rad}(X) < \frac{\pi}{2}$ and in [4, Thm. 1.7] for $\mathrm{diam}(X) = \frac{\pi}{2}$.

Proposition 1.2. *Let X be a finite-dimensional CAT(1) space. If $\mathrm{rad}(X) < \pi$, then we have $\mathrm{diam}(X) > \mathrm{rad}(X)$.*

From this, an easy limiting argument gives the following uniform result, which is a consequence of [8, Thm. B] in the case when $\mathrm{rad}(X) < \frac{\pi}{2}$:

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Theorem 1.3. *For each natural number n and each $r < \pi$ there is some $\varepsilon = \varepsilon(r, n) > 0$, such that for each $CAT(1)$ space X of dimension at most n, of radius at most r with at least two points, we have* $\mathrm{diam}(X)/\mathrm{rad}(X) \geq 1 + \varepsilon$.

Remark 1.1. For $\frac{\pi}{2} \leq r < \pi$, this result can not be achieved by the methods of [8]. It is not clear to us what the optimal value of ε should be and what the spaces with smallest possible ratio $\mathrm{diam}(X)/\mathrm{rad}(X)$ look like.

In [8, Thm. B] it is shown that a CAT(1) space with radius $< \frac{\pi}{2}$ has a unique circumcenter, that is therefore fixed by the whole isometry group. In the compact case the next result is proved in [4, Prop. 5.7].

Proposition 1.4. *Let X be a CAT(1) space of finite dimension and of radius $r \leq \frac{\pi}{2}$. Then X has a circumcenter which is fixed by every isometry of X.*

Remark 1.2. In both propositions the assumption of finite dimensionality is essential, compare [4, pg. 5] or Example 3.1.

We explain the idea of the proof of Theorem 1.1 in the case where X is compact. In this case we consider a smallest closed convex subset Y of X invariant under the isometry group of X and one can assume that Y is not a building. Since Y is minimal one can not cut off any small neighborhood of any point $y \in Y$, such that the remaining subset is still convex (see Section 4 for more on the notion of non-removable points). In dimension 2 this implies that each point is an inner point of some geodesic. Now we consider two points $y, z \in Y$ with maximal distance in Y. If $d(y, z) = \pi$ then Y contains a circle and one can deduce that $\mathrm{rad}(Y) = \frac{\pi}{2}$. If $d(y, z) < \pi$, we show that one can choose geodesics through y resp. through z. Using these, we construct a spherical quadrangle in which one of the diagonals is longer than $d(y, z)$, a contradiction. The last argument in the proof uses spherical quadrangles (not triangles as usual) and breaks down if the dimension is bigger than 2.

We thank Bernhard Leeb for posing the question.

2. Preliminaries

2.1. Notations

By d we denote distances in metric spaces. By $B_r(x)$ we will denote the closed metric ball of radius r around the point x. For a point x

in a metric space X we set $\mathrm{rad}_x(X) := \sup_{z \in X} d(x,z)$. By the radius resp. the diameter of X we denote $\mathrm{rad}(X) := \inf_{x \in X} \mathrm{rad}_x(X)$ resp. $\mathrm{diam}(X) := \sup_{x \in X} \mathrm{rad}_x(X)$.

Since we are dealing with non-proper spaces we will use the concept of ultraconvergence (instead of the Gromov-Hausdorff convergence) with respect to a fixed non-principal ultrafilter ω. We refer to [3, pp. 77-80] for details, see also [9, sect. 11]. We will denote by $\lim_\omega(X_i, x_i)$ the ultralimit of pointed metric spaces X_i with respect to ω. By X^ω we will denote the ultraproduct of X, i.e. the ultralimit of the constant sequence (X, x).

2.2. CAT(1) spaces

A complete metric space is called CAT(1) if each pair of points with distance $< \pi$ is connected by a geodesic and all triangles of perimeter less than 2π are not thicker than in S^2. We refer to [3, ch. II] for the theory of such spaces. In a CAT(1) space X we will denote by S_x the link at the point x. A subset C of a CAT(1) space is convex if all points in C with distance $< \pi$ are joined by a geodesic in C.

By $\dim(X)$ we denote the geometric dimension of X studied in [5]. Ultralimits of CAT(1) spaces are CAT(1) and the dimension does not increase by this procedure ([9, L. 11.1]).

A subset T of a CAT(1) space X is spherical if it can be isometrically embedded into some Euclidean sphere. See [2] for more on this.

2.3. Buildings and their convex subsets

We refer to [6, sect. 3] for an account on spherical buildings. From the geometry of buildings we will only use the following basic property:

Let G be a building. Then for each $x \in X$ there is a number $r_x > 0$ (set r_x so that any point $\bar{x} \in X$ with $d(\bar{x}, x) < r_x$ lies in a Weyl chamber containing x), such that for all $x, y \in X$ and all $\bar{x} \in B_{r_x}(x)$ and $\bar{y} \in B_{r_y}(y)$ the convex hull of the four points $x, \bar{x}, y, \bar{y}$ is a spherical (usually three-dimensional) subset of G. Remark that this property is inherited by convex subsets of buildings. We refer to [2] for a more thorough study.

We will use the following result derived in [2]:

Lemma 2.1 ([2, 6.3 + 6.4]). *Let X be a convex subset of some building G. If X is not a building, then* $\mathrm{rad}(X) < \pi$. *If X contains an isometrically embedded S^{n-1} and* $\dim(X) \leq n$, *then either X is a building or* $\mathrm{rad}(X) = \frac{\pi}{2}$.

3. Radii of CAT(1) spaces

3.1. Generalities

The following lemma is a consequence of [9, L. 11.7], but we give a direct proof here:

Lemma 3.1. *Let X be an n-dimensional CAT(1) space, and let $S \subset X$ be an embedded S^n. Then for each $x \in X$ there is an antipode $y \in S$, i.e. a point satisfying $d(x, y) \geq \pi$. Therefore we have* $\mathrm{rad}(X) \geq \pi$.

Proof. If $\dim(X) = \dim(S) = 0$, the claim is clear. Choose an arbitrary $y \in S$. Let $v \in S_y$ be the starting direction of yx. By induction, we can choose an antipode w of v in $S^{n-2} = S_y S \subset S_y X$. Now we obtain an antipode of x by extending xy inside S in the direction of w. □

The proof of the next lemma is a typical application of ultraproducts.

Lemma 3.2. *Let X be a CAT(1) space. If $r = \mathrm{diam}(X) = \mathrm{rad}(X) < \pi$, then for each point $x \in X^\omega$ one has* $\mathrm{rad}(S_x(X^\omega)) \leq \frac{\pi}{2}$.

Proof. Let $x = (x_i)$. Choose $z_i \in X$ with $d(z_i, x_i) \geq r - \frac{1}{i}$. For $z = (z_i) \in X^\omega$ we get $d(z, x) = r = \mathrm{diam}(X) = \mathrm{diam}(X^\omega)$. From the CAT(1) property we immediately obtain that the starting direction $v \in S_x$ of the geodesic xz satisfies $d(v, w) \leq \frac{\pi}{2}$ for each $w \in S_x$. □

Proof of Proposition 1.2. In [5, Thm. B] it is shown that in each CAT(1) space X of dimension $n + 1$, some link S_x contains an n-dimensional Euclidean sphere. Using Lemma 3.2 and Lemma 3.1, we deduce Proposition 1.2. □

Proof of Theorem 1.3. Assume the contrary and choose a sequence of spaces X_i with $\mathrm{diam}(X_i)/\mathrm{rad}(X_i) \to 1$. Since we assume that the spaces have more than one point we may rescale X_i (without leaving the category of at most n-dimensional $CAT(1)$ spaces) and assume that the radius of all the spaces X_i is equal to r. For the ultralimit space $X = \lim_\omega(X_i)$ we obtain $\mathrm{rad}(X) = \mathrm{diam}(X) = r$ in contradiction to Proposition 1.2, since the dimension of X is bounded by n too. □

Example 3.1. The subset P of points in the Hilbertsphere with all coordinates non-negative satisfies $\mathrm{rad}(P) = \mathrm{diam}(P) = \frac{\pi}{2}$. Moreover, since permuting coordinates is an isometry, the isometry group of P has no fixed points, thus the assumption of finite-dimensionality is essential in Proposition 1.2 and Proposition 1.4. (This was also observed in [4, pg. 5]). Moreover P contains points x with $\mathrm{rad}(S_x P) = \pi$ (let x be a point with all components positive).

3.2. CAT(1) SPACES OF RADIUS $\frac{\pi}{2}$

We recall

Definition 3.1. A point x in a metric space X is a *circumcenter* of X if $\mathrm{rad}_x(X) = \mathrm{rad}(X)$ holds.

Due to [8, Thm. B] a CAT(1) space X of radius $< \frac{\pi}{2}$ has a unique circumcenter. As a first step, we extend the existence result:

Lemma 3.3. *Let X be a CAT(1) space of finite dimension and radius $\frac{\pi}{2}$. Then X has a circumcenter.*

Proof. By definition, we find a sequence of points $x_i \in X$ such that $\mathrm{rad}_{x_i}(X) \to \frac{\pi}{2}$. This sequence defines a circumcenter $x = (x_i)$ for the ultraproduct X^ω, which is a space of the same dimension as X ([9, Cor. 11.2]). We have $d(x, X) \leq \frac{\pi}{2}$. In the case of equality we have $d(x, y) = \frac{\pi}{2}$ for all $y \in X$, and the convex hull of x and X in X^ω would be isometric to the spherical join $X * \{x\}$ ([9, L. 4.1]). This contradicts $\dim(X^\omega) = \dim(X)$.

Hence, there is a unique projection x' of x to X ([3, II.2.6.1]). Triangle comparison shows that x' is a circumcenter of X. □

Proof of Proposition 1.4. The proof can be concluded in the same manner as in the proof of [4, Thm. 1.3]. The set X_1 of all circumcenters of X is (by the above) a non-empty subset of X, which is closed, convex and invariant under the isometry group. Moreover by definition the diameter of X_1 is not not bigger than $\mathrm{rad}(X) = \frac{\pi}{2}$.

Due to Proposition 1.2, the radius of X_1 is smaller than $\frac{\pi}{2}$ and due to [8, Thm. B], the space X_1 has a unique circumcenter, that is therefore fixed by the isometry group of X. □

4. Groups acting on a CAT(1) space

4.1. REMOVABLE POINTS

For $r > 0$ we will denote by $S_r(x)$ the sphere of radius r around x, i.e. the set of all points y with $d(y, x) = r$.

Remark 4.1. Do not confuse the sphere $S_r(x)$ with the link S_x.

Definition 4.1. Let X be a CAT(1) space and $r > 0$. A point $x \in X$ is called *r-removable*, if the closed convex hull of the sphere $S_r(x)$ does not contain x. It is called *removable* if it is r-removable for all $r > 0$. A point which is not removable is called *non-removable.*

Remark that if $x \in X$ is r-removable then it is r'-removable for all $r' > r$. A point $x \in X$ is removable iff it is removable inside some ball $B_r(x)$. The following examples should only illustrate the notion and will not be used in the proof of Theorem 1.1.

Example 4.2. In every geodesically complete CAT(1) space each point is non-removable.

Example 4.3. A point $x \in X$ is non-removable if it is inner point of some geodesic or, more general, if it is an inner point of some geodesic in the ultraproduct X^ω. If X has dimension 1 the last condition is also necessary.

Example 4.4. For every CAT(1) space Z, each point in the spherical join $X = Z * S^1$ is a midpoint of a geodesic of length π, hence it is not r-removable for $r < \frac{\pi}{2}$.

Example 4.5. A closed convex subset S of a finite-dimensional sphere has no removable points iff S has the form $S = Z * S^1$.

Example 4.6. Let X be a locally conical space ([2, sect. 3.2]) with $\dim(X) \leq 2$. A point $x \in X$ is non-removable iff x is an inner point of some geodesic $\gamma \subset X^\omega$ in the ultraproduct X^ω of X.

Example 4.7. Let X be a closed convex subset in a Riemannian manifold of constant curvature. A point $x \in X$ is not removable iff it is the inner point of some geodesic. We conjecture this statement to be true also if X is a Riemannian manifold with variable curvature.

Unfortunately it is difficult to say much about the behavior of convex hulls and therefore of removable points under ultralimits. The following easy observation will be used below

Lemma 4.1. *Let (X_i, o_i) be CAT(1) spaces, $(X, o) = \lim_\omega(X_i, o_i)$. Consider a point $x = (x_i) \in X$ such that for some fixed $s > 0$ the point x_i is not s-removable in X_i.*

Then for each $m \in X$ with $l = d(m, x) \leq \frac{\pi}{2}$, there is a point $q \in X$ with $d(q, x) = s$ and $d(m, q) \geq d(m, x)$.

Proof. We can represent m as a sequence of points $m = (m_i)$, with $m_i \in X_i$, $l_i = d(m_i, x_i) < \frac{\pi}{2}$. We may assume that $m \neq x$ and hence, that $l_i \not\to 0$. Since x_i is not s-removable, there must be at least one point q_i with $d(q_i, x_i) = s$, that is not contained in the closed convex ball $B_{l_i - \rho_i}(m_i)$, for each $\rho_i > 0$. If we choose $\rho_i \to 0$, then the point $q = (q_i) \in X$ has the desired properties. □

4.2. Minimal invariant subsets

Non-removable points are related to minimal invariant subsets by the following observation:

Lemma 4.2. *Let X be a CAT(1) space, $x \in X$, Γ a group of isometries of X. Assume that there is no proper closed, convex, and Γ-invariant subset of X containing x. Then each removable point of X is contained in the closure of the orbit $\overline{\Gamma x}$. More precisely if $d(z, \Gamma x) = \varepsilon$, then z is not r-removable for $r < \varepsilon$.*

Proof. Assume the contrary and consider the closed convex convex hull C of $X \setminus B_r(z)$. Then C does not contain z but $gx \in C$ for $g \in \Gamma$. Considering $\cap_{g\in\Gamma} gC$ we get a contradiction to the minimality of X. □

Remark 4.8. If X is a minimal closed convex subset invariant under Γ, then either X has no removable points or Γ has a dense orbit. If X is finite dimensional and not discrete one can show that in the latter case each point is non-removable as well.

4.3. Centers

We will stick to the following

Definition 4.2. We say that a CAT(1) space X has a *center* if its group of isometries $\Gamma^X = Iso(X)$ has a fixed point.

A group operating isometrically on a CAT(1) space X has a fixed point iff it has an orbit of diameter $< \frac{\pi}{2}$. This implies

Lemma 4.3. *Let X_i be a sequence of CAT(1) spaces. If none of the X_i has a center, then $X = \lim_\omega X_i$ has no center either.*

5. The Proof of the Main Theorem

For the proof of Theorem 1.1 the following lemma is crucial:

Lemma 5.1. *Let Y be a convex subset of a building,* $\dim(Y) \leq 2$. *Let $y, z \in Y$ be inner points of some geodesics in Y. If $d(y, z) = \operatorname{diam}(Y)$, then $d(y, z) = \pi$ and Y contains an isometric copy of S^1.*

Proof. Let y resp. z be inner points of geodesics p_1p_2 resp. q_1q_2. We may assume that $p_i \in B_{r_y}(y)$ resp. $q_i \in B_{r_z}(z)$ for $i = 1, 2$, where r_y resp. r_z are the conicality radii at y resp. at z (see Subsection 2.3). If $d(y, z) = \pi$, the convex hull of q_1q_2 and y is a circle S^1 (by the lune lemma [1, L. 2.5]).

Thus assume that $d(y, z) < \pi$. For the geodesic $\eta = yz$ we deduce from the assumption $d(y, z) = \operatorname{diam}(Y)$, that η meets yp_i and zq_j

orthogonally. From the geometry of buildings we derive (cf. Subsection 2.3 again) that the convex hulls of yp_i and zq_j are spherical for $i, j = 1, 2$. Since $\dim(Y) \leq 2$ these spherical hulls are 2-dimensional spherical quadrangles.

Now in the two dimensional sphere the following statement holds: For $i = 1, 2$ let $\bar{\gamma}_i$ be geodesics in S^2 starting in x_i with $d(x_1, x_2) < \pi$. Assume that $\bar{\eta} = x_1x_2$ meets $\bar{\gamma}_1$ and $\bar{\gamma}_2$ orthogonally. If $\bar{\gamma}_1$ and $\bar{\gamma}_2$ are on different sides of η, i.e. if $\angle_{x_1}(\bar{\gamma}_1(\varepsilon), \bar{\gamma}_2(\varepsilon)) > \frac{\pi}{2}$, then $d(\bar{\gamma}_1(\varepsilon), \bar{\gamma}_2(\varepsilon)) > d(x_1, x_2)$ for small $\varepsilon > 0$.

Therefore we get a contradiction to $d(y, z) = \operatorname{diam}(Y)$ as soon as we can verify that $\angle_y(p_i, q_j) > \frac{\pi}{2}$ for some i and j. However from the one-dimensionality of S_yY we directly obtain that $\angle_y(p_1, q_j) > \frac{\pi}{2}$ for $j = 1$ or $j = 2$. □

Now we are going to prove Theorem 1.1 along the line explained in the introduction. The non-compactness causes additional difficulties.

Proof of Theorem 1.1. Let G be a spherical building modelled on the Coxeter group W. Let $X \subset G$ be a convex subset of dimension at most 2 and without center. Assume finally that X is not a building. Due to Lemma 2.1 we have $\bar{r} := \operatorname{rad}(X) < \pi$.

Let $C_W^{\bar{r}}$ be the class of all CAT(1) spaces that have dimension at most 2, no center, whose radius is at most $\bar{r}$ and that admit an isometric embedding into some spherical building modelled on the Coxeter group W. Since an ultralimit of buildings modelled on W is again a building modelled on W, the class $C_W^{\bar{r}}$ is closed under ultralimits (by Lemma 4.3). Therefore we can find a space Z in $C_W^{\bar{r}}$, whose radius $r \leq \bar{r}$ is the smallest possible for spaces in $C_W^{\bar{r}}$ (let $r_i \to r$ be radii of elements of $C_W^{\bar{r}}$, and let Z be the ultralimit of the corresponding spaces).

Replacing Z by its ultraproduct Z^ω we may assume that there is a point $x \in Z$ with $\operatorname{rad}_x(Z) = \operatorname{rad}(Z)$. Let Y be the smallest closed convex subset of Z containing x and invariant under the isometry group Γ^Z of Z. Clearly, a center for Y would also be a center for Z, thus Y is in C_W^r too. Moreover we have $\operatorname{rad}(Y) = \operatorname{rad}_x(Y) = r$ by the minimality of r, and $r > \frac{\pi}{2}$ by Prop. 1.4 since Y has no center.

From Proposition 1.2 we derive $\operatorname{diam}(Y) - \operatorname{rad}_x(Y) = 2\varepsilon > 0$. Therefore for points $y_i, z_i \in Y$ with $d(y_i, z_i) \to \operatorname{diam}(Y)$ we can apply Lemma 4.2 and see that for all big i the points y_i and z_i are not ε-removable.

Consider $y = (y_i), z = (z_i) \in Y^\omega$. We have $d(y, z) = \operatorname{diam}(Y^\omega) = \operatorname{diam}(Y)$. Assume that y and z are inner points of some geodesics in Y^ω. Then from Lemma 5.1 we derive that Y^ω contains a circle, from Lemma 2.1 we see that $\operatorname{rad}(Y^\omega) = \frac{\pi}{2}$ and by Proposition 1.4 the space Y^ω must have a center - a contradiction.

By symmetry it is enough to prove that y is an inner point of a geodesic. Decreasing ε we may assume that it is smaller than the conicality radius r_y of y in Y^ω.

Denote by η the geodesic between y and z and let $v \in S_y$ be its starting direction. From $d(y,z) = \mathrm{diam}(Y^\omega)$ we derive $\mathrm{rad}_v(S_y) \leq \frac{\pi}{2}$.

Set $m = \eta(\frac{\pi}{2})$. By Lemma 4.1 there is a point q with $d(q,m) \geq \frac{\pi}{2}$ and $d(y,m) = \varepsilon$. Since the triangle ymq is spherical, for the starting direction w of yq we obtain $d(v,w) = \frac{\pi}{2}$.

Now we consider the point z_δ on the geodesic qz with $d(z, z_\delta) = \delta \to 0$. As above we derive from Lemma 4.1 that there is some $q_\delta \in Y^\omega$ such that $d(q_\delta, y) = \varepsilon$ and such that for the starting directions v_δ of yz_δ resp. w_δ of yq_δ the inequality $d(v_\delta, w_\delta) \geq \frac{\pi}{2}$ holds.

Since the triangle yzq is spherical we know $d(w, v_\delta) + d(v_\delta, v) = d(v,w) = \frac{\pi}{2}$. Therefore from $\mathrm{rad}_v(S_y) = \frac{\pi}{2}$ and the fact that $S_y Y^\omega$ is one-dimensional we deduce that $d(w, w_\delta) \to \pi$. Now using a diagonal argument in Y (or going to another ultraproduct $(Y^\omega)^\omega$) we obtain a point q_0 such that y is an inner point of the geodesic qq_0.

This finishes the proof of Theorem 1.1. $\square$

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