THE DE RHAM DECOMPOSITION THEOREM FOR METRIC SPACES

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Abstract. We generalize the classical de Rham decomposition theorem for Riemannian manifolds to the setting of geodesic metric spaces of finite dimension.

1. Introduction

The direct product of metric spaces Y and Z is the Cartesian product $X = Y \times Z$ with the metric given by $d((y,z),(\bar{y},\bar{z})) = \sqrt{d^2(y,\bar{y}) + d^2(z,\bar{z})}$. Call a metric space X irreducible if for each decomposition $X = Y \times Z$ one of the factors Y or Z must be a point. It is a very natural question if a given metric space has a "unique" decomposition as a product of finitely many irreducible spaces.

In general, no finite decomposition as a product of irreducible spaces may exist, as an infinite product (for instance, a Hilbert cube) shows. On the other hand, there is no uniqueness in general, even for some subsets of the Euclidean plane ([Fou71], [Her94]).

In the realm of Riemannian geometry this question is answered by the classical Theorem of de Rham ([dR52]). It says that for a simply connected, complete Riemannian manifold M and each point $x \in M$, subspaces of the tangent space T_xM that are invariant under the action of the holonomy group Hol_x are in one-to-one correspondence with the factors of M. As a consequence one derives that each simply connected complete Riemannian manifold M admits a unique decomposition as a direct product $M = M_0 \times M_1 \times ... \times M_k$, where M_0 is a Euclidean space $\mathbb{R}^m$ (possibly a point) and all M_i, $i = 1, ..., k$, are irreducible Riemannian manifolds (with more than one point) not isometric to the real line. The uniqueness states that the factors are determined not only up to an abstract isometry, but that the M_i-fibers through a given point x (i.e. the fiber $(M_i)_x := P_i^{-1}(P_i(x))$ of the projection $P_i : M \to M_0 \times ... \times \hat{M}_i \times .... \times M_k$) are uniquely determined up to

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a permutation of indices. Observe that the Euclidean space plays a special role, since it has many different decompositions as a product of real lines.
In [EH98] the statement about the uniqueness of the decomposition of M was generalized to non simply connected complete Riemannian manifolds by studying the action of the fundamental group of M on the product decomposition of the universal covering.
Our main result presented in this paper is a broad generalization of de Rham's decomposition theorem. In order to state it precisely, recall that a geodesic in a metric space X is an isometric embedding of an interval into X. A metric space is called geodesic if each pair of its points is connected by a geodesic. We say that a metric space is affine if it is isometric to a linearly convex subset of a normed vector space. Given a metric space X we define its affine rank, $\mathrm{rank}_{\mathrm{aff}}(X)$, as the supremum over all topological dimensions of affine spaces that admit an isometric embedding into X. Note that $\mathrm{rank}_{\mathrm{aff}}(X)$ is bounded above by the topological dimension of X.
With this terminology our main result reads as follows.

Theorem 1.1. *Let X be a geodesic metric space of finite affine rank. Then X admits a unique decomposition as a direct product*

$$X = Y_0 \times Y_1 \times Y_2 \times ... \times Y_n,$$

where Y_0 is a Euclidean space (possibly a point), and where the Y_i, $i = 1, ..., n$, are irreducible metric spaces not isometric to the real line nor to a point. Thus, if there is another direct product decomposition $X = Z_0 \times Z_1 \times ... \times Z_m$ of this kind then we have $m = n$ and there exists a permutation s of $\{0, 1, ..., n\}$ such that for each point $x \in X$ the Y_i-fiber through x coincides with the $Z_{s(i)}$-fiber through x for all $i = 1, ..., n$.

Remark 1.1. Note that the Theorem 1.1 cannot be a consequence of some kind of "general nonsense", as the examples of [Fou71] and [Her94] demonstrate. Moreover, there is no similar decomposition theorem in many other categories. For instance, there are finitely generated groups that have many completely different decompositions as a direct sum (cf. [Bau75]) and there are manifolds (for instance the Euclidean space) that have completely different decompositions as products of irreducible manifolds.

Remark 1.2. For compact subsets X of a Euclidean space $\mathbb{R}^n$ the uniqueness of the decomposition of X was proved in [Mos92].

In the formulation of Theorem 1.1 the Euclidean spaces again play a special role. As a particular case of Theorem 1.1 (that is also an important step in the proof) this special role can be expressed as a funny rigidity statement:

Corollary 1.2. *Let X be a geodesic metric space of finite affine rank. Assume that X is decomposed in two different ways as $Y \times \bar{Y} = X = Z \times \bar{Z}$. Assume that the decompositions are transversal at some point $x \in X$, i.e. for the fibers $Y_x, \bar{Y}_x, Z_x, \bar{Z}_x$ we have: $Y_x \cap Z_x = Y_x \cap \bar{Z}_x = \bar{Y}_x \cap Z_x = \bar{Y}_x \cap \bar{Z}_x = \{x\}$. Then X is a Euclidean space $X = \mathbb{R}^{2m}$.*

Using Theorem 1.1 we can analyze the group of isometries of a product space. To appreciate the next result and the special role of Euclidean spaces, one should recall, that the isometry group $\mathrm{Iso}(\mathbb{R}^2)$of the plane $\mathbb{R}^2$ is 3-dimensional, and much larger than the two dimensional group $\mathrm{Iso}(\mathbb{R}) \times \mathrm{Iso}(\mathbb{R})$, i.e. the product $\mathbb{R} \times \mathbb{R}$ has many isometries that do not respect the product structure. On the other hand we have:

Corollary 1.3. *Let X be a geodesic space of finite affine rank and let $X = Y_0 \times Y_1 \times ... \times Y_n$ be its product decomposition as in Theorem 1.1. Denote by $\mathcal{P}$ the group of all permutations $s \in \sigma_n$, such that Y_i and $Y_{s(i)}$ are isometric for all $i = 1, ..., n$. Then there is a natural exact sequence:*

$$1 \to \mathrm{Iso}(Y_0) \times \mathrm{Iso}(Y_1) \times ... \times \mathrm{Iso}(Y_n) \xrightarrow{i} \mathrm{Iso}(X) \xrightarrow{p} \mathcal{P} \to 1.$$

Remark 1.3. In the formulation of the results above and below, the equality sign in $X = Y_0 \times ... \times Y_n$ or $X = Y \times \bar{Y}$ should be understood as a fixed isometry $I : X \to Y_0 \times ... \times Y_n$ or $I : X \to Y \times \bar{Y}$. This fixed isometry then defines projections $P_i : X \to Y_i$ and $P^Y : X \to Y$, respectively. It also defines Y-fibers, as $Y_x = (P^{\bar{Y}})^{-1}(P^{\bar{Y}}(x))$.

We are going to explain the basic idea behind the proof of our main theorem. The idea is to find a class $\mathcal{C}$ of metric spaces that is small enough, such that one can prove the main theorem for this class by some direct means, and that is large enough such that each metric space X as in Theorem 1.1 can be approximated by elements of $\mathcal{C}$ in some suitable sense.
We use the class $\mathcal{C}$ of affine metric spaces of finite dimension. To "approximate" general spaces by elements of $\mathcal{C}$ we prove that for each metric space X, each maximal affine subset C of X is "rectangular", i.e. for each decomposition $X = Y \times \bar{Y}$ we have $C = P^Y(C) \times P^{\bar{Y}}(C)$. This claim is a direct consequence of the following technical result:

Proposition 1.4. *Let $X = Y \times \bar{Y}$ be a direct product. If a subset $C \subset X$ is affine, then so is the projection $P^Y(C) \subset Y$.*

Remark 1.4. Essentially, the statement of Proposition 1.4 is that for $C \subset X$ as above, parallel linear geodesics in C have the same slope with respect to the product decomposition of X. If C is a normed vector space (i.e. if each geodesic in C is part of an infinite geodesic) then this claim is easy to verify and Proposition 1.4 for the case of normed vector spaces C was already proven in [FS02]. In our more general case, the result is more subtle and relies on the infinitesimal considerations in [HL]. In [FS02] it is shown by an example, that Proposition 1.4 becomes wrong if one replaces the word "affine" by "convex subset of a Euclidean space".

Now we explain the heart of the proof of Corollary 1.2. Under the assumptions of Corollary 1.2 we choose a largest affine subset C of X that contains x. We know that C is rectangular, hence it has two mutually transversal decompositions, as well. Assuming that we already know the result for affine spaces, we deduce that C is a Euclidean space $C = \mathbb{R}^{2m}$. It remains to prove that $C = X$. First we observe that each rectangular subset of X that contains $C \cap Y_x$ must contain the whole subset C (by easy linear algebra). Now take an arbitrary point $z \in \bar{Y}_x$. Then z and $C \cap Y_x$ "span" a flat subset of X. Let C_0 be a largest affine subset of X that contains this flat one. Then C_0 is rectangular, hence it contains C and by maximality of C we have $C = C_0$. Thus $Y_x \subset C$. Interchanging the roles of Y and $\bar{Y}$ we deduce the result.

Outline of the paper: In Section 2 we discuss basic facts about metric products. In Section 3 we discuss affine metric spaces and prove Proposition 1.4. In Section 4 we draw some direct consequences of the equality of slopes in an affine subset of a product. In Section 5 and Section 6 we prove that for different decompositions $Y \times \bar{Y} = X = Z \times \bar{Z}$ and each point $x \in X$ the intersection $Y_x \cap Z_x$ is a factor of $Z_x = Z$. In Section 7 we use this observation to reduce Theorem 1.1 to Corollary 1.2. Finally, in Section 8 we prove Corollary 1.2.

2. Preliminaries

2.1. **Notations and basic observations.** By d we will denote distances in metric spaces without an extra reference to the space.

For a direct product $X = Y \times \bar{Y}$ we will use the following notations.

By $P^Y : X \to Y$ and $P^{\bar{Y}} : X \to \bar{Y}$ we will denote the canonical projections to the factors. (In fact the equality $X = Y \times \bar{Y}$ just means that two maps $P^Y : X \to Y$ and $P^{\bar{Y}} \to \bar{Y}$ are given such that $d^2(x,z) = d^2(P^Y(x), P^Y(z)) + d^2(P^Y(x), P^{\bar{Y}}(z))$ holds for all $x, z \in X$).

For a point $x \in X$ we call the subset $(P^{\bar{Y}})^{-1}(P^{\bar{Y}}(x))$ the Y-fiber through x and denote it by Y_x. The restriction $P^Y : Y_x \to Y$ is an isometry and we will sometimes identify Y_x with Y via this isometry. The composition P^{Y_x} of $P^Y : X \to Y$ and the inverse of $P^Y : Y_x \to Y$ is the natural projection of X to Y_x. It sends a point $z \in X$ to the unique intersection of $\bar{Y}_z$ and Y_x. Each restriction $P^{Y_x} : Y_z \to Y_x$ is an isometry. For all $x, z \in X$ we have $d^2(x,z) = d^2(x, P^{Y_x}(z)) + d^2(P^{Y_x}(z), z)$.

2.2. **Recognition of products.** Assume on the other hand that a space X is given as a union $X = \cup_{i\in J} Y_i$ (We do not assume the union to be disjoint, moreover, Y_i and Y_j may coincide for different i and j). Assume that for all $i, j \in J$ a map $P_{ij} : Y_i \to Y_j$ is given such that for all $i, j, k \in J$ we have $P_{ij} \circ P_{ji} = Id$ and $P_{jk} \circ P_{ij} = P_{ik}$. Furthermore, assume that for all $i, j \in J$, all $x \in Y_i$ and $\bar{x} \in Y_j$ we have $d^2(x, \bar{x}) = d^2(x, P_{ij}(x)) + d^2(P_{ij}(x), \bar{x})$.
We are going to show that X splits as a product with fibers Y_i.
For all $x, \bar{x}$ as above we have:

$$d^2(x,\bar{x}) = d^2(x, P_{ij}(x)) + d^2(P_{ij}(x), \bar{x}) = d^2(\bar{x}, P_{ji}(\bar{x})) + d^2(P_{ji}(\bar{x}), x)$$

and

$$d^2(P_{ij}(x), P_{ji}(\bar{x})) = d^2(P_{ij}(x), \bar{x}) + d^2(\bar{x}, P_{ji}(\bar{x})) = d^2(P_{ij}(x), x) + d^2(x, P_{ji}(\bar{x})).$$

Subtracting these equalities from another we obtain $d^2(x, P_{ij}(x)) = d^2(\bar{x}, P_{ji}(\bar{x}))$. Therefore $d(x, P_{ij}(x)) = d(\bar{x}, P_{ji}(\bar{x}))$. In particular, P_{ij} is an isometry and we have $d(Y_i, Y_j) = d(x, P_{ij}(x))$ for each $x \in Y_i$.
Therefore, the Y_i define a so called equidistant decomposition of X and $d(i,j) := d(Y_i, Y_j)$ defines a pseudo metric on J, where two indices have distance 0 if and only if they define the same subset of X. We identify equal fibers and may assume that different fibers are disjoint, i.e. that J is a metric space.
For all $i, j \in J$ and all $x \in Y_i, \bar{x} \in Y_j$ we have: $d^2(x, \bar{x}) = d^2(P_{ij}(x), \bar{x}) + d^2(i,j)$.
Fix now a fiber Y_o for some $o \in J$ and consider the map $P : Y_0 \times J \to X$ given by $P(y, i) = P_{oi}(y)$. This map is surjective, by assumption. We claim that it is an isometry.
Indeed, for all $y, \bar{y} \in Y_o$ and all $i, j \in J$ we have $d^2(P_{oi}(y), P_{oj}(\bar{y})) = d^2(i,j) + d^2(P_{ij}(P_{oi}(y)), P_{oj}(\bar{y}))$.
But $P_{ij}(P_{oi}(y)) = P_{oj}(y)$ and $d(P_{oj}(y), P_{oj}(\bar{y})) = d(y, \bar{y})$. This finishes the proof.

2.3. **Intersections of different fibers.** Let $Y \times \bar{Y} = X = Z \times \bar{Z}$ be two decompositions of a space X. Fix a point $x \in X$ and set $F_x = Y_x \cap Z_x$. The following lemma together with the preceding subsection suggests that F_x has good chances to be a factor of Z_x.

Lemma 2.1. *For each point $p \in \bar{Y}_x$ and each point $q \in F_x$ we have $d^2(P^Z(p), P^Z(q)) = d^2(P^Z(p), P^Z(x)) + d^2(P^Z(x), P^Z(q))$.*

Proof. Identify Z with Z_x and denote by $\tilde{p}$ the projection of p onto Z_x. Observe that x and q are already in Z_x.
We have $d^2(q,p) = d^2(q,\tilde{p}) + d^2(\tilde{p},p)$ and $d^2(x,p) = d^2(x,\tilde{p}) + d^2(\tilde{p},p)$. Since $q \in Y_x$ and $p \in \bar{Y}_x$ we have $d^2(q,p) = d^2(q,x) + d^2(x,p)$. We insert the second equality in the third and the third in the first one and get $d^2(\tilde{p},q) = d^2(\tilde{p},x) + d^2(x,q)$. $\square$

2.4. **Geodesics in products.** Recall that geodesics (if not otherwise stated) are parameterized by the arclength. A subset C of a geodesic metric space X is called convex (totally convex, resp.) if for each pair of points $y, z \in C$ there is some geodesic in C between these two points (if each geodesic between y and z is contained in C, resp.).
Each geodesic γ in a product $X = Y \times \bar{Y}$ has the form $\gamma(t) = (\eta(at), \bar{\eta}(\bar{a}t))$ for some geodesics η in Y and $\bar{\eta}$ in $\bar{Y}$ and some real numbers $a, \bar{a}$ with $a^2 + \bar{a}^2 = 1$. The numbers a and $\bar{a}$ are called the slopes of the geodesic γ with respect to Y and to $\bar{Y}$, respectively. Note that the slope of γ with respect to Y is 0 if and only if γ is contained in some $\bar{Y}$-fiber.
A product $X = Y \times \bar{Y}$ is a geodesic space if and only if the factors Y and $\bar{Y}$ are geodesic. In this case each fiber Y_x is totally convex in X. The fact that geodesics project to geodesics implies that for each convex subset C in a product $X = Y \times \bar{Y}$ the projection $P^Y(C)$ is a convex subset of Y.

2.5. **Groups of isometries.** We are going to deduce Corollary 1.3 from Theorem 1.1. Thus let $X = Y_0 \times Y_1 \times ... \times Y_n$ be as in Corollary 1.3 and let $g : X \to X$ be an isometry. By Theorem 1.1 the isometry g must induce a permutation $s : \{0, 1, ..., n\} \to \{0, 1, ..., n\}$ such that $g((Y_i)_x) = (Y_{s(i)})_{g(x)}$. The restriction $g : (Y_i)_x \to (Y_{s(i)})_{g(x)}$ must be an isometry, hence Y_i and $Y_{s(i)}$ must be isometric. In particular, $s(0) = 0$. The assignment $g \to s$ is a homomorphism $p : \mathrm{Iso}(X) \to \mathcal{P}$. Interchanging isometric factors by some fixed isometry, one sees that p is surjective.
The map $i : \mathrm{Iso}(Y_0) \times ... \times \mathrm{Iso}(Y_n) \to \mathrm{Iso}(Y_0 \times ... \times Y_n)$ is given by $i(g_0, g_1, ..., g_n)(y_0, y_1, ..., y_n) = (g_0(y_0), g_1(y_1), ..., g_n(y_n))$. It is a well defined injection and satisfies $p \circ i = 1$. Let now g be an element in the kernel of p. Fix a point $x \in X$. Then g induces isometries $g : (Y_i)_x \to (Y_i)_{g(x)}$. Identifying $(Y_i)_x$ and $(Y_i)_{g(x)}$ with Y_i via the projection P^{Y_i} we obtain an isometry $g_i : Y_i \to Y_i$. Set $\tilde{g} = i(g_0, g_1, ..., g_n)$. We claim $g = \tilde{g}$. Consider the isometry $h = g \circ \tilde{g}^{-1}$. We have $h(x) = x$ and the restriction of h to $(Y_i)_x$ is the identity for all i. For each point

$\bar{x} \in X$ we must have $P^{(Y_i)_x}(\bar{x}) = P^{(Y_i)_x}(h(\bar{x}))$ for all i. But this means $P^{Y_i}(\bar{x}) = P^{Y_i}(h(\bar{x}))$, hence $\bar{x} = h(\bar{x})$. This shows $h = Id$ and proves that the sequence is exact.

3. Affine spaces

3.1. **Definitions.** We are going to prove Proposition 1.4 in this section. If we want to prove that projections of affine subsets are affine, we first have to find linear geodesics in the projection. Thus we need to work with a distinguished class of geodesics.

Definition 3.1. Let X be a metric space. A bicombing Γ on X assigns to each pair of points $(x, y) \in X \times X$ a geodesic γ_{xy} connecting x to y, such that $\gamma_{xy} = \gamma_{yx}$ as sets (where γ_{xy} and γ_{yx} have opposite orientation as curves) and such that for each $m \in \gamma_{xy}$ we have $\gamma_{my} \subset \gamma_{xy}$.

A space with a bicombing is per definitionem a geodesic metric space. Given a space with a bicombing Γ we will call the geodesics assigned by Γ the special geodesics.

Example 3.1. Let X be a geodesic metric space in which there is only one geodesic connecting each pair of points. Then X has a (unique) natural bicombing.

A map $I : X \to Y$ between two spaces with bicombings Γ and Γ' is called affine if it sends special geodesics to special geodesics parameterized proportionally to the arclength. Such a map I is called a (Γ, Γ')-isometric embedding ((Γ, Γ')-isometry, resp.) if it is an isometric embedding (an isometry, resp.) between the underlying metric spaces. A subset Y of a space X with a bicombing Γ will be called Γ-convex if for each pair of points $x, y \in Y$ the geodesic γ_{xy} is contained in Y. Observe that the image of each (Γ, Γ')-isometric embedding is a Γ'-convex subset and that each Γ-convex subset inherits a natural bicombing.
If (X_i, Γ_i) are spaces with bicombings for $i = 1, 2$, then the direct product $X = X_1 \times X_2$ has a unique bicombing such that the projections $P^{X_i} : X \to X_i$ are affine.
For us, the main example will be the following. Let V be a normed vector space. Then the linear geodesics (i.e. geodesics of the form $\gamma(t) = v + tw$) define a bicombing on V. Normed vector spaces will always be equipped with this particular bicombing.
A subset C of a normed vector space V is Γ-convex if and only if it is linearly convex (i.e. for each $x, y \in C$ and each $t \in [0, 1]$ the point $tx + (1 - t)y$ is in C).
The notion of affine maps in this setting coincides with the usual one. First, let V, W be normed vector spaces. Then each linear map A :

$V \to W$ is affine. On the other hand, let $F : V \to W$ be an affine map with $F(0) = 0$. Then $F(tx) = tf(x)$ and $F(x+y/2) = F(x+y)/2$, for each $t \in \mathbb{R}$ and all $x, y \in V$, hence F is linear in this case. By adding a translation we deduce that a map $F : V \to W$ is affine if and only if it has the form $F(v) = w_0 + A(v)$ for some $w_0 \in W$ and some linear map $A : V \to W$.
Let now X be a linearly convex subset of a normed vector space V and let $f : X \to W$ be an affine map into a normed vector space W. We first assume that $0 \in X$ and that $f(0) = 0$. Denote by C the cone over X, i.e. $C = \{tx | t \geq 0, x \in X\}$. Then C is linearly convex in V and $\bar{f}(tx) = tf(x)$ is a well defined extension of f to C that is again affine. Denote by V_0 the linear hull of X. Then $V_0 = \{x - y | x, y \in C\}$. The map $\tilde{f}(x-y) = \bar{f}(x) - \bar{f}(y)$ is well defined and affine. By the above observation the map $\tilde{f}$ is linear and can be extended to a linear map $F : V \to W$, by linear algebra. We deduce, that a map $f : X \to W$ is affine if and only if it is the restriction of an affine map $F : V \to W$.

Definition 3.2. We call a space X with a bicombing Γ affine if there is a Γ-isometric embedding $I : X \to V$ into a normed vector space.

Thus a metric space is affine in the sense of the introduction if and only if it has a bicombing Γ such that (X, Γ) is affine.

Remark 3.2. Each metric space X has an isometric embedding into a Banach space. If X is geodesic then the image of X is a convex subset of X. In order to make the above definition not trivial it is necessary to distinguish (as we did) between convex and linear convex subsets.

3.2. **Three points characterization.**

Proposition 3.1. *Let X be a space with a bicombing Γ. If for all points $x, y, z \in X$ there is a Γ-convex subset $C_{x,y,z}$ of X that is affine and contains the three points x, y and z, then X is affine.*

Proof. Fix a point o in X. Consider the space $Y = X \times [0, \infty)$ and identify points (x, t) and (z, s) if we have $t \cdot d(o, x) = s \cdot d(o, z)$ and the geodesics γ_{ox} and γ_{oz} initially coincide. Moreover we identify the subsets $X \times \{0\}$ and $\{o\} \times [0, \infty)$ with a unique point in Y that we denote by 0 and call the origin.
Denote by Z the arising set. On Z we define a metric by setting $d((x,t),(y,s)) := \frac{1}{\epsilon} d(\gamma_{ox}(\epsilon t), \gamma_{oy}(\epsilon s))$ for a sufficiently small number $\epsilon > 0$. This quantity is well defined (i.e. does not depend on ϵ) as one sees by making the computations in the linear subset $C_{o,x,y}$ as in the assumptions. Moreover d is a metric as one derives from the triangle inequality in X.

We have a multiplicative operation of $[0, \infty)$ on Z by $\lambda\cdot(x,t) := (x, \lambda t)$. For $z_1, z_2 \in Z$ we have by definition $d(\lambda z_1, \lambda z_2) = \lambda d(z_1, z_2)$.
Next the space X has an isometric embedding I into Z defined through $I(x) = (x, 1)$. We identify X with its image in Z. By our definition we see that for each finite subset D of Z there is some $\lambda > 0$ such that D is contained in λX.
Taking two points $z_1, z_2 \in Z$ we find some $\lambda > 0$ and points $x_1, x_2 \in X$ with $\lambda x_i = z_i$. We extend the bicombing from X to Z by letting the geodesic $\gamma_{z_1 z_2}$ be the (reparameterized) curve $\lambda\gamma_{x_1 x_2}$. Again considering the triangle $C_{x_1,x_2,o}$ we see that this definition does not depend on the choice of λ (i.e. on the choice of x_1 and x_2) and that this is indeed a bicombing on Z. Moreover, for three points $z_1, z_2, z_3 \in Z$ we define a Γ-convex set C_{z_1,z_2,z_3} by choosing x_i, $i = 1, 2, 3$, and λ as above, setting $C_{z_1,z_2,z_3} := \lambda C_{x_1,x_2,x_3}$. It follows that C_{z_1,z_2,z_3} is affine.
Hence Z again has the same 3-point property as X and, since X isometrically embeds in Z, it is enough to prove that Z is affine.
For this purpose we define an addition on Z in the following way. For $x, y \in Z$ we set $x + y := 2m$ where m is the midpoint of γ_{xy}. By definition we have $x + 0 = x$ and $x + y = y + x$ for all $x, y \in Z$. Considering the subset $C_{o,x,y}$ we see that for all $x, y \in Z$ we have $\lambda(x + y) = \lambda x + \lambda y$ for each $0 \leq \lambda \leq 1$ and therefore for each $\lambda \geq 0$.
Let $x, y \in Z$ be arbitrary. Considering the affine space $C_{o,2x,2y}$ we see that for all $0 \leq t \leq 1$ we have $tx + (1 - t)y = \gamma_{xy}(td(x, y))$.
Take now three points $x, y, z \in Z$. Then $u := \frac{1}{3}(((x + y) + z) = \frac{2}{3}(\frac{1}{2}(x + y)) + \frac{1}{3}z$. This shows that $u \in C_{x,y,z}$. The same is true for $\bar{u} := \frac{1}{3}(x + (y + z))$. Making the computations in $C_{x,y,z}$ we derive that $u = \bar{u}$. This implies that the addition is associative.
Finally considering for three arbitrary points $x, y, z \in Z$ the subset $C_{x,y,z}$ we see that $\frac{x+y}{2} = \frac{x+z}{2}$ implies $y = z$.
Hence Z with the defined addition is a commutative semi-group in which $x + y = x + z$ implies $y = z$. Let W denote the abelian group defined by Z, i.e. W is defined as the set of all pairs $(x, y) \in Z^2$ modulo the equivalence relation $(x, y) \sim (\bar{x}, \bar{y})$ if and only if $x + \bar{y} = \bar{x} + y$. The addition on W is defined by $(x, y) + (\bar{x}, \bar{y}) = (x + \bar{x}, y + \bar{y})$.
The set W with this addition is an abelian group and Z has the canonical embedding $i : Z \to W$ defined by $i(x) := (x, 0)$. The operation of $\mathbb{R}^+$ on Z extends to an operation on W by setting $\lambda(x, y) := (\lambda x, \lambda y)$. Finally we set $(-1) \cdot (x, y) := (y, x)$ and for each positive λ and each $w \in W$ we set $(-\lambda)\cdot w := (-1)\lambda w$. Taking this together we see that W is a vector space over $\mathbb{R}$. Moreover, the embedding $i : Z \to W$ sends Γ-geodesics to linear intervals.

Finally we define the function $|\cdot|$ on W by setting $|(x,y)| := d(x,y)$. It remains to prove that $|\cdot|$ is well defined and that it is a norm.
To see that it is well defined let $x, y, \bar{x}, \bar{y} \in Z$ be such that $(x,y) \sim (\bar{x}, \bar{y})$, i.e. $x + \bar{y} = \bar{x} + y$. Then the midpoint m of $\gamma_{x\bar{y}}$ is also the midpoint of $\gamma_{\bar{x}y}$. Take the midpoint p between x and y. Considering the space $C_{x,y,\bar{x}}$ we see that $\gamma_{p\bar{x}}$ intersects $\gamma_{x\bar{y}}$ in some point p_1 that is between x and m. Similarly $\gamma_{p\bar{y}}$ intersects γ_{ym} in some point p_2. Therefore the subspace $C_{p,\bar{x},\bar{y}}$ contains p_1, p_2 and m. Hence for some small $\epsilon > 0$ we can find points q_1 and $\bar{q}_1$ on γ_{p_2m} and $\gamma_{m\bar{x}}$ and points q_2 and $\bar{q}_2$ on γ_{p_1m} and $\gamma_{m\bar{y}}$ such that $d(m,q_1) = d(m,\bar{q}_1) = \epsilon d(m,y)$ and $d(m,q_2) = d(m,\bar{q}_2) = \epsilon d(m,x)$.
In $C_{p,\bar{x},\bar{y}}$ we deduce from this that $d(q_1,q_2) = d(\bar{q}_1,\bar{q}_2)$. Now looking at the space $C_{x,y,m}$ ($C_{\bar{x},\bar{y},m}$, resp.) we see that $d(q_1,q_2) = \epsilon d(x,y)$ ($d(\bar{q}_1,\bar{q}_2) = \epsilon d(\bar{x},\bar{y})$, resp.). This shows that $|(x,y)| = |(\bar{x},\bar{y})|$ and $|\cdot|$ is well defined.
From the definition we immediately conclude that $|\lambda w| = |\lambda||w|$ for each $\lambda \in \mathbb{R}$ and each $w \in W$. Finally, given two points $w = [(x,y)]$ and $v = [(\bar{x},\bar{y})]$ in W we can write $w = [(x+\bar{x}, y+\bar{x})]$ and $v = [(\bar{x}+x, \bar{y}+x)]$, i.e. we may assume that $x = \bar{x}$. In this case one deduces from $C_{x,y,\bar{y}}$ that $d(x,y) + d(x,\bar{y}) \geq 2d(x, \frac{y+\bar{y}}{2})$ which implies $|v| + |w| \geq |v+w|$. This finishes the proof. □

As a corollary we obtain:

Corollary 3.2. *Let X be a metric space and let Y be an affine subset of X with a bicombing Γ. Then there is a maximal affine subset Y' of X that contains Y and the bicombing of which extends Γ.*

Proof. By Zorn's lemma it is enough to prove that for a chain Y_i of affine subsets such that the bicombing of Y_i extends the bicombing of Y_j for $i \geq j$ their union Y' is affine.
This union Y' has a natural bicombing that extends the bicombings of all Y_i. The three points property from Proposition 3.1 is satisfied by Y', since it is satisfied by all Y_i. Thus Y' is affine. □

3.3. **Invariance under products.** Since a projection of a product onto a factor sends geodesics to geodesics parameterized proportionally to the arclength, Proposition 1.4 is a direct consequence of the following result:

Proposition 3.3. *Let (X,Γ) be an affine metric space with a bicombing. Let $f : X \to Y$ be a surjective continuous map that sends each special geodesic to a geodesic parameterized proportionally to the arclength. Then Y is affine. More precisely, there is a unique bicombing*

Γ' on Y, such that (Y,Γ') is affine and such that $f:(X,\Gamma)\to(Y,\Gamma')$ is affine.

Before embarking on the proof we will cite an important special case. Assume namely that f is bijective. Then the images of special geodesics (reparameterized to have speed 1) define a (unique) bicombing Γ' on Y, such that f is affine with respect to this bicombing. The inverse $f^{-1}:(Y,\Gamma')\to(X,\Gamma)$ is an (a priori not continuous) affine equivalence. In this case Theorem 1.3 of [HL] says that (Y,Γ') is in fact affine. Thus Proposition 3.3 is in fact the extension of Theorem 1.3 of [HL] to the non-injective case.

Proof. First observe that the restriction $f:C\to f(C)$ of f to a Γ-convex subset of X again satisfies the assumption.
If a required bicombing Γ' exists we must have $\gamma_{f(x_1)f(x_2)}=f(\gamma_{x_1x_2})$ (as sets) for all $x_1,x_2\in X$. In particular, if such a bicombing exists, it is unique. We only need to prove that the above assignment is well defined and that Y equipped with this bicombing is affine.
Assume first that X is a triangle, i.e. X is the linearly convex hull of three points in a two dimensional normed vector space V. If the triangle is degenerate, i.e. a point or an interval, then $Y=f(X)$ is a point or an interval as well and the conclusion is clear. Thus we may assume that X is non-degenerate. Denote by X_0 the set of inner points of X (with respect to V). Observe that X_0 is linearly convex and dense in X. Set $Y_0=f(X_0)$. Each fiber of f in X_0 is linearly convex. More precisely, it is the intersection of X_0 with an affine subspace of X. Hence there are three cases.
1) There is a point $x\in X_0$ such that $f^{-1}(f(x))=x$. Denote by U the set of all such points and let O be a connected component of U.
For a point $z\in X_0$ and a unit vector $v\in V$ we set $|v|_z$ to be the speed of the geodesic $\gamma(t)=f(z+tv)$. By continuity of f the function $|v|_z$ is continuous in z and in v. Moreover, a point z is in U if and only if $|v|_z>0$ for all unit vectors $v\in V$. By continuity of $|v|_z$ the subset U (and therefore O) is open.
On each convex subset O_1 of O the restriction $f:O_1\to f(O_1)$ is bijective. Thus one can apply [HL], and Proposition 3.3 is true for this restriction. Hence $f:=O_1\to f(O_1)$ is the restriction of an affine map between normed vector space in this case. Thus $|v|_{z_1}=|v|_{z_2}$ for all unit vectors $v\in V$ and all $z_1,z_2\in O_1$. By connectedness and local convexity of O we deduce that $|v|_z$ is constant on O. Hence, by continuity of $|v|_z$, at each boundary point z of O we still have $|v|_z>0$ for all unit vectors $v\in V$. By connectedness of X_0 this implies $X_0=U$. Hence $F:X_0\to Y_0$ is bi-Lipschitz in this case. Hence the continuous

extension $F : X \to F(X)$ is bi-Lipschitz as well. Thus we can apply [HL] again and obtain the validity of the statement in this case.
2) Each fiber of f is an interval. In this case one can rearrange each triple of fibers I_1, I_2, I_3, such that I_2 is between I_1 and I_2, i.e. for each point $x_1 \in I_1$ and $x_3 \in I_3$ the geodesic between x_1 and x_3 intersects I_2. This implies that for each triple of points in Y_0 one can rearrange them such that one point is on a geodesic between the other two. But this implies that Y_0 is an interval (i.e. a subset of a real line). In this case Y is an interval, too (since Y_0 is dense in Y) and we are done.
3) There is only one fiber of f. In this case $Y_0 = f(X_0)$ is a point. Hence Y is a point and there is nothing to prove.
Therefore the statement is true if X is a triangle.
Let now X be arbitrary. To prove that the bicombing is well defined choose points $x_i, z_i \in X$, for $i = 1, 2$ with $f(x_i) = f(z_i)$. Considering the triangles spanned by x_1, x_2, z_1 and x_1, z_1, z_2 and using the fact that the statement is true for triangles we deduce that $f(\gamma_{x_1 x_2}) = f(\gamma_{x_1 z_2}) = f(\gamma_{z_1 z_2})$. This shows that the bicombing on Y making f an affine map is well defined.
For an arbitrary triple $y_1, y_2, y_3 \in Y$ choose $x_1, x_2, x_3 \in X$ with $f(x_i) = y_i$. Let C be the triangle spanned by x_i. Then $f(C)$ is an affine Γ'-convex subset that contains the points y_i. Applying Proposition 3.1 we deduce that Y is in fact affine. $\square$

4. First applications

4.1. **Rectangular subsets.** We will call a subset S of a metric space X rectangular with respect to the product decomposition $X = Y \times \bar{Y}$ of X if $S = P^Y(S) \times P^{\bar{Y}}(S)$ holds. We will say that S is a rectangular subset of X if it is rectangular with respect to each product decomposition of X.

Example 4.1. If X is irreducible, then each subset S of X is rectangular. If $X = \mathbb{R}^n$, then the only rectangular subsets of X are the whole space, the empty set and subsets with only one point.

Let now X be a metric space and let C be a maximal affine subset of X (i.e. C with some bicombing Γ is affine and there is no larger affine subset of X the bicombing of which extends that of C). If X is decomposed as a direct product $X = Y \times \bar{Y}$, then due to Proposition 1.4 the projections $P^Y(C)$ and $P^{\bar{Y}}(C)$ are affine and therefore so is the product $\tilde{C} = P^Y(C) \times P^{\bar{Y}}(C) \subset Y \times \bar{Y}$. Moreover, the restrictions of the projections to C are affine (Proposition 3.3), thus the natural bicombing of $P^Y(C) \times P^{\bar{Y}}(C)$ extends the bicombing of C. By maximality of C we deduce $C = P^Y(C) \times P^{\bar{Y}}(C)$. This shows:

Corollary 4.1. *Let X be a metric space and C be a maximal affine subset of X. Then C is a rectangular subset of X.*

Since the dimension of a product of two affine spaces equals the sum of the dimension of the factors we get (compare [FS02]):

Corollary 4.2. *For each decomposition $X = Y \times \bar{Y}$ we have*

$$\mathrm{rank}_{\mathrm{aff}}(X) = \mathrm{rank}_{\mathrm{aff}}(Y) + \mathrm{rank}_{\mathrm{aff}}(\bar{Y}).$$

4.2. **Equality of slopes.** From Proposition 3.3 we deduce that if C is an affine subset of a product $X = Y \times \bar{Y}$ then each pair of (linear) geodesics in C that are parallel in C have the same slope with respect to Y (and to $\bar{Y}$).
From this we derive the following. Let $X = Y \times \bar{Y}$ and $X = Z \times \bar{Z}$ be two decompositions of a geodesic metric space X. Let γ be a geodesic in Y. Then the slope of the geodesic $\gamma \times \{\bar{y}\} \subset X$ with respect to $\bar{Z}$ does not depend on the point $\bar{y} \in \bar{Y}$. Indeed, for different points $\bar{y}_1, \bar{y}_2 \in \bar{Y}$ choose a geodesic η connecting them. Then $\gamma \times \eta \subset X$ is a flat rectangle and the $\gamma \times \bar{y}_i$ are parallel sides of it.
This observation allows us to speak of the slope of a geodesic $\gamma \in Y$ with respect to $\bar{Z}$. Observe that this slope is 0 if and only if for the endpoints y_1, y_2 of γ and some (and therefore each) point $\bar{y} \in \bar{Y}$ the points $(y_i, \bar{y}) \in X$ are contained in the same Z-fiber.
In particular, we see that if Y_x is contained in Z_x for some point $x \in X$ then $Y_{\bar{x}}$ is contained in $Z_{\bar{x}}$ for each point $\bar{x} \in X$.

5. Intersections and Projections

Let X be a geodesic metric space with two decompositions $Y \times \bar{Y} = X = Z \times \bar{Z}$. Let $x \in X$ be a point and set $F_x = Y_x \cap Z_x$. Consider $T = T_x := P^Y(F_x) \times \bar{Y} \subset Y \times \bar{Y} = X$. Then we have:

Lemma 5.1. *In the notations above the image $P^Z(T) \subset Z$ splits as $P^Z(T) = P^Z(F_x) \times P^Z(\bar{Y}_x)$.*

Proof. Set $o = P^Y(x)$ and $F = P^Y(F_x)$. From the equality of slopes we deduce $F \times \{\bar{y}\} = F_p$ with $p = (o, \bar{y}) \in Y \times \bar{Y} = X$. This implies that $T_p = T_x$ for each point $p \in T_x$.
For all points $q, p \in T$ the fiber $\bar{Y}_q$ intersects F_p in a unique point. Moreover, for this intersection point $\bar{p}$ we have $d^2(P^Z(q), P^Z(p)) = d^2(P^Z(q), P^Z(\bar{p})) + d^2(P^Z(\bar{p}), P^Z(p))$, due to Lemma 2.1.

Now consider Z as the union $Z = \cup_{p\in T} P^Z(F_p)$ and define the map $P_{pq} : P^Z(F_p) \to P^Z(F_q)$ by sending $P^Z(\bar{p})$ to $P^Z(\bar{q})$, where $\bar{q}$ is the unique intersection of $\bar{Y}_{\bar{p}}$ with F_q.
The uniqueness of the intersection shows that $P_{pq} \circ P_{qp} = Id$ and that $P_{pq} \circ P_{rp} = P_{rq}$ for all $r, q, p \in T$. From the above equality we deduce that for all $z \in P^Z(F_p)$ and all $\bar{z} \in P^Z(F_q)$ we have $d^2(z, \bar{z}) = d^2(z, P_{pq}(z)) + d^2(P_{pq}(z), \bar{z})$.
Hence we can apply the considerations in Subsection 2.2 and get the desired conclusion. $\square$

We believe that the image $P^Z(T) \subset Z$ in the last lemma always coincides with Z. We will prove it below under the assumption of the finiteness of the affine rank. First we are going to reduce it to the affine case, which then will be finished in the next section.
We say that X has the property O if for all decompositions $X = Y \times \bar{Y} = Z \times \bar{Z}$ and each point $x \in X$ the projection $P^Z : T \to Z$ is surjective, where $T = T_x$ is defined as above.

Lemma 5.2. *If each finite dimensional affine metric space has property O then so does each geodesic metric space of finite affine rank.*

Proof. Let X be decomposed as $Y \times \bar{Y} = X = Z \times \bar{Z}$. We identify Z with Z_x and use the notations from above. Choose a point $z \in Z_x$. Consider a geodesic γ from x to z. Let C be a maximal affine subset of X that contains γ. Due to Corollary 3.2 such a subset exists and due to Corollary 4.1 it is rectangular. Therefore, $C = A \times \bar{A} = B \times \bar{B}$, where $A = P^Y(C), \bar{A} = P^{\bar{Y}}(C), B = P^Z(C)$ and $\bar{B} = P^{\bar{Z}}(C)$. By our assumption C has the property O. Therefore, for $\tilde{T} = P^A(A_x \cap B_x) \times \bar{A}$ we get $z \in P^{B_x}(\tilde{T})$. Since $\tilde{T} \subset T$ we deduce that z is contained in $P^{Z_x}(T)$. Since z was chosen arbitrary, we deduce $P^{Z_x}(T) = Z_x$. $\square$

6. Linear algebra

6.1. **Euclidean case.** Let V be a finite dimensional vector space. We will write convex for linearly convex below. We assume that all convex subsets that appear below contain the origin 0.
Let $0 \in C$ be a convex subset of V. We denote by $H(C)$ the linear hull of C. By $L(C)$ we denote the largest linear subspace that is contained in C. Observe that $L(C)$ is well defined, actually, $L(C)$ is the union of all linear lines in C. In particular, $L(C) = 0$ if and only if C does not contain a line.
For convex subsets $C_1, C_2 \subset V$ their sum is defined as $C_1 + C_2 = \{c_1 + c_2 | c_i \in C_i\}$. We say that the sum is direct and write $C_1 \oplus C_2$

if $H(C_1) \cap H(C_2) = 0$. We say that C is indecomposable if for each decomposition $C = C_1 \oplus C_2$ one of the summands is 0.
We recall the result of Gruber ([Gru70]):

Theorem 6.1. *Let C be a convex subset of V that does not contain lines. Then C has a unique decomposition $C = C_1 \oplus C_2 \oplus \dots \oplus C_k$ with indecomposable C_j.*

Let now V be a Euclidean vector space and $C \subset V$ be a convex subset. Then C has a unique decomposition $C = L(C) \oplus C_1 \oplus \dots \oplus C_k$, where all C_i are indecomposable and do not contain lines and where all C_i are orthogonal to $L(C)$ (compare [Gru70]).
For a decomposition $C = C_1 \oplus C_2$ we have well defined projections $P^{C_i} : C \to C_i$ defined by $P^{C_i}(c_1 + c_2) = c_i$, for all $c_1 \in C_1, c_2 \in C_2$. We call the decomposition $C = C_1 \oplus C_2$ orthogonal if $H(C_1)$ and $H(C_2)$ are orthogonal. Observe, that the decomposition $C = C_1 \oplus C_2$ is orthogonal if and only if the natural map $I : C_1 \times C_2 \to C$ given by $I(c_1, c_2) = c_1 + c_2$ is an isometry.

Lemma 6.2. *Let V be a Euclidean space and C be a convex subset of V. Let $C = A \oplus \bar{A} = B \oplus \bar{B}$ be two orthogonal decompositions. Then $P^B((A \cap B) + \bar{A}) = B$.*

Proof. Decompose A as $A = L(A) \oplus A_0$, where A_0 is orthogonal to $L(A)$. In the same way decompose $\bar{A} = L(\bar{A}) \oplus \bar{A}_0$, $B = L(B) \oplus B_0$ and $\bar{B} = L(\bar{B}) + \bar{B}_0$. We have $L(C) = L(A) \oplus L(\bar{A}) = L(B) \oplus L(\bar{B})$. Moreover, by the orthogonality assumption $A_0 \oplus \bar{A}_0$ is the orthogonal complement of $L(C)$ in C. Since the same is true for $B_0 \oplus \bar{B}_0$ we have $C_0 = A_0 \oplus \bar{A}_0 = B_0 \oplus \bar{B}_0$. From the uniqueness of the decomposition of C_0 into a direct sum of indecomposable parts, we deduce $(A_0 \cap B_0) + (\bar{A}_0 \cap B_0) = B_0$. In particular, $P^B((A \cap B) + \bar{A})$ contains B_0.
On the other hand, the equality $L(B) = L(A) \cap L(B) + P^{L(B)}(L(\bar{A}))$ is an exercise in linear algebra. (We may assume $C = L(C)$. Considering the quotient space $C/((A \cap B) + (\bar{A} \cap \bar{B}))$ one is reduced to the case $A \cap B = \bar{A} \cap \bar{B} = \{0\}$. In this case we must have $\dim(A) = \dim(\bar{B})$ and $\dim(B) = \dim(\bar{A})$. Since the kernel of the projection $P^B : \bar{A} \to B$ is $\bar{A} \cap \bar{B} = \{0\}$, we deduce that $P^B : \bar{A} \to B$ is injective and therefore surjective as well.)
Combining the both equalities we arrive at $P^B((A \cap B) + \bar{A}) = B$. $\square$

In the notations used in the last lemma, we have seen $B_0 = B_0 \cap A_0 + B_0 \cap \bar{A}_0$. From this we deduce:

Lemma 6.3. *Let $C = A \oplus \bar{A} = B \oplus \bar{B}$ be two orthogonal decompositions of a convex subset $C \subset V = \mathbb{R}^n$. If $B \cap A = B \cap \bar{A} = \{0\}$, then $B = L(B)$, i.e. B is a linear space in this case.*

6.2. **Banach spaces.** Let V be a finite dimensional real vector space. An ellipsoid in V is the image of the unit ball in some Euclidean space $\mathbb{R}^N$ under a linear map $A : \mathbb{R}^N \to V$.
Recall that the map that assigns to a norm on V the unit ball K of the norm is a one-to-one correspondence between norms on V and centrally symmetric convex subsets of V with a non-empty interior. Recall further, that the norm stems from a scalar product if and only if the unit ball K of this norm is an ellipsoid.
For each norm $|| \cdot ||$ on V, there is a unique ellipsoid E of maximal volume that is contained in the unit ball K of V (see for instance [Ami86] or [Tho96]). This ellipsoid (called the Löwner ellipsoid) defines a scalar product on V. Hence we have defined an assignment $(V, K) \to (V, E)$ that assigns to a norm on V a scalar product on V. We will denote this Euclidean space arising from a normed space V by V^e.
The following easy observation is probably well known. Since we could not find a precise reference,we include a short proof:

Lemma 6.4. *If a finite dimensional normed vector space V is a direct product $V = V_1 \times V_2$ of its subspaces V_1, V_2 then we have $V^e = V_1^e \times V_2^e$, i.e. V_1 and V_2 are orthogonal in the Euclidean space V^e.*

Proof. Let K be the unit ball of V and let E be the ellipsoid in K of maximal volume. Denote by E_i the projection of E to V_i with respect to the decomposition $V = V_1 \oplus V_2$. Set $\tilde{E} := (E_1 \times E_2) \cap K$. By construction we have $E \subset \tilde{E} \subset K$. On the other hand, E_1 and E_2 are ellipsoids in V_i and $x = (x_1, x_2) \in E_1 \times E_2$ is in K if and only if $||x_1||^2 + ||x_2||^2 \leq 1$, since V is the direct product of V_1 and V_2. This shows that $\tilde{E}$ is the unit ball of the scalar product on V, such that V_1 and V_2 are orthogonal with respect to this product and such that E_i is the intersection of the unit ball of this scalar product with V_i.
By maximality of E we have $E = \tilde{E}$ and we are done. $\square$

6.3. **General case.** Let now V be a finite dimensional normed vector space. Let C be a (linearly) convex subset of V with $0 \in C$. Assume that C is decomposed as a direct product $C = C_1 \times C_2$. Identify C_i with the C_i-fiber through 0. Then C_i is again linearly convex and the assumption $C = C_1 \times C_2$ is equivalent to the statement that C is a direct sum $C = C_1 \oplus C_2$ and that for all $c_i \in C_i$ we have $||c_1 + c_2|| = \sqrt{||c_1||^2 + ||c_2||^2}$.
We claim that $H(C) = H(C_1) \times H(C_2)$. To see this, one can add a translation and assume that 0 is an inner point of C in $H(C)$. In this case, we deduce $||c_1 + c_2|| = \sqrt{||c_1||^2 + ||c_2||^2}$ for all c_i in a small

neighborhood of 0 in C_i. By homogeneity of the norm we deduce $||v_1 + v_2|| = \sqrt{||v_1||^2 + ||v_2||^2}$ for all $v_i \in H(C_i)$.

Without loss of generality we will assume $V = H(C)$. From Lemma 6.4 we deduce that if one replaces $H(C) = V$ by V^e, i.e. if one equips V with the canonical scalar product defined in the previous subsection and considers C with the new metric coming from V^e, then we still have $C = C_1 \times C_2$. Observe finally, that the projections $P^{C_i} : C \to C_i$ remain unchanged under this procedure since they are defined in purely linear terms.

These observations together with Lemma 6.2 show that each finite dimensional affine metric space has the property O, as defined in Section 5. From Lemma 5.1 and Lemma 5.2 we deduce:

Corollary 6.5. *Let X be a geodesic metric space of finite affine rank. If $Y \times \bar{Y} = X = Z \times \bar{Z}$ are two decompositions of X, then for each point $x \in X$ the intersection $Y_x \cap Z_x$ is a direct factor of Z_x.*

This reduction of the general affine case to the Euclidean one and Lemma 6.3 imply the following:

Corollary 6.6. *Let C be an affine metric space with two decompositions $C = A \times \bar{A} = B \times \bar{B}$. If $B \cap A = B \cap \bar{A} = 0$ then $B = L(B)$, i.e. B is a Banach space.*

7. Reduction

Now we are in position to reduce Theorem 1.1 to Corollary 1.2.

Observe that each metric space has non-negative affine rank and each space that contains at least one non-constant geodesic has rank ≥ 1. Hence the affine rank of each geodesic metric space with at least two points is at least 1. Due to Corollary 4.2, each geodesic metric space of finite affine rank m can have a decomposition in at most m non-trivial factors. Therefore each such space has a decomposition $X = Y_1 \times Y_2 \times ... \times Y_l$ with irreducible factors Y_i. Rearranging the factors and taking together all factors that are isometric to $\mathbb{R}$, we get at least one decomposition as required in Theorem 1.1.

It remains to prove the uniqueness. We proceed by induction on the affine rank m. The case $m = 1$ is clear, since in this case the space X is irreducible. We assume that the uniqueness holds true for all $m' \leq m$. Take a space X of affine rank m and consider a decomposition $X = Y_0 \times Y_1 \times ... \times Y_n$ such that Y_0 is a Euclidean space and such that the Y_i are irreducible spaces of positive rank that are non-isometric to $\mathbb{R}$, for all $i \geq 1$. Let $X = Z_0 \times Z_1 \times ... \times Z_k$ be another such decomposition.

We may assume that X is not a Euclidean space and not irreducible, since the result is clear in these cases.

Fix a point $x \in X$. First, assume $(Z_j)_x = (Y_i)_x$ for some $i, j \geq 1$. Renumerating we may assume $i = j = 1$. Then $(Y_0 \times Y_2 \times Y_3 \times ... \times Y_k)_x = (Z_0 \times Z_2 \times Z_3 \times ... \times Z_l)_x \subset X$, since this is the subset of all points $\bar{x}$ of X with $d(\bar{x}, (Z_1)_x) = d(\bar{x}, (Y_1)_x) = d(\bar{x}, x)$.

Applying our inductive assumption to the space $(Y_0 \times Y_2 \times ... \times Y_k)_x$ we deduce that $k = l$ and that after a renumeration we have $(Y_i)_x = (Z_i)_x$ for all i. From the equality of slopes (Subsection 4.2) we deduce that $(Y_i)_{\bar{x}} = (Z_i)_{\bar{x}}$ for all $0 \leq i \leq k$ and all $\bar{x} \in X$.

From Corollary 6.5 and the irreducibility of the Y_i for $i \geq 1$ we know that for all $0 \leq i, j \leq k$ we either have $(Y_i)_x \cap (Z_j)_x = \{x\}$ or $(Y_i)_x = (Z_j)_x$.

Assume now that the intersection $G_x = (Y_0)_x \cap (Z_0)_x$ contains more than one point. Then G_x is a Euclidean space and a direct factor of $(Y_0)_x$ and of $(Z_0)_x$, due to Corollary 6.5. Write Y_0 as $G \times \tilde{Y}_0$ and Z_0 as $G \times \tilde{Z}_0$. Then we have $(\tilde{Y}_0 \times Y_1 \times Y_2 \times ... \times Y_n)_x = (\tilde{Z}_0 \times Z_1 \times ... \times Z_k)_x$. By our inductive assumption, we deduce $(Y_i)_x = (Z_j)_x$ for some $i, j \geq 1$. The argument above shows that the decompositions $X = Y_0 \times ... \times Y_n$ and $X = Z_0 \times ... \times Z_k$ coincide up to a reindexing.

Taking both observations together we may assume that $(Y_i)_x \cap (Z_j)_x = \{x\}$ for all $0 \leq i \leq n, 0 \leq j \leq k$.

Under this assumptions we set $Y = Y_n$, $\bar{Y} = Y_0 \times Y_1 \times ... \times Y_{n-1}$, $Z = Z_k$ and $\bar{Z} = Z_0 \times Z_1 \times ... \times Z_{k-1}$ and consider the decompositions $Y \times \bar{Y} = X = Z \times \bar{Z}$.

By assumption we have $Y_x \cap Z_x = \{x\}$. If $F_x = \bar{Y}_x \cap \bar{Z}_x$ has more than one point, then F_x is a non-trivial factor of $\bar{Y}_x$ and of $\bar{Z}_x$, due to Corollary 6.5. Take an irreducible factor G of F_x. Applying the inductive assumption to $\bar{Z}$ and to $\bar{Y}$, we deduce that either $G_x = (Y_i)_x = (Z_j)_x$ for some $i, j \geq 1$, or that G is a Euclidean line and that $G_x \subset (Y_0)_x \cap (Z_0)_x$. Both conclusions contradict our assumption. Therefore $\bar{Z}_x \cap \bar{Y}_x = \{x\}$.

In the same way we deduce $Y_x \cap \bar{Z}_x = \bar{Y}_x \cap Z_x = \{x\}$. Therefore the decompositions $Y \times \bar{Y} = X = Z \times \bar{Z}$ satisfy the assumptions of Corollary 1.2. From Corollary 1.2 (that we will prove in the next section) we deduce that X is a Euclidean space which is in contradiction with our assumption.

8. Euclidean rigidity

We are going to prove Corollary 1.2 in this section. Thus let $Y \times \bar{Y} = X = Z \times \bar{Z}$ be two decompositions of a geodesic space X of finite affine rank such that $Y_x \cap Z_x = Y_x \cap \bar{Z}_x = \bar{Y}_x \cap Z_x = \bar{Y}_x \cap \bar{Z}_x = \{x\}$.

8.1. **Reduction to Banach spaces.** We claim that X is a Banach space. Indeed, let C be a maximal affine subset of X that contains the point x. Due to Corollary 4.1, the subset C is rectangular. Hence $A \times \bar{A} = C = B \times \bar{B}$, where $A = P^Y(C)$, $\bar{A} = P^{\bar{Y}}(C)$, $B = P^Z(C)$ and $\bar{B} = P^{\bar{Z}}(C)$. Since $A_x \cap B_x = A_x \cap \bar{B}_x = \bar{A}_x \cap \bar{B}_x = \bar{A}_x \cap B_x = \{x\}$, we deduce from Corollary 6.6 that B and $\bar{B}$, and therefore also C, are Banach spaces.

We may assume that x is the origin of the Banach space C and identify $A, \bar{A}, B, \bar{B}$ with their fibers through $x = 0$. From the transversality of the decompositions we deduce that $\dim(A) = \dim(\bar{A}) = \dim(B) = \dim(\bar{B})$. Since the projection of A to B is injective it is also surjective, i.e. $P^Z(A_x) = B_x$. Similarly $P^{\bar{Y}}(B_x) = \bar{A}_x$. Therefore, each rectangular subset C_0 of X that contains A_x must contain C.

We claim $\bar{A}_x = \bar{Y}_x$. Take an arbitrary point $z \in \bar{Y}_x$. Connect x and z by a geodesic γ. Then $\tilde{C} = A \times P^{\bar{Y}}(\gamma) \subset X$ is an affine space. Take a maximal affine subset C_0 of X that contains $\tilde{C}$. Then C_0 is rectangular (by Corollary 4.1) and contains A_x. Therefore C_0 contains C and by maximality of C we have $C_0 = C$. Therefore, $z \in C \cap \bar{Y}_x = \bar{A}_x$. In the same way we see $A_x = Y_x$. Hence $C = X$.

8.2. **Strategy.** Thus we may assume that X is a finite dimensional Banach space. It remains to prove the following claim:

Claim 8.1. *Let C be a Banach space of finite dimension. If C has two decompositions as a direct product $A \times \bar{A} = C = B \times \bar{B}$ such that $A \cap B = A \cap \bar{B} = \bar{A} \cap B = \bar{A} \cap \bar{B} = \{0\}$ then C is a Euclidean space.*

We are going to prove the claim in the following manner. If B and A enclose a unique angle, i.e. if each line of B has the same slope with respect to A, then the relation between the both decompositions induce an easy condition on the norm, that turns out to be equivalent to the parallelogram equality (Subsection 8.3). For the general case, we prove in the remaining part of this section that there is a rectangular subspace L of C that is as non-Euclidean as C and such that its decompositions "enclose a unique angle". This subspace L is constructed as the projection of some extremal subsets in the product space C^2, the extremality being described as the maximal possible violation of the parallelogram equality.

Before we embark on the proof, note that a subspace V of a Banach space C is rectangular with respect to a decomposition $C = C_1 \times C_2$ if and only if $V = V \cap C_1 + V \cap C_2$. This implies that $V + W$ is rectangular with respect to the decomposition $C = C_1 \times C_2$ if V and W are rectangular.

8.3. **Projections and their compositions.** Let C be decomposed as above. Consider the bijective linear projections $P^A : B \to A$ and $P^B : A \to B$, and let $Q := P^A \circ P^B : A \to A$ be their composition.

Lemma 8.2. *Under the above assumptions Q has an eigenvector.*

Proof. Let $x \in A$ be arbitrary. The vectors $P^B(x)$ and $P^{\bar{B}}(x)$ span a two-dimensional Euclidean subspace F of C that contains x. If we denote by α the angle between x and $P^B(x)$ then by definition $||P^B(x)|| = \cos(\alpha)||x||$. On the other hand, the projection $\bar{x}$ of $P^B(x)$ to the line spanned by x in F satisfies $||\bar{x}|| = ||P^B(x)|| \cos(\alpha)$. This implies $\frac{||P^B(x)||}{||x||} \leq \frac{||P^A(P^B(x))||}{||P^B(x)||}$ and equality holds if and only if x is an eigenvector of Q.
Similarly, we have $\frac{||P^A(\bar{x})||}{||\bar{x}||} \leq \frac{||P^B(P^A(\bar{x}))||}{||P^A(\bar{x})||}$ for all $\bar{x} \in B$. Thus, taking a point x in the unit sphere of A such that $\frac{||P^B(x)||}{||x||}$ is maximal, we deduce that x must be an eigenvector of Q. $\square$

Note that if some $a \in A$ satisfies $Q(a) = \lambda a$, then for $b = P^B(a)$ we have $(P^B \circ P^A)(b) = \lambda b$. Moreover, we have $P^A \circ P^{\bar{B}}(a) = (1 - \lambda)a$. This shows that if $a \in A$ is an eigenvector of Q, then $P^B(a)$ and $P^{\bar{B}}(a)$ generate a two-dimensional Euclidean subspace F of C that is rectangular with respect to both decompositions of C. Making the computations in the Euclidean space F, one concludes that $0 < \lambda < 1$ and that $||P^B(a)|| = \sqrt{\lambda}||a||$ holds.

Lemma 8.3. *Under the above assumptions let in addition Q be a multiple of the identity: $Q = \lambda \mathrm{Id}_A$. Then C is Euclidean.*

Proof. We have seen that in this case $P^{\bar{A}} \circ P^B = (1-\lambda)\mathrm{Id}_{\bar{A}}$. Moreover, $P^A : C \to A$ satisfies $||P^B(a)|| = \sqrt{\lambda}||a||$ for all $a \in A$. In the same way, $||P^{\bar{A}}(b)|| = \sqrt{1-\lambda}||b||$ for all $b \in B$.
Consider the map $I : A \to \bar{A}$ given by $I(a) = \frac{1}{\sqrt{\lambda(1-\lambda)}}(P^{\bar{A}} \circ P^B)(a)$. Then I is an isometry. Identify A and $\bar{A}$ via this isometry. Then the space C is identified with $A^2 = A \times A$ and the subspaces B and $\bar{B}$ of C are given as $B = \{(x, sx)|x \in A\}$ and $\bar{B} = \{(-sx, x)|x \in A\}$, where with $s = \sqrt{\frac{1-\lambda}{\lambda}}$.

The assumption $A^2 = C = B\times\bar{B}$ now reads as $||(x, sx)||^2+||(-sy, y)||^2 = ||(x - sy, sx + y)||^2$, for all $x, y \in A$. Thus we have $(1 + s^2)||x||^2 + (1 + s^2)||y||^2) = ||x - sy||^2 + ||sx + y||^2$ for all $x, y \in A$.
For $s = 1$ this is just the usual parallelogram equality which implies that A is Euclidean. For general $s \neq 0$ it is shown in [Car62] (compare also [Ami86], 1.16) that this equality also implies that A is Euclidean. Since $C = A \times A$, the conclusion follows. □

8.4. **Projections and squares.** Let a Banach space C be decomposed as $C = A \times \bar{A}$. Consider the Banach space $C^2 = C \times C$ and its decomposition $C^2 = A^2 \times \bar{A}^2$. Let L be a subspace of C^2 that is rectangular with respect to this decomposition. Then for each point $(v, w) \in L \subset C^2$ there are unique points $(v_1, w_1) \in L\cap A^2$ and $(v_2, w_2) \in L\cap\bar{A}^2$ such that $(v, w) = (v_1, w_1)+(v_2, w_2) = (v_1+v_2, w_1+w_2)$. Denote by V and W the projection of $L \subset C^2$ onto the first and the second C-factor, respectively (i.e. V is the set of all $v \in C$, such that for some $w \in C$, the point $(v, w) \in C^2$ is contained in L). Then $V, W \subset C$ satisfy $V = V \cap A + V \cap \bar{A}$ and $W = W \cap A + W \cap \bar{A}$. Therefore, V, W and $V + W$ are rectangular subspaces with respect to $C = A \times \bar{A}$.
Assume now that C has two transversal decompositions $A \times \bar{A} = C = B \times \bar{B}$ as above. Consider the corresponding decompositions $A^2 \times \bar{A}^2 = C^2 = B^2 \times \bar{B}^2$ and let $L \subset C^2$ be a subspace that is rectangular with respect to both decompositions. Let V (W, resp.) be the projection of L onto the first (the second, resp.) C-factor of $C^2 = C \times C$.
Consider the orthogonal projections $P^{A^2} : B^2 \to A^2$ and $P^{B^2} : A^2 \to B^2$ and the composition $\tilde{Q} = P^{A^2} \circ P^{B^2}$.
In the same way define $Q_V : V \cap A \to V \cap A$ and $Q_W : W \cap A \to W \cap A$.
From the definition we conclude that the equality $\tilde{Q} = \lambda Id_{L\cap A^2}$ for some $\lambda \in \mathbb{R}$ implies that $Q_V = \lambda Id_{V\cap A}$ and $Q_W = \lambda Id_{W\cap A}$.
In this case we see that $(V + W) \cap A$ is contained in the λ-eigenspace of $P^A \circ P^B : A \to A$. From Lemma 8.3 we deduce that $V + W$ is a Euclidean space.

8.5. **Extremal points.** Recall that a Banach space C satisfies the parallelogram inequality $||x + y||^2 + ||x - y||^2 \leq 2(||x||^2 + ||y||^2)$ for all $x, y \in C$ if and only if C is Euclidean. In this case the above inequality is in fact an equality for all pairs $(x, y) \in C^2$.
We may reformulate this condition as follows. Consider the Banach space C^2 with the product norm and the linear map $D : C^2 \to C^2$ defined by $D((x, y)) = \frac{1}{\sqrt{2}}(x + y, x - y)$ for all $x, y \in C$. Then C is a Euclidean space if and only if the bijection D has norm 1. In this case D is in fact an isometry.

We define $M(C)$ to be the norm of the linear map D and we denote by $E(C)$ the set of all $v \in C^2$ for which $||D(v)|| = M(C)||v||$ holds. By definition λv is contained in $E(C)$ for all $v \in E(C)$ and all $\lambda \in \mathbb{R}$.
Let now C be decomposed as $C = A \times \bar{A}$ and consider the induced decomposition $C^2 = A^2 \times \bar{A}^2$. The subsets A^2 and $\bar{A}^2$ are invariant under the map D defined above. Therefore for arbitrary $v \in A^2$ and $w \in \bar{A}^2$, we get $||D(v + w)||^2 = ||D(v) + D(w)||^2 = ||D(v)||^2 + ||D(w)||^2 \leq M(C)||v||^2 + M(C)||w||^2 = M(C)||v + w||^2$. Moreover, equality holds if and only if v and w are contained in $E(C)$.
This shows that the subset $E(C)$ of C^2 is rectangular with respect to the decomposition $C^2 = A^2 \times \bar{A}^2$.

8.6. **Extremal subspaces.** Now we are in position to finish the proof of Claim 8.1. Thus let $A \times \bar{A} = C = B \times \bar{B}$ be two transversal decompositions. Let $M(C) \in \mathbb{R}^+$ and $E(C) \subset C^2$ be defined as in the last subsection.
Choose a largest linear subspace L_0 of C^2 that is contained in $E(C)$. By definition $E(C)$ contains at least one line, hence $\dim(L_0) \neq 0$. Since $E(C)$ is rectangular with respect to the decompositions $C^2 = A^2 \times \bar{A}^2$ and $C^2 = B \times \bar{B}^2$, the largest subspace L_0 of $E(C)$ is rectangular with respect to these decompositions as well.
Denote by $\tilde{Q} : L_0 \cap A^2 \to L_0 \cap A^2$ the composition of projections $\tilde{Q} = P^{A^2} \circ P^{B^2}$. Due to Lemma 8.2 there is an eigenvector v of $\tilde{Q}$. Then the two-dimensional subspace L of L_0 that is generated by $P^{B^2}(v)$ and $P^{\bar{B}^2}(v)$ is rectangular with respect to the decompositions $A^2 \times \bar{A}^2 = C^2 = B^2 \times \bar{B}^2$.
Since $L \cap A^2$ is one-dimensional, the restriction of $\tilde{Q}$ to $L \cap A^2$ is a multiple of the identity. As in Subsection 8.4, denote by V and W the projections of $L \subset C \times C$ onto the first and the second C-factor, respectively. Finally, set $\tilde{C} = V + W$. In Subsection 8.4 we have seen that $\tilde{C}$ is a Euclidean space.
On the other hand, $L \neq \{0\}$, hence either V or W are not $\{0\}$. Without loss of generality assume $V \neq \{0\}$ and choose some $v \in V$, $v \neq 0$. By construction there is some $w \in W$ with $(v, w) \in L \subset E(C)$. Hence $||D((v, w))|| = E(C)||(v, w)||$ and, since $(v, w) \in \tilde{C}^2 \subset C^2$ we deduce that $E(C) = E(\tilde{C})$. But $\tilde{C}$ is Euclidean, hence $E(\tilde{C}) = 1$. Thus $E(C) = 1$ and C is a Euclidean space.

References

[Ami86] D. Amir. *Characterizations of Inner Product Spaces.* Birkhäuser Verlag, Basel, 1986.

[Bau75] G. Baumslag. Direct decompositions of finitely generated torsion-free nilpotent groups. *Math. Z.*, 145:1–10, 1975.

[Car62] S. Carlsson. Orthogonality in normed linear spaces. *Ark. Mat.*, 4:297–318, 1962.

[dR52] G. de Rham. Sur la reductibilité d'un espace de Riemann. *Comment. Math. Helv.*, 26:328–344, 1952.

[EH98] J. Eschenburg and E. Heintze. Unique decomposition of Riemannian manifolds. *Proc. Amer. Math. Soc*, 126:3075–3078, 1998.

[Fou71] G. Fournier. On a problem of S. Ulam. *Proc. Amer. Math. Soc*, 29:622, 1971.

[FS02] T. Foertsch and V. Schroeder. Minkowski versus Euclidean rank for products of metric spaces. *Adv. Geom.*, 2:123–131, 2002.

[Gru70] P. Gruber. Zur Charakterisierung konvexer Körper. über einen Satz von Rogers und Shephard. *Math. Ann.*, 184:79–105, 1970.

[Her94] I. Herburt. There is no cancellation law for metric products. In *Intuitive geometry (Szeged, 1991)*. North-Holland, Amsterdam, 1994.

[HL] P. Hitzelberger and A. Lytchak. Spaces with many affine functions. Preprint: Arxiv: *math.MG/0511583* To appear in Proc. Amer. Math. Soc.

[Mos92] M. Moszyńska. On the uniqueness problem for metric products. *Glas. Mat. Ser. III*, 47:145–158, 1992.

[Tho96] A. C. Thompson. *Minkowski Geometry.* Cambridge University Press, Cambridge, 1996.

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