ISOPERIMETRIC CHARACTERIZATION OF UPPER CURVATURE BOUNDS

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Abstract. We prove that a proper geodesic metric space has non-positive curvature in the sense of Alexandrov if and only if it satisfies the Euclidean isoperimetric inequality for curves. Our result extends to non-geodesic spaces and non-zero curvature bounds.

1. Introduction

1.1. **Main result.** We say that a metric space X satisfies the Euclidean isoperimetric inequality for curves if any closed Lipschitz curve $\gamma : S^1 \to X$ bounds a Lipschitz map of the unit disc $v : \bar{D} \to X$ whose parametrized Hausdorff area is at most $\frac{1}{4\pi}\ell_X^2(\gamma)$. Here, $\ell_X(\gamma)$ denotes the length of γ in X. We refer to the first two sections below for the notion of parametrized Hausdorff area and other basic notions of metric geometry involved in the following main theorem of the present paper.

Theorem 1.1. *Let X be a proper metric space in which any pair of points is connected by a curve of finite length. Let X^i denote the set X with the induced length metric. The space X^i is* $\mathrm{CAT}(0)$ *if and only if X satisfies the Euclidean isoperimetric inequality for curves.*

This result provides an analytic access to upper curvature bounds and can be used to recognize upper curvature bounds without being able to identify geodesics or angles, a situation often appearing in metric constructions, cf. [AB04]. For instance, it is used in [LS17] to control upper curvature bounds under conformal changes of the distance, a result inaccessible by purely geometric means. Theorem 1.1 admits a natural generalization to non-zero curvature bounds, see Theorem 1.4 below.

The "only if part" of our theorem is folklore and follows as an easy consequence of Reshetnyak's majorization theorem, [Res68]. With a different definition of area the "only if part" already appears in [Ale57] at the very origin of the theory of spaces with upper curvature bounds.

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Results closest to the much subtler “if part” of our theorem have been proven in [BR33] and [Res61] in the case of surfaces. E. Beckenbach and T. Rado proved in [BR33] our Theorem 1.1 for smooth 2-dimensional Riemannian manifolds, finding a connection between log-subharmonicity, isoperimetric inequalities and curvature bounds. In [Res61] the result of [BR33] was extended to some singular surfaces and non-zero curvature bounds.

Our theorem is motivated by Gromov’s characterization [Gro87] of Gromov hyperbolic spaces by subquadratic or small quadratic isoperimetric inequalities for large curves. Theorem 1.1 can be viewed as the borderline case and the non-rough version of Gromov’s theorem and its optimal improvement in [Wen08]. In fact, Theorem 1.1 admits a large scale version discussed in [Wen17]. On the technical side, our result and its proof has some similarities with the work by Petrunin and Stadler in [PS17], [Pet99] on the curvature of discs satisfying some minimality property.

1.2. **Main idea.** Essentially, the strategy of our proof of the ”if part” is to reduce the problem in a general metric space to the situation considered in [Res61]. Recall the following simple consequence of the Gauss equation in Riemannian geometry: a minimal surface has curvature no larger than the ambient space. We reverse this idea and find a curvature bound for the total space by proving that all minimal discs have the corresponding curvature bound:

Theorem 1.2. *Let X be a proper metric space which satisfies the Euclidean isoperimetric inequality for curves. Let Γ be a Jordan curve of finite length in X and let $u : \bar{D} \to X$ be a solution of the Plateau problem in the space X for the boundary curve Γ. Then the intrinsic minimal disc Z associated with u is a* $\mathrm{CAT}(0)$ *space.*

We refer to [LW17b], [LW18] and Section 6 below for the notion of a solution of the Plateau problem and the associated intrinsic minimal disc. By definition of the intrinsic minimal disc Z, the solution of the Plateau problem u in Theorem 1.2 factorizes as $u = \bar{u} \circ P$ for a surjective map $P : \bar{D} \to Z$ and a 1-Lipschitz map $\bar{u} : Z \to X$. Moreover, $\bar{u}$ sends the boundary circle ∂Z of Z in an arclength preserving way onto Γ.

It is not difficult to see that Theorem 1.2 implies Theorem 1.1. Assume for simplicity that the proper space X with the Euclidean isoperimetirc inequality for curves is a length space. The existence of a solution u of the Plateau problem for any rectifiable Jordan curve Γ in X is proved in [LW17b], generalizing [Mor48] to the setting of metric spaces. In order to prove that X is $\mathrm{CAT}(0)$, one needs to prove that any Jordan triangle in X is thin, cf. Section 3 below. However, Theorem 1.2 implies that Γ is thin in the intrinsic minimal disc Z. Using the map $\bar{u}$ (which majorizes Γ in the sense of [Res68] and [AKP16, Section 9.8]) this easily implies that Γ is thin in X as well.

1.3. **Main steps.** The proof of Theorem 1.2 involves several steps. First, a special case of the Blaschke-Santalo inequality implies that, among normed planes, only the Euclidean plane satisfies the Euclidean isoperimetric inequality, [Tho96]. The quasi-convexity of the Hausdorff area proved in [BI12] and a natural blow-up argument imply that X has only "Euclidean tangent spaces", at least as far as infinitesimal properties of Sobolev maps with values in X are concerned. This is the property (ET) introduced in [LW17b], which greatly simplifies the description of Sobolev maps and solutions of the Plateau problem.

In particular, the solution of the Plateau problem u in Theorem 1.2 is a conformal map, see Section 4. Thus, there exists a non-negative Borel function $f \in L^2(D)$, *the conformal factor of* u, such that for *almost all curves* γ in D the length of the image of γ under u is controlled by f:

$$\ell_X(u \circ \gamma) = \int_\gamma f. \tag{1.1}$$

The next step goes back to [BR33] and shows that the isoperimetric inequality forces f to be log-subharmonic.

The subsequent step, contained in [Res61], relates log-subharmonicity of conformal factors to non-positive curvature in the sense of Alexandrov. More precisely, the length metric defined on D by setting the length of *every* rectifiable curve $\gamma \subset D$ to be $\int_\gamma f$ is locally CAT(0). The metric space Y defined in this way is intimately related to the intrinsic minimal disc Z. The only difference is that (1.1) holds in Y for *all* and in Z for *almost all* rectifiable curves γ. In particular, we have a 1-Lipschitz map $I : Y \to Z$ which preserves the length of almost all curves.

The final, rather subtle step is devoted to the proof that the spaces Z and (the completion of) Y are identical. While the analytically defined conformal factor f controls the lengths of *almost all* curves in Z, it cannot control the lengths of all curves: contracting one interval in D to a point does not change the conformal factor. In particular, f does not a priori control the most important boundary curve. Applying some cutting and pasting tricks we reduce the final step to the question whether the length of the boundary curve "is controlled by the conformal factor". Using general structural results about the intrinsic minimal discs obtained in [LW18], the final step reduces to the following:

Theorem 1.3. *Let Z be a geodesic metric space homeomorphic to the closed disc $\bar{D}$. Denote by ∂Z the boundary circle and assume that $Z \setminus \partial Z$ is locally* CAT(0)*. Then the following are equivalent.*

(1) *Z is* CAT(0)*.*
(2) *$Z \setminus \partial Z$ with the metric induced from Z is a length space.*
(3) *For any Jordan curve $\eta \subset Z$ the open disc J_η enclosed by η in Z satisfies $\mathcal{H}^2(J_\eta) \leq \frac{1}{4\pi} \cdot \ell_Z^2(\eta)$.*

Throughout the text, we denote by $\mathcal{H}^2$ the 2-dimensional Hausdorff measure. Thus, condition (3) in Theorem 1.3 is the geometric (unparametrized) version of the Euclidean isoperimetric inequality. This theorem does not sound very surprising. The implication from (1) to (3) is an easy consequence of Reshetnyak's majorization theorem. The equivalence of (2) and (1) is not very difficult either. In contrast, the proof of the main implication from (3) to (1) is rather long and technical and comprises one half of this paper. One might have the following example in mind in order to grasp the problem one faces when trying to prove this implication. Start with a complicated Jordan curve Γ in $\mathbb{R}^2$, for example Koch's snowflake. Define Z as the closure of the Jordan domain of Γ but let the boundary curve $\Gamma \subset Z$ have some finite length without changing the lengths outside of Γ. (To make the picture more complicated, change the metric inside the Jordan domain by a smooth conformal factor in such a way that the curvature is everywhere non-positive and tends to $-\infty$ in the neighborhood of Γ). The arising space Z is not CAT(0). Therefore, the proof of Theorem 1.3 must detect in this space Z Jordan curves which violate the Euclidean isoperimetric inequality. Since $Z \setminus \partial Z$ is locally CAT(0), parts of these curves must be contained in the boundary Γ, where the geometry is particularly complicated.

1.4. **Generalization to non-zero curvature bounds.** Theorem 1.1 generalizes to other curvature bounds. The extension is achieved along the same route and involves only minor difficulties of notational and technical nature. In order to formulate the statement we introduce the notion of a *Dehn function*. Let X be a metric space. Let $\delta : (0,\infty) \to [0,\infty]$ be a function. We say that X satisfies the δ-isoperimetric inequality, if for any $r > 0$, any Lipschitz curve $\gamma : S^1 \to X$ of length $\leq r$ bounds a Lipschitz disc $u : \bar{D} \to X$ of parametrized Hausdorff area $\leq \delta(r)$. The Dehn function δ_X of X (with respect to Lipschitz discs) is the infimum of all functions $\delta : (0, \infty) \to [0, \infty]$ for which X satisfies the δ-isoperimetric inequality.

For any real number κ we consider the simply connected space form M^2_κ of curvature κ and denote by $R_\kappa \in (0, \infty]$ twice the diameter of M^2_κ. Let δ_κ be the Dehn function of M^2_κ. Now we can state the generalization of Theorem 1.1 to non-zero curvature bounds.

Theorem 1.4. *Let X be a proper metric space in which any pair of points is connected by a curve of finite length. Let X^i be the set X with the induced length metric. The space X^i is* CAT(κ) *if and only if the Dehn function δ_X of X satisfies $\delta_X \leq \delta_\kappa$ on the interval $(0, R_\kappa)$.*

1.5. **Structure of paper and final comments.** The paper consists of two parts and one appendix. The first part, which relies heavily on the existence and regularity theory of solutions of the Plateau problem, reduces Theorem 1.1 and Theorem 1.2 to Theorem 1.3. We closely follow the plan sketched above. The second part is devoted to the proof of Theorem 1.3. It consists of purely 2-dimensional metric geometry. The structure of this part is explained in Section 10. In the appendix we explain the minor additional difficulties arising in the proof of Theorem 1.4 and sketch the solutions of these problems.

Remark 1.1. Once Theorem 1.1 has been proven, the statements of Theorem 1.2 and Theorem 1.3 can be strengthened, see [Pet99], [PS17] and [LW17a].

Remark 1.2. In convex and metric geometry there are many natural ways to measure area of 2-rectifiable sets and Lipschitz discs besides the Hausdorff area. The most famous among such definitions of area are Gromov's mass* and the Holmes-Thompson definition of area. We refer to [Tho96] and [LW18] for lengthy discussions on definitions of area. As explained above, the first step of the proof of our main theorem uses an inequality from convex geometry to exclude all non-Euclidean tangent planes. The argument applies to all quasi-convex definitions of area μ with the following property: among all normed planes only the Euclidean plane satisfies the Euclidean isoperimetric inequality with respect to μ. For the Holmes-Thompson definition of area μ^{ht} the Euclidean isoperimetric inequality holds sharply for all normed planes. Thus, Theorem 1.1 is valid for any quasi-convex definition of area μ which satisfies $\mu \geq \mu^{ht}$ on all normed planes with equality only on the Euclidean plane. In particular, Theorem 1.1 remains valid for Gromov's mass* and for Ivanov's "inscribed Riemannian" definition of area.

The validity of the Euclidean isoperimetric inequality for curves with respect to the Holmes-Thompson definition of area might be related to other forms of convexity beyond CAT(0).

Remark 1.3. If the constant $\frac{1}{4\pi}$ in the formulation of the Euclidean isoperimetric inequality is replaced by any smaller constant, then the space X^i in Theorem 1.1 turns out to be a tree, [LWY16], [Wen08].

Remark 1.4. As a consequence of Theorem 1.1, a proper geodesic metric space with Euclidean isoperimetric inequality for curves must be contractible. It would be interesting to know whether any topological conclusions can be drawn, if the constant $\frac{1}{4\pi}$ is replaced by a slightly larger constant $\frac{1}{4\pi} < C < \frac{1}{2\pi}$.

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Part I. Structure of minimal discs.

2. Basics on metric spaces

2.1. **Notation.** The Euclidean norm of a vector $v \in \mathbb{R}^2$ is denoted by $|v|$. We denote the open unit disc in $\mathbb{R}^2$ by D. Connected open subsets of $\mathbb{R}^2$ will be called domains. A metric space is called proper if its closed bounded subsets are compact. We will denote distances in a metric space X by d or d_X. Let $X = (X, d)$ be a metric space. The open ball in X of radius r and center $x_0 \in X$ is denoted by

$$B(x_0, r) = B_X(x_0, r) = \{x \in X : d(x_0, x) < r\}.$$

A Jordan curve in X is a subset $\Gamma \subset X$ which is homeomorphic to S^1. Given a Jordan curve $\Gamma \subset X$, a continuous map $\gamma\colon S^1 \to X$

is called a weakly monotone parametrization of Γ if γ is a uniform limit of homeomorphisms $\gamma_i\colon S^1 \to \Gamma$. For $m \geq 0$, the m-dimensional Hausdorff measure on X is denoted by $\mathcal{H}^m = \mathcal{H}^m_X$. The normalizing constant is chosen in such a way that on Euclidean $\mathbb{R}^m$ the Hausdorff measure $\mathcal{H}^m$ equals the Lebesgue measure $\mathcal{L}^m$.

If no confusion is possible we will identify parametrized curves and their unparametrized images and denote them by the same symbol. The length of a curve γ in a metric space X will be denoted by $\ell_X(\gamma)$ or simply by $\ell(\gamma)$. A continuous curve of finite length is called *rectifiable.* A (local) *geodesic* in a space X is a (locally) isometric map from an interval to X. A space X is called *a geodesic space* if any pair of points in X is connected by a geodesic. A space X is a *length space* if for all $x, y \in X$ the distance $d(x, y)$ equals $\inf\{\ell_X(\gamma)\}$, where γ runs over the set of all curves connecting x and y.

2.2. **Length metric associated with a map.** We refer the reader to [BBI01], [Pet10], [LW18] for discussions of the following construction and related topics. Let X', X be metric spaces. Let $u : X' \to X$ be a continuous map. Assume that for any $y_1, y_2 \in X'$ there exists a continuous curve $\gamma : I \to X'$ connecting y_1 and y_2 such that the curve $u\circ\gamma$ has finite length. Then we let $d_u(y_1, y_2) \in [0, \infty)$ be the infimum of lengths of all such curves $u \circ \gamma$. The so defined function $d_u : X' \times X' \to [0, \infty)$ is a pseudo-distance on the set X'. The corresponding metric space Z_u, which arises from X' by identifying pairs of points with d_u-distance 0, is a length space. We will call it the length *metric space associated with the map u.*

By construction, the space Z_u associated with the map u comes with a canonical, possibly non-continuous, surjective projection $P : X' \to Z_u$ and a 1-Lipschitz map $\bar{u} : Z_u \to X$ such that $u = \bar{u} \circ P$.

The most prominent example of this construction is given as follows. Let X be a metric space in which any pair of points is connected by a curve of finite length. Then the *length space* X^i *associated to* X is the special case $X^i = Z_u$ of the above construction for the identity map $u = \mathrm{Id} : X \to X$. If X is proper then, due to the theorem of Arzela-Ascoli, any pair of points in X is connected by a curve of shortest length. Therefore the space X^i is a geodesic space. The completeness of X implies that X^i is complete as well. The 1-Lipschitz map $\bar{u} : X^i \to X$ from above is the identity in this case. The map $P = \bar{u}^{-1} : X \to X^i$ need not be continuous, but it sends curves of finite length in X to continuous curves of the same length in X^i.

2.3. **Polygons and triangles.** A polygon in a metric space X is a closed curve $\gamma : [a, b] \to X$ such that for some $a = t_1 \leq .... \leq t_n = b$ all restrictions $\gamma : [t_i, t_{i+1}] \to X$ are geodesics. A Jordan curve Γ which can be parametrized as a polygon will be called a *Jordan polygon*. If $n = 3$ we obtain the notions of a *triangle* and *Jordan triangle*, respectively.

2.4. **Parametrized area of Lipschitz maps.** Let K be a Borel subset of $\mathbb{R}^2$ and let $u : K \to X$ be a Lipschitz map into some metric space X. The set K can be decomposed as a disjoint union $K = \cup_{i=1}^{\infty} K_i \cup A$ in such a way that all K_i are compact and A has measure 0, and such that $u : K_i \to X$ is either injective or $\mathcal{H}^2(u(K_i)) = 0$, see [Kir94]. The parametrized area of u, which generalizes the classical parametrized area of smooth maps is given by (see [LW18, Subsection 2.4]):

$$\text{Area}(u) := \sum_{i=1}^{\infty} \mathcal{H}^2(u(K_i)) = \int_{u(K)} N(x)\, d\mathcal{H}^2(x)\,,$$

where $N(x)$ is the cardinality of $u^{-1}(x)$. Alternatively, the parametrized area can be computed by a metric transformation formula cf. [LW17b, p.3].

For any biLipschitz homeomorphism $F : K_0 \to K \subset \mathbb{R}^2$ the parametrized areas of $u : K \to X$ and $u_0 = u \circ F : K_0 \to X$ coincide.

3. Upper curvature bounds

3.1. **Definition.** For a triangle Γ in a metric space X we consider the (unique up to Euclidean motions) comparison triangle $\Gamma_0 \subset \mathbb{R}^2$ with the same side-lengths as Γ. The triangle Γ is called *thick* (more precisely 0-thick) if there are points on Γ_0 which have smaller distance than the corresponding points on Γ, cf. [Bal04]. Otherwise the triangle is called *thin* (or CAT(0)-*triangle* in the terminology of [Bal04]).

A complete geodesic metric space X is CAT(0) if there are no thick triangles in X. The following observation allows the restriction to Jordan triangles:

Lemma 3.1. *Let X be a complete geodesic metric space. If X is not* CAT(0) *then there exists a thick Jordan triangle in X.*

Proof. If there are two different geodesics between a pair of points, then we find parts of these geodesics that build a Jordan curve. This Jordan curve is a geodesic bigon, a degenerate case of a triangle, which is automatically thick.

Otherwise geodesics are uniquely determined by their endpoints. Given a thick triangle with vertices A_1, A_2, A_3, we find a uniquely determined Jordan triangle with vertices A_1', A_2', A_3' in the union of the sides, by taking A_i' to be the last common point of the sides A_iA_j and A_iA_k. If the triangle $A_1'A_2'A_3'$ is thin, then so is the triangle $A_1A_2A_3$

by Alexandrov's lemma, cf. [Bal04, Lemma 3.5]. Thus we have found a thick Jordan triangle $A_1'A_2'A_3'$ in X. $\square$

3.2. **Majorization theorem.** Let X be a CAT(0) space. Due to the majorization theorem of Reshetnyak, [Res68], any closed curve $\gamma : [0,l] \to X$ parametrized by arclength is *majorized* by a closed convex set $\bar{\Omega} \subset \mathbb{R}^2$ in the following sense, cf. [AKP16] and [Bal04]. There exists a simple closed parametrization by arclength $\eta : [0,l] \to \mathbb{R}^2$ of the boundary $\partial\Omega$ and a 1-Lipschitz map $M : \bar{\Omega} \to X$ such that $M \circ \eta = \gamma$. Then, for any biLipschitz parametrization $F : \bar{D} \to \bar{\Omega}$, the area of the Lipschitz disc $M \circ F$ is bounded by $\text{Area}(M \circ F) \leq \mathcal{H}^2(\Omega)$. The isoperimetric inequality in $\mathbb{R}^2$ yields $\text{Area}(M \circ F) \leq \frac{1}{4\pi} l^2$. Now it is easy to deduce:

Lemma 3.2. *Let X be a* CAT(0) *space. Then any Lipschitz curve $\gamma : S^1 \to X$ of length l is the boundary of a Lipschitz map $u : \bar{D} \to X$ with* $\text{Area}(u) \leq \frac{1}{4\pi} l^2$.

Proof. Let $\gamma_0 : S^1 \to X$ be a parametrization of γ proportional to arclength. The existence of a Lipschitz map $u_0 : \bar{D} \to X$ extending γ_0 with the right bound on the area follows from the paragraph preceding the lemma. We attach to u_0 a Lipschitz annulus of zero area connecting γ_0 and γ by a linear reparametrization, cf. [LWY16, Lemma 3.6]. The arising Lipschitz disc u has the same area as u_0 and provides the required filling of γ. $\square$

3.3. **Curvature bounds via majorization.** The majorization theorem is closely related to the following observation.

Lemma 3.3. *Let X be a complete geodesic metric space. The space X is* CAT(0) *if and only if for any Jordan triangle $\Gamma \subset X$ there exists a* CAT(0) *space Z and a 1-Lipschitz map $F : Z \to X$ which sends some closed rectifiable curve $\Gamma' \subset Z$ in an arclength preserving way onto Γ.*

Proof. If X is CAT(0) then, for any triangle $\Gamma \subset X$, we can take $Z = X$ and $F = \text{Id} : Z \to X$. Now assume that any Jordan triangle in X is majorized by a CAT(0) space Z as in the formulation of the lemma. In order to prove that X is CAT(0), we only need to prove that any Jordan triangle Γ is thin. Consider a majorization $F : Z \to X$ of the triangle Γ. Then the preimage in Γ' of any geodesic contained in Γ is a geodesic in Z of the same length, cf. [AKP16, p.88]. Hence Γ' is a geodesic triangle in Z with the same side-lengths as Γ, thus Γ and Γ' have the same comparison triangle Γ_0 in $\mathbb{R}^2$. Since Z is CAT(0), the triangle Γ' is thin. Since $F : \Gamma' \to \Gamma$ is 1-Lipschitz we deduce that Γ is thin as well. $\square$

3.4. **Local curvature bounds.** A metric space X has non-positive curvature if any point in X has a CAT(0) neighborhood. A complete geodesic metric space X of non-positive curvature is CAT(0) if and only if X is simply connected, by a version of the theorem of Cartan-Hadamard [Bal04, Section 6].

3.5. **Reshetnyak's gluing theorem.** Let $X^{\pm}$ be CAT(0) spaces with closed convex subsets $A^{\pm} \subset X^{\pm}$. If $G : A^{+} \to A^{-}$ is an isometry then the space X arising from gluing X^{+} and X^{-} along the isometry G is CAT(0), cf. [BBI01, Theorem 9.1.21]. Localizing the statement we see that a gluing of two spaces of non-positive curvature along isometric locally convex subsets is again a space of non-positive curvature.

4. Generalities on Sobolev maps

We assume some knowledge of Sobolev maps with values in a metric space and refer to [HKST15], [Res97], [KS93] [LW17b], [LW18] and references therein for explanations. Let Ω be a bounded domain in $\mathbb{R}^2$ and X be a complete metric space. A map $u \in L^2(\Omega, X)$ is contained in the (Newton-) Sobolev space $N^{1,2}(\Omega, X)$ if there exists a Borel function $\rho \in L^2(\Omega)$ such that for 2-*almost all Lipschitz curves* $\gamma : [a, b] = I \to \Omega$ the composition $u \circ \gamma$ is continuous and

$$\ell_X(u \circ \gamma) \leq \int_\gamma \rho := \int_a^b \rho(\gamma(t)) \cdot |\gamma'(t)| \, dt \,. \tag{4.1}$$

We refer to [HKST15] for a thorough discussion of the notion of 2-almost all curves. For the present paper it is sufficient to know that for any biLipschitz embedding $F : I \times I \to \Omega$ and almost all $t \in I$ inequality (4.1) holds true for the curve $\gamma_t(s) = F(t, s)$. There exists a minimal function $\rho = \rho_u$ satisfying the condition above, uniquely defined up to sets of measure 0. It will be called the *generalized gradient* of u. The integral $\int_\Omega \rho_u^2(z) dz$ coincides with the Reshetnyak energy, see [Res97], [LW17b], which we denote by $E_+^2(u)$.

Let $u \in N^{1,2}(\Omega, X)$ be arbitrary. For almost all $z \in \Omega$ there exists a seminorm $\operatorname{ap}\operatorname{md} u_z$ on $\mathbb{R}^2$ called the approximate metric differential, such that the following conditions hold true, [Kar07], [LW17b, Section 4], [LW18, Lemma 3.1]. The map $z \mapsto \operatorname{ap}\operatorname{md} u_z$ into the space of seminorms has a Borel measurable representative. For 2-almost all curves $\gamma : I \to \Omega$ we have:

$$\ell_X(u \circ \gamma) = \int_I \operatorname{ap}\operatorname{md} u_{\gamma(t)}(\gamma'(t)) dt. \tag{4.2}$$

Moreover, for almost any $z \in \Omega$ we have $\rho_u(z) = \sup_{v \in S^1} \operatorname{ap}\operatorname{md} u_z(v)$.

There is a countable, disjoint decomposition $\Omega = S \cup_{1\le i<\infty} K_i$ into a set S of measure zero and compact subsets K_i such that the restriction of u to any K_i is Lipschitz continuous. The (parametrized Hausdorff) area of the Sobolev map u is defined to be $\mathrm{Area}(u) := \sum_{i=1}^{\infty} \mathrm{Area}(u_i)$, where u_i denotes the Lipschitz continuous restriction of u to K_i. This number $\mathrm{Area}(u)$ is finite, independent of the decomposition and generalizes the area of Lipschitz discs, cf. [LW18, Subsection 3.6].

A map $u \in N^{1,2}(\Omega, X)$ is called conformal if at almost all $z \in \Omega$ the seminorm $\mathrm{ap\,md}\, u_z$ is a multiple $f(z)\cdot s_0$ of the standard Euclidean norm s_0 on $\mathbb{R}^2$. In this case, $f \in L^2(\Omega)$ will be called the conformal factor of u. The conformal factor f of a conformal map $u \in N^{1,2}(\Omega, X)$ coincides with the generalized gradient ρ_u. In the conformal case, equation (4.2) therefore simplifies to

$$\ell_X(u \circ \gamma) = \int_\gamma f \,, \tag{4.3}$$

valid for 2-almost all curves γ in Ω. Moreover, the restriction of u to any subdomain $O \subset \Omega$ satisfies

$$\mathrm{Area}(u|_O) = \int_O f^2 \,. \tag{4.4}$$

Any map $u \in N^{1,2}(D, X)$ has a well-defined trace $\mathrm{tr}(u) \in L^2(S^1, X)$. If $u \in N^{1,2}(D, X)$ has a representative with a continuous extension to $\bar{D}$, then $\mathrm{tr}(u)$ is the restriction of this extension to the boundary circle.

5. Excluding non-Euclidean norms in tangent spaces

5.1. **Isoperimetric sets in normed planes.** Let V be a 2-dimensional normed space. There exists a convex subset $\mathbb{I}_V \subset V$ with the largest possible area among all convex sets with the same length of the boundary $\partial\mathbb{I}_V$. This subset is unique up to translations and dilations and is called the isoperimetric set, [Tho96]. The following reformulation of the Blaschke-Santalo inequality shows that the Euclidean isoperimetric inequality never holds in V unless V is Euclidean, cf. Remark 1.2:

Lemma 5.1. *In the notations above*

$$\mathcal{H}^2(\mathbb{I}_V) \ge \frac{1}{4\pi} \ell_V^2(\partial\mathbb{I}_V), \tag{5.1}$$

with equality if and only if V is Euclidean.

Proof. After rescaling (cf. [Tho96, (4.10)]) we may assume $2\mathcal{H}^2(\mathbb{I}_V) = \ell_V(\partial\mathbb{I}_V)$. Then (5.1) is equivalent to $\mathcal{H}^2(\mathbb{I}_V) \le \pi$ with equality if and only if V is Euclidean. However, due to [Tho96, (4.14)], this is exactly the statement of the 2-dimensional Blaschke-Santalo inequality [Tho96, Theorem 2.3.3]. □

5.2. **Formulation of the claim.** A complete metric space X has property (ET), if for any map $u \in N^{1,2}(D, X)$ almost all approximate metric differentials $\operatorname{ap}\operatorname{md} u_z$ are Euclidean norms or degenerate semi-norms, [LW17b, Section11]. Examples of spaces with property (ET) are spaces with one-sided curvature bounds and sub-Riemannian manifolds. We refer to [LW17b] for a thorough discussion of this property. The aim of this section is to prove the following:

Theorem 5.2. *Let X be a proper metric space with Euclidean isoperimetric inequality for curves. Then X has property (ET).*

5.3. **Sobolev-Dehn function.** For the limiting arguments used in the proof of Theorem 5.2 it is better to use a variant of the Dehn function with Sobolev instead of Lipschitz discs, due to better stability properties. For a complete metric space X we let the *Sobolev-Dehn function of X* be the minimal function $\delta_X^{Sob} : (0, \infty) \to [0, \infty]$ for which the following holds true. For any Lipschitz curve $\gamma : S^1 \to X$ of length at most r and any $\epsilon > 0$ there exists a Sobolev map $u \in N^{1,2}(D, X)$ with $\operatorname{tr}(u) = \gamma$ and $\operatorname{Area}(u) \leq \delta_X^{Sob}(r) + \epsilon$.

Since any Lipschitz disc is contained in $N^{1,2}(D, X)$ the Sobolev-Dehn function δ_X^{Sob} is bounded from above by the (Lipschitz-) Dehn function δ_X of the space X. If the space X is Lipschitz 1-connected, for instance a Banach or a CAT(0) space, then $\delta_X = \delta_X^{Sob}$, [LWY16, Propostion 3.1]. For any space X which satisfies the Euclidean isoperimetric inequality for curves, we have $\delta_X^{Sob}(r) \leq \frac{1}{4\pi} r^2$.

5.4. **Limiting arguments.** Property (ET) can be thought of as an infinitesimal property of the space, informally expressed by the condition that tangent spaces do not contain non-Euclidean normed planes. This idea can be made precise using blow-ups of metric spaces as a special case of ultralimits of the rescaled original space. We refer to [LW17b, Section 11] for details and just recall the following fact:

Lemma 5.3. *Let X be a complete metric space and ω a non-principal ultrafilter on $\mathbb{N}$. Assume that for all $x \in X$ and all sequences t_i of positive real numbers converging to 0 the following holds true: any normed plane V contained in the blow-up $B = \lim_\omega(\frac{1}{t_n} X, x)$ is Euclidean. Then X has property (ET).*

The only property of blow-ups needed for the proof of Theorem 5.2 is the following stability of quadratic isoperimetric inequalities from

[LWY16, Theorem 1.8]. Here it is crucial to work with Sobolev-Dehn functions and properness is used in an essential way.

Lemma 5.4. *Let X be a proper geodesic metric space with $\delta_X^{Sob}(r) \leq \frac{r^2}{4\pi}$. Then for any blow-up B of X as in Lemma 5.3, we have $\delta_B^{Sob}(r) \leq \frac{r^2}{4\pi}$ for all $r \geq 0$.*

5.5. **Quasi-convexity of the Hausdorff area.** Using Lemma 5.1 and [BI12] we readily obtain:

Proposition 5.5. *Assume that a complete metric space B contains a non-Euclidean normed plane V. Then the Sobolev-Dehn function of B satisfies $\delta_B^{Sob}(r) > \frac{r^2}{4\pi}$ for all $r > 0$.*

Proof. Let $\mathbb{I}_V \subset V$ be an isoperimetric set of V whose boundary $\partial \mathbb{I}_V$ has length r. The quasi-convexity of the Hausdorff area proved in [BI12] together with Lemma 5.1 implies that

$$\delta_B^{Sob}(r) \geq \mathcal{H}^2(\mathbb{I}_V) > \frac{r^2}{4\pi},$$

see also [LW17b, Section 2.4] and [LW18, Section 10.2]. This finishes the proof. □

Combining Lemma 5.3, Lemma 5.4 and Lemma 5.5, we finish the proof of Theorem 5.2.

Remark 5.1. A more direct but slightly more technical proof of Theorem 5.2 can be provided along the lines of [Wen08, Theorem 5.1], also including the case of non-proper target spaces X.

6. Solutions of the Plateau problem

6.1. **Solution of the Plateau problem.** Let X be a proper metric space with the Euclidean isoperimetric inequality for curves. Due to Theorem 5.2, the space X satisfies property (ET). Let Γ be a Jordan curve in X of finite length. Consider the non-empty set $\Lambda(\Gamma, X)$ of all maps $v \in N^{1,2}(D, X)$ whose trace is a weakly monotone parametrization of Γ. A *solution of the Plateau problem* for the boundary curve Γ is a conformal map $u \in \Lambda(\Gamma, X)$ which has smallest area among all maps in $\Lambda(\Gamma, X)$. Equivalently, u is a map with minimal Reshetnyak energy $E_+^2(u)$ among all maps in $\Lambda(\Gamma, X)$, [LW17b, Theorem 11.4]. Due to [LW17b, Corollary 11.5], a solution of the Plateau problem exists for every Jordan curve Γ of finite length in X. Any such solution of the Plateau problem has the following property, [LW17b, Theorem 1.4], [LW18, Proposition 1.8]:

Theorem 6.1. *Let Γ be a Jordan curve of finite length in X and let u be a solution of the Plateau problem for the curve Γ. Then u has a representative, again denoted by u, which continuously extends to $\bar{D}$. For any Jordan curve $\eta \subset \bar{D}$ with Jordan domain $J \subset D$*

$$\operatorname{Area}(u|_J) \leq \frac{1}{4\pi} \ell_X^2(u \circ \eta).$$

In fact, from [LW15,Theorem 1.4] one can conclude that u is locally Lipschitz on D.

6.2. **Intrinsic minimal disc.** Let X, Γ, u be as in Theorem 6.1. In [LW18] it was shown that the intrinsic pseudo-metric d_u on $\bar{D}$ described in Subsection 2.2 is well defined, finite-valued and continuous with respect to the Euclidean metric. As in Subsection 2.2, denote by $Z = Z_u$ the associated metric space. Then the following holds true, see [LW18, Theorem 1.1, Theorem 1.2, Theorem 1.4, Theorem 1.5]:

Theorem 6.2. *Let Γ be a Jordan curve in X of finite length and let $u : \bar{D} \to X$ be a continuous solution of the Plateau problem with boundary Γ. Let $Z = Z_u$ be the associated length metric space, $P : \bar{D} \to Z$ the canonical projection and $\bar{u} : Z \to X$ the unique map with $u = \bar{u} \circ P$. Then we have:*

1. *Z is a geodesic space homeomorphic to $\bar{D}$ and P is continuous. The preimage $P^{-1}(Z \setminus \partial Z)$ is homeomorphic to D.*
2. *The map $\bar{u} : Z \to X$ is 1-Lipschitz and sends ∂Z in an arclength preserving way onto Γ.*
3. *For any curve $\gamma \subset \bar{D}$ we have $\ell_X(u \circ \gamma) = \ell_Z(P \circ \gamma)$.*
4. *For any open $V \subset D$ we have $\operatorname{Area}(u|_V) = \mathcal{H}^2_Z(P(V))$.*
5. *For any Jordan curve $\eta \subset Z$ and the corresponding Jordan domain $O \subset Z$ we have $\mathcal{H}^2(O) \leq \frac{1}{4\pi}\ell_Z^2(\eta)$.*

The space Z in Theorem 6.2 will be called the intrinsic minimal disc associated with u.

6.3. **Reduction to Theorem 1.2.** Now we can prove:

Proposition 6.3. *Theorem 1.2 implies Theorem 1.1.*

Proof. The "only if part" of Theorem 1.1 has already been verified in Lemma 3.2, since the identity map $\mathrm{Id} : X^i \to X$ is 1-Lipschitz and lengths of curves in X and in X^i coincide. Let now X be a proper metric space which satisfies the Euclidean isoperimetric inequality for curves. Assume in addition that any pair of points in X is connected by a curve of finite length. Consider the induced length space X^i. Since X is proper, the space X^i is a complete geodesic metric space. We are going to prove that X^i is CAT(0). We take an arbitrary Jordan triangle $\Gamma \subset X^i$ and need to majorize it by some CAT(0) space in the sense of Lemma 3.3. Now Γ has the same length when viewed as a curve in X. We find a solution u of the Plateau problem for the curve $\Gamma \subset X$ and apply Theorem 6.1 and Theorem 6.2 to $\Gamma \subset X$. As in Theorem 6.2, we denote by Z the intrinsic minimal disc associated with u. Thus, Z is a compact geodesic metric space homeomorphic to $\bar{D}$ and there exists a 1-Lipschitz map $\bar{u} : Z \to X$ which maps the boundary ∂Z in an arclength preserving way to Γ. Since Z is a geodesic space, the map $\bar{u}$ considered as a map to X^i is still 1-Lipschitz and arclength preserving on ∂Z. Assuming that Theorem 1.2 holds true, the space Z is CAT(0). Thus, Lemma 3.3 implies that X^i is CAT(0) as well. □

7. The conformal factor

7.1. **An integral inequality.** Let X be a complete metric space which satisfies the Euclidean isoperimetric inequality for curves. Let Γ be a Jordan curve of finite length in X and let $u : \bar{D} \to X$ be a solution of the Plateau problem as in Theorem 6.1. Let $f \in L^2(D)$ be a conformal factor of u. Applying Theorem 6.1 to concentric circles and using (4.4) and (4.3) we deduce:

Lemma 7.1. *The conformal factor $f \in L^2(D)$ satisfies the inequality*

$$\int_{B(z,r)} f^2 \leq \frac{1}{4\pi} \cdot \left(\int_{\partial B(z,r)} f \right)^2 , \tag{7.1}$$

for any $z \in D$ and almost any $0 < r < 1 - |z|$.

7.2. **Log-subharmonic functions.** Recall that a function $f : U \to [-\infty, \infty)$ defined on a domain $U \subset \mathbb{R}^2$ is called *subharmonic* if f is upper semi-continuous, contained in L^1_{loc} and satisfies $f(z) \leq \fint_{B(z,s)} f$ for all $z \in U$ and all $s > 0$ with $B(z,s) \subset U$. Here and below, we denote by $\fint_T g = \fint_T g \, d\mu$ the integral mean value $\frac{1}{\mu(T)} \int_T g \, d\mu$ of a function integrable with respect to a measure μ. A function $f \in L^1_{loc}(U)$ has a subharmonic representative if and only if $\Delta f \geq 0$ in the distributional sense. This representative is uniquely defined at each point by $f(z) = \lim_{s \to 0} \fint_{B(z,s)} f$.

A function $f : U \to [0, \infty)$ is called *log-subharmonic* if $\log(f)$ is a subharmonic function. Any log-subharmonic function is locally bounded. Log-subharmonic functions are intimately related to non-positive curvature.

Inequality (7.1) turns out to imply log-subharmonicity (the other implication is true as well, cf. [BR33], but will not be needed here).

Proposition 7.2. *Any non-negative function $f \in L^2(D) \setminus \{0\}$ which satisfies* (7.1) *has a log-subharmonic representative.*

Proof. We can rewrite (7.1) in terms of the integral averages as

$$\fint_{B(z,r)} f^2 \leq \left(\fint_{\partial B(z,r)} f \right)^2 . \tag{7.2}$$

For continuous positive functions f satisfying (7.2), the log-subharmonicity is proved in [BR33, Lemma on p. 665]. The general case reduces to the case of smooth positive functions as follows, cf. [EKMS09]. Applying Hoelder's inequality to (7.2) we infer

$$\fint_{B(z,r)} f \leq \fint_{\partial B(z,r)} f . \tag{7.3}$$

Thus, f has a subharmonic representative, see [EKMS09, Lemma 4.6, Remark 4.8]. In particular, f is locally bounded.

A combination of (7.2) and (7.3) directly implies that for any $\delta > 0$ the function $f^\delta(z) := f(z) + \delta$ satisfies (7.2) as well. If f^δ has a log-subharmonic representative for all $\delta > 0$ then (after changing to the subharmonic representative) we obtain f as a limit of locally uniformly bounded log-subharmonic functions. Then, by the classical convergence theorems for subharmonic functions (cf. [AG01, Section 3.7, Exercise 3.15]), the function f has a log-subharmonic representative. Therefore, it suffices to prove the proposition under the assumption that f is everywhere positive.

For this, we take any $\epsilon > 0$ and consider the usual mollified functions $f_\epsilon : B(0, 1-\epsilon) \to [0, \infty)$ obtained by convolutions with a standard (Friedrichs) mollifier. We use the observation of [EKMS09, Lemma 4.5] (going back to the proof in [BR33]) that the smooth function f_ϵ still satisfies (7.2) for all balls contained in $B(0, 1-\epsilon)$. Due to [BR33], f_ϵ is log-subharmonic. By the limiting argument as above f has a log-subharmonic representative as well. This finishes the proof. □

7.3. **Conclusion.** Taking Proposition 7.2 and Lemma 7.1 together we have shown that the conformal factor $f \in L^2(D)$ of our minimal disc u has a log-subharmonic representative $\bar{f}$. From now on we will replace f by $\bar{f}$ and assume that f is log-subharmonic.

8. Metric defined by a conformal factor

We refer to [Res93] and the references therein for a detailed description of the theory, a special case of which is sketched here. Let U be a domain in $\mathbb{R}^2$ and let $f : U \to [0, \infty)$ be a log-subharmonic function.

For a Lipschitz curve $\gamma : [a, b] \to U$ define the f-length of γ to be

$$L_f(\gamma) := \int_\gamma f = \int_a^b f(\gamma(t))\,|\gamma'(t)|\,dt.$$

The f-length does not change if γ is reparametrized. Since f is locally bounded the f-length of any Lipschitz curve is finite.

Define $d_f : U \times U \to [0, \infty)$ by

$$d_f(z, z') := \inf\{L_f(\gamma) \mid \gamma \text{ Lipschitz curve between } z \text{ and } z'\}. \tag{8.1}$$

This function d_f defines a metric on U and the identity map $i : U \to (U, d_f)$ from the Euclidean subset U to the new metric space is a homeomorphism, [Res93, Theorem 7.1.1]. Denote by Y the metric space (U, d_f).

The metric d_f does not change if in (8.1) the infimum is taken over all injective curves of bounded variation of turn instead over all Lipschitz curves, [Res93, p.101]. If injective curves γ_n of uniformly bounded variation of turn converge pointwise to the curve γ in U then, due to [Res93, Theorem 8.4.4],

$$\lim L_f(\gamma_n) = L_f(\gamma)\,. \tag{8.2}$$

Thus, the distance $d_f(z_1, z_2)$ can be defined by the formula (8.1), where the infimum is taken over the set of all polygonal curves $\gamma \subset U$ between z_1 and z_2.

For any compactly contained subdomain $V \subset U$, the restriction $i : V \to Y$ is Lipschitz continuous, since f is locally bounded. For any Lipschitz curve γ in U we have $\ell_Y(i \circ \gamma) = L_f(\gamma)$, [Res63]. We deduce from (4.2) that $i : V \to Y$ is conformal with conformal factor f. Therefore:

$$\mathcal{H}^2(i(V)) = \int_V f^2\,. \tag{8.3}$$

The main results of Reshetnyak's analytic theory of Alexandrov surfaces of (integral) bounded curvature [Res93, Theorems 7.1 and 7.2] take in our case the following form:

Theorem 8.1. *The space $Y = (U, d_f)$ constructed above has non-positive curvature. Conversely, for any space M of non-positive curvature which is homeomorphic to a surface without boundary the following is true. For any point $x \in M$ there exists a neighborhood of x isometric to some $Y = (U, d_f)$, where U is a domain in $\mathbb{R}^2$ and $f : U \to [0, \infty)$ is log-subharmonic.*

Proof. We merely explain why Theorem 8.1 is a special case of Reshetnyak's results, relying on the results presented in [Res93].

Recall that a locally compact length space X has non-positive curvature if and only if any point $x \in X$ has a neighborhood U such that any triangle Γ in U has a non-positive excess, [Ale57, p.36]. (Here the excess of a triangle is the sum of its angles minus π.) Thus a length space X homeomorphic to a surface without boundary has non-positive curvature if and only if it has bounded (integral) curvature in the sense of Aleksandrov and the (signed) curvature measure of X is non-positive.

Now both claims of Theorem 8.1 are exactly Theorems 7.1 and 7.2 in [Res93]. □

9. Reduction to Theorem 1.3

9.1. **Formulation.** The aim of this section is to prove:

Proposition 9.1. *Theorem 1.3 implies Theorem 1.2.*

Thus we assume that Theorem 1.3 is true. Let X be a proper metric space which satisfies the Euclidean isoperimetric inequality for curves. Let Γ be a Jordan curve of finite length in X. Let $u : \bar{D} \to X$ be a solution of the Plateau problem in X for the boundary curve Γ. Denote by Z the intrinsic minimal disc associated with u as in Theorem 6.2 and let $P : \bar{D} \to Z$ be the canonical surjective map. Let Z_0 be the open disc $Z \setminus \partial Z$ and denote by D_0 the preimage $P^{-1}(Z_0)$. Then D_0 is homeomorphic to the open disc and, in particular, $D_0 \subset D$.

Let f denote the conformal factor of u, which is log-subharmonic due to Section 7. Denote by Y_0 the open disc D_0 equipped with the length metric d_f as introduced in the previous section. Theorem 8.1 implies that Y_0 has non-positive curvature. Let $i : D_0 \to Y_0$ denote the canonical homeomorphism (identity map). Let $I : Y_0 \to Z_0 \subset Z$ denote the composition $P \circ i^{-1}$. We now easily conclude, using Theorem 1.3:

Lemma 9.2. *If $I : Y_0 \to Z_0$ is a local isometry then Z is* CAT(0)*.*

Proof. If I is a local isometry then it is locally injective. By the invariance of domains we see that I is an open map. Since I is surjective and Y_0 non-positively curved, the space Z_0 has non-positive curvature. Due to Theorem 6.2, the space Z satisfies the isoperimetric inequality for Jordan curves as required in Theorem 1.3, (3). From Theorem 1.3 we deduce that Z is CAT(0). □

Therefore, in order to prove Proposition 9.1, we only need to show that $I : Y_0 \to Z_0$ is a local isometry.

9.2. **Properties of the map I.** We claim:

Lemma 9.3. *The map $I : Y_0 \to Z_0$ is* 1*-Lipschitz.*

Proof. Since the metric in Y_0 can be defined using f-lengths of polygonal curves we only need to prove $L_f(\gamma) \geq \ell_Z(P \circ \gamma) = \ell_X(u \circ \gamma)$ for any straight segment $\gamma : [a,b] \to D$. Consider the variation γ_s, for $-\epsilon < s < \epsilon$, of $\gamma_0 = \gamma$ through segments parallel to γ. For any s, we have $L_f(\gamma_s) = \int_{\gamma_s} f$. Moreover, by the definition of the conformal factor, we have $\ell_X(u \circ \gamma_s) = \int_{\gamma_s} f$ for *almost all* $s \in (-\epsilon, \epsilon)$. Thus, we find a sequence $s_n \to 0$ such that $L_f(\gamma_{s_n}) = \ell_X(u \circ \gamma_{s_n})$. The result follows from

$$L_f(\gamma) = \lim_{n\to\infty} L_f(\gamma_{s_n}) = \lim_{n\to\infty} \ell_X(u \circ \gamma_{s_n}) \geq \ell_X(u \circ \gamma_0) = \ell_X(u \circ \gamma),$$

where we have used (8.2) for the first equality. □

By construction, (4.4), (8.3) and Theorem 6.2 (4), we obtain:

Lemma 9.4. *The map I preserves the Hausdorff measure $\mathcal{H}^2$. More precisely, for any open subset $V \subset D_0$ we have*

$$\mathcal{H}^2(i(V)) = \int_V f^2 = \mathcal{H}^2(P(V)).$$

9.3. **The conclusion.** Now we can finish the proof of the main result of this section.

Proof of Proposition 9.1. By the definition of the metrics on Y_0 and Z it suffices to show that for any simple curve $\eta : [a,b] \to Y_0$ we have $\ell_{Y_0}(\eta) = \ell_Z(I \circ \eta)$. Assume the contrary and consider a curve η with

$$\ell_{Y_0}(\eta) > \ell_Z(I \circ \eta).$$

In order to obtain a contradiction we will roughly proceed as follows. We will first complement a subcurve of η to a Jordan curve and equip the closure of the corresponding Jordan domain with a new metric, to which Theorem 1.3 will be applied. Inside the domain, the new metric will come from that of Y_0, while the length of the boundary will come from that of its image in Z.

To be more concrete, we first replace the curve η by a subcurve and may assume that either $I \circ \eta$ is a point or that no subarc of η is mapped by I to a point. Further replacing η by a slightly smaller subcurve, we may assume that η is part of a Jordan curve T such that the closure of $T \setminus \eta$ is a rectifiable arc η' in Y_0. Let $J \subset Y_0$ denote the Jordan domain of T whose closure is $\bar{J} = J \cup T$.

We call admissible any curve in $\bar{J}$ that is a finite concatenation of simple curves either completely contained in η or intersecting η at most in a finite set of points. Define the new length $\mathcal{L}^+(\gamma)$ of such an admissible curve γ to be the sum of the Y_0-lengths of arcs outside of η and the lengths of the images in Z of the subarcs contained in η. The pseudo-distance $d^+ : \bar{J} \times \bar{J} \to [0, \infty]$ associated with the length functional $\mathcal{L}^+$ is finite-valued and continuous with respect to the Euclidean topology on $\bar{J}$ since η' and $I \circ \eta$ have finite length. Denote the corresponding metric space by S and let $Q : \bar{J} \to S$ be the canonical projection.

Then S is a compact length space, hence it is a geodesic space. The map Q is a local isometry outside T and a homeomorphism outside η (here and below we consider $\bar{J}$ with the metric restricted from Y_0, not with a length metric!). If I does not send η to a point (hence is bijective by assumption) then Q is bijective, hence a homeomorphism. If I sends η to a point then Q collapses η to a point in S. In both cases, S is homeomorphic to $\bar{D}$. The restriction $I : \bar{J} \to Z$ factorizes through Q and for any curve $k \subset \bar{J}$ we have $\ell_S(Q \circ k) \geq \ell_Z(I \circ k)$ by Lemma 9.3.

We claim that S is CAT(0). Indeed, $S \setminus \partial S$ is locally isometric to J. Thus $S \setminus \partial S$ has non-positive curvature. Due to Theorem 1.3 we only need to verify the isoperimetric inequality for Jordan curves $c \subset S$. For any Jordan curve c in S there is a unique Jordan curve $\hat{c}$ in $\bar{J}$ which is mapped by Q to c. Let $G \subset J$ denote the Jordan domain of $\hat{c}$. Then $Q(G) \subset S$ is the Jordan domain of c and has the same Hausdorff measure as $G \subset Y_0$ since Q is a local isometry outside T.

We have:

$$\ell_S(c) = \ell_S(Q \circ \hat{c}) \geq \ell_Z(I \circ \hat{c}) = \ell_X(u \circ i^{-1} \circ \hat{c}) \ .$$

Moreover,

$$\mathcal{H}^2_S(Q(G)) = \mathcal{H}^2_{Y_0}(G) = \mathcal{H}^2_Z(I(G)) = \mathrm{Area}(u|_{i^{-1}(G)}) \ .$$

Since $i^{-1}(G) \subset D$ is the Jordan domain of $i^{-1}(\hat{c})$, the desired inequality

$$\mathcal{H}^2_S(Q(G)) \leq \frac{1}{4\pi} \cdot \ell^2(c)$$

follows from Theorem 6.1. Therefore, an application of Theorem 1.3 finishes the proof of the claim that S is CAT(0).

Another application of Theorem 1.3 implies that the metric on $S \setminus \partial S$ is a length metric. Therefore, the length preserving map $Q^{-1} : S \setminus \partial S \to J$ is 1-Lipschitz. Thus, it extends to a 1-Lipschitz map $\hat{Q}^{-1} : S \to \bar{J}$. By continuity, the composition $\hat{Q}^{-1} \circ Q$ must be the identity on $\bar{J}$.

By assumption, $\ell_S(Q \circ \eta) < \ell_{Y_0}(\eta)$. We obtain a contradiction to $\eta = \hat{Q}^{-1} \circ Q \circ \eta$, since the 1-Lipschitz map $\hat{Q}^{-1}$ cannot increase lengths. $\square$

Thus Theorems 1.1 and 1.2 are reduced to Theorem 1.3.

Part II. Geometry of strange metric discs.

10. Plan of the second part of the paper

The second part of the paper is devoted to the proof of Theorem 1.3. Before we start with the actual proof of the theorem, we recollect in Section 11 some basic observations about the local geometry of non-positively curved surfaces. In Section 12 we prove that the completion of a non-positively curved open disc is CAT(0) and homeomorphic to a closed disc whenever compact. All but the main implication (3) to (1) of Theorem 1.3 turn out to be relatively simple. The proofs of these simple implications are provided in Section 13.

In Section 14 we embark on the proof of the implication (3) to (1) of Theorem 1.3, thus assuming the isoperimetric inequality and trying to prove that the disc Z is CAT(0). We use subdomains of Z in order to reduce the situation to the case where the "problematic" part of ∂Z consists of a geodesic $c \subset \partial Z \subset Z$. We consider the complement $Y_0 = Z \setminus \partial Z$ equipped with the induced length metric and the completion Y of Y_0, which is CAT(0) and compact. The space Y comes along with a canonical 1-Lipschitz map $I : Y \to Z$ which is a local isometry in Y_0. This reduces the question to the situation described in the introduction (the example of Koch's snowflake): the disc Z arises from a CAT(0) disc Y by possibly shortening or collapsing a part of the boundary curve $\eta \subset Y$.

If I is an isometry then we are done. Otherwise, some part η of the boundary curve ∂Y is sent by I to a curve in ∂Z of smaller length. In order to obtain a contradiction to the isoperimetric inequality, it suffices to extend small parts of η to Jordan curves T in Y which bound almost optimal isoperimetric regions in Y. The map I then shortens the length of T but leaves the area of the enclosed Jordan domain unchanged, thus $I(T) \subset Z$ violates (3) of Theorem 1.3.

If η is rectifiable we approximate η by geodesics and use the approximation of Y by flat cones in order to reduce the problem to the situation where η is a line in $\mathbb{R}^2$. In that case one can take as suitable Jordan curves T large parts of sufficiently large circles complemented by a short chord contained in η. This is carried out in Section 15. The more difficult case of a non-rectifiable curve η is carried out in Section 16. Here the non-rectifiability of η provides parts which are contracted by an arbitrary large amount. This allows sufficient flexibility in the choice of critical Jordan curves in Y.

11. Geometry of non-positively curved surfaces

11.1. **Basic geometric features.** Let M be a metric space of non-positive curvature homeomorphic to D. We refer to [Res93] and [BB98] for a deep analysis of such and related more general spaces. Let $x \in M$ be arbitrary. We find a small open metric ball $U = B(x, \epsilon)$ around x whose closure is a compact CAT(0) space. Any geodesic starting at x can be extended to a geodesic in $\bar{U}$ of length 2ϵ, [BB98, 1.B.7]. The space of directions Σ_x is a circle of some length $\alpha \geq 2\pi$. By definition, the tangent cone $T_x M$ at the point x is the Euclidean cone over Σ_x.

11.2. **Hinges.** Let $M, x \in U = B(x, \epsilon) \subset M$ be as above and let $\gamma_{1,2}$ be geodesics of length $\geq \epsilon$ starting in x and having only the point x in common. Then $\gamma_{1,2}$ intersects the boundary circle of $B(x, \epsilon)$ at exactly one point. Denote by Γ the union of (the images of) γ_1 and γ_2 inside $B(x, \epsilon)$. Then Γ divides $B(x, \epsilon)$ into two closed subsets $H_\pm$ homeomorphic to closed half-planes intersecting in their common boundary Γ. We call $H_\pm$, equipped with the induced length metric, the *hinges* (of size ϵ) defined by $\gamma_{1,2}$.

We claim that both hinges have non-positive curvature. If γ_1 and γ_2 concatenate to a geodesic then $H_\pm$ are convex in $B(x, \epsilon)$ and the claim follows. Otherwise, we extend γ_1 by a geodesic γ_1^+ of length ϵ starting in x to a geodesic γ of length 2ϵ. Then γ divides $B(x, \epsilon)$ into two convex subsets $A_\pm$ with common boundary γ. Without loss of generality, we may assume that H_+ is contained in A_+. Then $\gamma \cup \gamma_2$ divide $B(x, \epsilon)$ into 3 convex hinges H_+, A_- and a third hinge H' (between γ_2 and γ_1^+). The hinge H_+ has non-positive curvature by convexity and H_- is the result of gluing H' and A_- along the geodesic γ_1^+, hence it has non-positive curvature by Reshetnyak's gluing theorem. This finishes the proof of the claim.

Lemma 11.1. *Let M be a space of non-positive curvature homeomorphic to the open disc. Let Γ be a Jordan polygon in M. Then the closed Jordan domain $\bar{J}$ of Γ with its intrinsic metric is* CAT(0).

Proof. The space $\bar{J}$ is compact and simply connected. At any point $x \in \bar{J}$ a small ball around x is either open in M or isometric to a hinge described above. Therefore, $\bar{J}$ is non-positively curved. The lemma follows from the theorem of Cartan-Hadamard. □

11.3. **Approximation by flat cones.** For the next result we will use Theorem 8.1 from above and a theorem about the approximation of Alexandrov surfaces by their tangent cones.

Lemma 11.2. *Let X be a space of non-positive curvature which is homeomorphic to a surface (possibly with boundary). Let $x \in X$ be a point. Let $\gamma_{1,2}$ be geodesics starting at x and enclosing a positive angle. Then there exists a closed interval $T \subset \mathbb{R}$, a ball O around the vertex o of the Euclidean cone CT over T and a biLipschitz map $E : O \to E(O) \subset X$ with the following properties.*

(1) *The map E sends the vertex o of CT to x.*
(2) *E sends initial parts of the boundary rays of the flat hinge CT isometrically onto the initial parts of $\gamma_{1,2}$.*
(3) *We have $\lim_{v,w \to o} \frac{d(E(v),E(w))}{d(v,w)} \to 1$.*

Thus, the biLipschitz constant of the restriction of E to small balls around o goes to 1 with the radius of the balls tending to 0. Clearly, the length of the interval T, hence the total angle of the Euclidean hinge CT, is not less than the angle between γ_1 and γ_2.

Proof of Lemma 11.2. We first assume that x is not on the boundary ∂X. The existence of a biLipschitz map $E' : O \to M$ from a ball O around the origin $0 \in T_x M$ such that (1) and (3) of Lemma 11.2 hold true is the content of a theorem of Y. Burago, [Bur65], stated in [Res93, Theorem 9.10]. Note that this theorem is applicable due to Theorem 8.1 above. Composing E' with a self-isometry of $T_x M$ we can assume that the initial part of a given ray η starting at the origin 0 is sent by E' to a curve $E' \circ \eta$ whose starting direction coincides with the starting direction of η. Consider the rays $\eta_{1,2}$ in $T_x M$ whose starting directions are $\gamma_{1,2}'(0) \in \Sigma_x \subset T_x M$. We now find a biLipschitz map of $T_x M$ to itself, which fixes 0, has at 0 the identity as its differential, and which sends the initial part of η_i to $(E')^{-1}(\gamma_i)$. Composition of E' with this biLipschitz map provides the required map E upon restriction to the smaller hinge in O between the rays η_i.

Let us now assume that x is contained in the boundary ∂X. We choose any simple arc γ connecting a point on γ_1 with a point on γ_2 and such that γ and the corresponding parts of γ_1 and γ_2 constitute together a Jordan curve T. Consider the union V of the corresponding Jordan domain and the curve $T \setminus \gamma$. Since $\gamma_{1,2}$ are geodesics and $x \in \partial X$, the set V is locally convex in X by topological reasons. Thus, in order to find a biLipschitz embedding required in Lemma 11.2 we may replace X by V and therefore assume that ∂X is the union of the geodesics $\gamma_{1,2}$. But (a small ball around x in) such a space is isometric to a hinge in some manifold M *without boundary* which still has non-positive curvature, as we see by applying Reshetnyak's gluing theorem twice. (First we glue to V along γ_1 a hinge of large angle from $\mathbb{R}^2$. In the so arising space the boundary is a geodesic and we may double the arising space to obtain the required manifold M). By construction V is the smaller hinge of the two hinges determined by $\gamma_{1,2} \subset M$. Thus, the map E constructed above (for the space M) has its image in V. This finishes the proof of Lemma 11.2. □

Using the notations of Lemma 11.2, we will call *hinge* between two geodesics $\gamma_{1,2}$ the intersection of $E(O)$ with a small metric ball $B(x,r)$. This provides an extension of the definition of a hinge to the case of points at the boundary.

12. Completions of 2-dimensional open discs

Before embarking on the proof of Theorem 1.3 we will study the interiors of discs appearing in that theorem and their completions. The following basic result generalizes [Bis08].

Proposition 12.1. *Let Y_0 be a length space homeomorphic to the open disc. If Y_0 is non-positively curved then the completion Y of Y_0 is* CAT(0)*.*

Proof. Choose Jordan curves Γ'_n with increasing Jordan domain whose union is Y_0. Approximate Γ'_n by Jordan polygons Γ_n. Let J_n be the corresponding Jordan domains and denote by Y_n the closure of these Jordan domains, equipped with their intrinsic metric.

By Lemma 11.1, every Y_n is CAT(0). The completion Y isometrically (and canonically) embeds into the CAT(0) space Y', obtained as an ultralimit of the Y_n (choosing the same fixed point lying in $Y_1 \subset Y_n$ as the base point of the spaces Y_n). Hence Y is isometric to a subset of the CAT(0) space Y'. Due to completeness, Y is a closed subset of Y'. Since Y_0 is a length space, its completion Y is a length space as well. Therefore, Y must be convex in Y'. Thus Y is CAT(0). □

The reader should consult [AG99, p.1270] for references on homology manifolds used in the next lemma.

Lemma 12.2. *Let Y_0 be a non-positively curved length metric space homeomorphic to D. If the completion Y of Y_0 is compact then Y is homeomorphic to $\bar{D}$.*

Proof. By construction, Y_0 is dense in Y. Since Y_0 is locally complete, Y_0 is open in Y. The space Y is a separable CAT(0) space, hence it is contractible and locally contractible. The topological dimension of Y coincides with its geometric dimension, see [Kle99]. But Y embeds isometrically into an ultralimit of 2-dimensional CAT(0)-spaces, therefore the geometric dimension of Y is at most 2, see [Kle99], [Lyt05, Lemma 11.1]. Thus Y has topological dimension 2.

Set $\partial Y := Y \setminus Y_0$. We claim that for any $z \in \partial Y$ the local homology with integer coefficients $H_*(Y, Y \setminus \{z\})$ vanishes. In order to see this, note that $Y \setminus \{z\}$ is connected since Y_0 is connected. By dimensional reasons, the contractibility of Y, and the long exact sequence of the pair $(Y, Y \setminus \{z\})$, we only need to prove that $H_1(Y \setminus \{z\}) = 0$. Since Y is locally contractible and Y_0 is contractible it is sufficient to prove that any closed curve $\gamma : S^1 \to Y \setminus \{z\}$ can be approximated by closed curves with images in Y_0. Covering γ by small metric balls $B(x, r)$, we observe that it is sufficient to prove that $B(x, r) \cap Y_0$ is connected for any $x \in Y$ and $r > 0$. But this follows from the fact that Y_0 is a length space and Y is the completion of Y_0. This finishes the proof of the claim.

Thus Y is a homology 2-manifold with boundary ∂Y. Therefore, ∂Y is a homology 1-manifold and, due to [Ray60], the doubling Y^+ of Y along the boundary ∂Y is a homology 2-manifold without boundary. Due to [Wil49, IX.5.9 and IX.5.10], for $n = 1, 2$, any homology n-manifold without boundary is a manifold without boundary. Thus Y^+ is 2-manifold and ∂Y is a closed 1-submanifold. We deduce that Y is a manifold with boundary. Since Y_0 is homeomorphic to D, the space Y must be homeomorphic to $\bar{D}$. □

13. Simple implications

We now embark on the proof of Theorem 1.3. Thus, from now on, let Z be a geodesic metric space homeomorphic to $\bar{D}$ and such that the space $Z \setminus \partial Z$ is non-positively curved.

13.1. **(1) implies (3).** Suppose the space Z is CAT(0) and consider a Jordan curve $\eta \subset Z$ of finite length ℓ. Then Reshetnyak's majorization theorem (Lemma 3.2) provides a Lipschitz disc $u : \bar{D} \to Z$ filling η of area at most $\frac{1}{4\pi} \cdot \ell^2$. For topological reasons, the image of u must contain the Jordan domain J of η. Therefore, $\mathcal{H}^2(J) \leq \text{Area}(u) \leq \frac{1}{4\pi} \cdot \ell^2$.

13.2. **(2) implies (1).** Note that Z is the completion of $Z \setminus \partial Z$. Thus, if $Z \setminus \partial Z$ is a length space then Z is CAT(0) by Proposition 12.1.

13.3. **(1) implies (2).** Thus, we assume that Z is CAT(0) and claim that $Z \setminus \partial Z$ is a length space. The proof of this implication (probably well-known to experts) is slightly more involved.

Consider arbitrary points $y_+, y_- \in Z \setminus \partial Z$. Let $\gamma : [a, b] \to Z$ be a geodesic between y_+ and y_-. Fix a positive $\epsilon > 0$. We need to find a curve γ_ϵ in $Z \setminus \partial Z$ which connects y_+ and y_- and has length at most $(1+2\epsilon) \cdot \ell(\gamma)$.

For any $y = \gamma(t)$ we claim the existence of an open ball W^t around y with the following property. For any $z_1, z_2 \in W^t \cap \gamma$ there exists a curve $\eta \subset W^t$ connecting z_1 and z_2, with $\eta \cap \partial Z \subset \{z_1, z_2\}$ and such that

$$(1+\epsilon) \cdot d_Z(z_1, z_2) \geq \ell(\eta).$$

Indeed, for $y \notin \partial Z$, in particular for $y = y_\pm$, the claim is evident. For any $y \in \partial Z$ we apply Lemma 11.2 to both parts of γ emanating from y and deduce the claim from the corresponding result in the flat hinge CT, where the claim is clear as well.

We can cover γ by a finite number of open balls $U_i = W^{t_i}$. By choosing an appropriate subsequence and rearrangement, we may assume that any two consecutive balls in the sequence U_i intersect. Choose arbitrary points y_i on γ in the intersection of U_i and U_{i+1}, with the only requirement that $y_i \notin \partial Z$ whenever $U_i \cap U_{i+1} \cap \gamma$ is not completely contained in ∂Z. We connect y_i and y_{i+1} by a curve $\eta_{i+1} \subset U_{i+1}$ provided by the definition of W^{t_i}. We may assume the first y_0 and the last y_m to be the ends $y_\pm$ of γ. Denote by η the concatenation of all η_i. Then η is a curve in Z between y_+ and y_- which has length at most $(1+\epsilon) \cdot \ell(\gamma)$. Moreover, η intersects ∂Z at most at finitely many points $y'_j = \eta(s_j)$.

For any such y'_j, a neighborhood of y'_j in ∂Z is contained in γ. We again apply Lemma 11.2 to both parts of γ emanating from y'_j and note that in this case the image of the map E must be *open* in Z by the invariance of domains. Using the corresponding property in the flat cone CT, we find an arbitrarily short curve k_j with the following property. The curve k_j connects two points $\eta(a_j), \eta(b_j)$ with some $a_j < s_j < b_j$ and does not intersect ∂Z. Replacing $\eta|_{[a_j, b_j]}$ by k_j we obtain the desired short connection between y_+ and y_-.

14. Some simplifications

The rest of the paper is devoted to the proof that (3) implies (1) in Theorem 1.3. Thus, from now on we assume that Z is a geodesic metric space homeomorphic to $\bar{D}$, such that $Z \setminus \partial Z$ is non-positively curved and such that Z satisfies the isoperimetric inequality as stated in Theorem 1.3, (3). We need to prove that Z is CAT(0).

14.1. **Subdomains.** Let T be a Jordan curve of finite length in Z with Jordan domain J. Consider the closure $\bar{J} = J \cup T \subset Z$ with the induced length metric. Since T has finite length, the topologies induced by the length metric and by the induced metric coincide. Thus $\bar{J}$ is a compact length metric space homeomorphic to $\bar{D}$. The compactness implies that $\bar{J}$ is geodesic. The identity map $F : \bar{J} \to Z$ preserves the lengths of all curves and the $\mathcal{H}^2$-area of all domains. Moreover, the restriction of F to $J = \bar{J} \setminus T$ is a local isometry. Therefore, the assumptions of Theorem 1.3, (3) are valid for the space $\bar{J}$ as well. As a consequence, we may reduce the problematic part of Z to a single geodesic:

Lemma 14.1. *We may assume in addition that*

(1) $\mathcal{H}^2(Z)$ *is finite.*
(2) *There is a geodesic* $c : [a_0, b_0] \to \partial Z \subset Z$ *and some* $a_0 < a < b < b_0$*, such that* $Z \setminus c([a, b])$ *has non-positive curvature.*

Proof. Assume some Z satisfies the assumption of Theorem 1.3,(3) but is not CAT(0). We are going to find a Jordan domain Z^- in Z which satisfies both assumptions of the lemma, but which is not CAT(0) either.

Due to Subsection 13.2, $Z \setminus \partial Z$ cannot be a length space. Thus we find points $x, y \in Z \setminus \partial Z$ and $\epsilon > 0$ such that for any curve $\gamma \subset Z \setminus \partial Z$ connecting x and y we have $\ell(\gamma) > d_Z(x, y) + \epsilon$.

Consider a geodesic $c : [a_0, b_0] \to Z$ in Z between x and y. Connect further x and y by some simple piece-wise geodesic curve $\hat{c}$ in $Z \setminus \partial Z$ disjoint from c outside the endpoints. Consider the arising Jordan curve $T = c \cup \hat{c}$ and the corresponding closed Jordan domain Z^- with its induced length metric. Since T has finite length we have $\mathcal{H}^2(Z^-) < \infty$ by the isoperimetric inequality. Outside the intersection of T with ∂Z, the space Z^- has non-positive curvature by Lemma 11.1. Thus Z^- satisfies (1) and (2) required in the lemma, where $a < b$ can be chosen arbitrary in (a_0, b_0), sufficiently close to a_0 and b_0, respectively.

Assume that Z^- is CAT(0). Then $Z^- \setminus T$, with the metric induced from Z^-, is a length space due to Subsection 13.3. Therefore, we find a curve γ in Z^- connecting x and y, such that γ does not intersect $T \cap \partial Z$ and such that the length of γ is arbitrary close to $\ell(c) = d_Z(x, y)$. This contradicts our assumption on x and y, shows that Z^- cannot be CAT(0), and finishes the proof of the lemma. ∎

14.2. **Simple setting.** We may and will assume from now on that Z satisfies the assumption of Lemma 14.1. Thus Z has non-positive curvature outside a geodesic c contained in ∂Z. We consider the complement $Z \setminus \partial Z$ and call the associated length space Y_0. The identity map $I : Y_0 \to Z \setminus \partial Z$ is a 1-Lipschitz homeomorphism. The length space Y_0 has non-positive curvature, thus, by Proposition 12.1, the completion Y of Y_0 is CAT(0). Moreover, $I : Y_0 \to Z$ extends to a

1-Lipschitz map $I : Y \to Z$. Under these assumptions we obtain the following uniform area estimate for balls in Y_0 near c.

Lemma 14.2. *For any $y \in Y_0$ and all r such that $\overline{I(B_{Y_0}(y,r))} \cap \partial Z \subset c$, the area of the r-ball in Y_0 around y satisfies:*

$$\mathcal{H}^2(B_{Y_0}(y,r)) \geq \frac{\pi}{4} r^2 \ .$$

Proof. We proceed similarly to [LW18, Section 9.2], following the standard arguments leading to the boundary regularity of minimal surfaces under a chord-arc condition on the boundary.

Let $y \in Y_0$ and $r > 0$ be as in the statement of the lemma. Since ∂Z is not completely contained in c, the assumption on r implies that there exists some "remote" point $x \in Y_0$ with the following property: $d(x,y) > r$ and for the connected component U of x in $Y_0 \setminus B(y,r)$ the closure of $I(U)$ in Z contains a point in $\partial Z \setminus c$.

We now argue by contradiction and assume that $\mathcal{H}^2(B_{Y_0}(y,r)) < \frac{\pi}{4} r^2$. For any $t < r$, we consider the ball $B_{Y_0}(y,t)$ around y in Y_0, denote by $v(t)$ its area and by L_t its boundary in Y_0. Note that L_t separates x and y in Y_0. Set $w(t) = \mathcal{H}^1(L_t)$. By the co-area inequality we have $\int_0^t w(s) \ ds \leq v(t)$ for all $t < r$. (See [LW18, Lemma 2.3] and note that Y_0 is countably 2-rectifiable, no non-Euclidean planes appear as tangent spaces of Y_0 and the distance function to any point is 1-Lipschitz). The contradiction to $v(r) < \frac{\pi}{4} r^2$ follows by integration, once we have verified for almost all $t < r$ the inequality

$$v(t) \leq \frac{1}{\pi} w^2(t) \ . \tag{14.1}$$

We claim that for all t with finite $w(t) = \mathcal{H}^1(L_t)$, the set L_t contains a closed subset L_t' still separating x and y in Y_0, and such that L_t' is either a Jordan curve or homeomorphic to an open interval. Indeed, consider the topological sphere $S^2 = Z/\partial Z$ obtained from Z by collapsing ∂Z to a point, equipped with the quotient metric. Let K be the closure of the image of $I(L_t)$ in S^2. Then K separates the (images of the) points x and y in S^2. Since $w(t) < \infty$ it follows that K contains a Jordan curve K_0 which separates x and y, cf. [LW18, Corollary 7.5]. The preimage L_t' of K_0 in Y_0 is either a Jordan curve or an open interval, it is contained in L_t and separates x and y in Y_0.

Assume that L_t' is a Jordan curve. By the choice of x, the Jordan domain of L_t' contains y and therefore the whole ball $B_{Y_0}(y,t)$. Therefore, $v(t) \leq \frac{1}{4\pi} \ell_Z^2(I(L_t')) \leq \frac{1}{4\pi} w^2(t)$, hence (14.1).

Assume now that L_t' is an open interval, which has finite length by assumption. Then the closure of $I(L_t')$ consists of $I(L_t')$ and one or two points on ∂Z. By construction and assumption, these points are contained in $\overline{I(B_{Y_0}(y,r))} \cap \partial Z \subset c$. We consider the Jordan curve $T_t \subset Z$ given by $I(L'_t)$ and the part of c between the endpoints of $I(L'_t)$. Again, by the choice of x, the Jordan domain of T_t contains $I(y)$ and therefore the image $I(B_{Y_0}(y,t))$. On the other hand, the length of T_t is at most $2w(t)$: since c is a geodesic in Z, the curve $I(L'_t)$ has at least half of the length of T_t. Thus, the isoperimetric inequality gives us $v(t) \leq \frac{1}{4\pi}(2w(t))^2 = \frac{1}{\pi}w^2(t)$, finishing the proof of (14.1) and of the lemma. □

Under the assumptions of Lemma 14.1 we conclude:

Proposition 14.3. *The space Y is compact and homeomorphic to $\bar{D}$. The restriction $I : \partial Y \to \partial Z$ is weakly monotone, in particular, the preimage $\eta = I^{-1}(c)$ is an arc. The restriction $I : Y \setminus \eta \to Z \setminus c$ is a bijective local isometry. The map $I : Y \to Z$ is an isometry if and only if I maps η to c in an arclength preserving way.*

Proof. Assume that Y is not compact. Then we find some $\epsilon > 0$ and an infinite sequence of points $y_n \in Y_0$ with pairwise distance at least 2ϵ. After choosing a subsequence we may assume that $I(y_n)$ converges to a point $z \in Z$. If the point z has a CAT(0) neighborhood in Z, then for a small ball B around z the metric of $B \cap (Z \setminus \partial Z)$ is a length metric, due to Subsection 13.3. From this we get $d_Y(y_n, y_m) = d_Z(I(y_n), I(y_m))$ for all n, m large enough. Therefore, the points y_n cannot be 2ϵ-separated.

Therefore, we may assume that z does not have a CAT(0) neighborhood. Hence $z \in c([a,b]) \subset c([a_0,b_0])$. Thus, we find some small $r < \epsilon$ such that $B(z,3r) \cap \partial Z \subset c$. This allows us to apply Lemma 14.2 to y_n and deduce, that for all sufficiently large n the area of the ball $B_{Y_0}(y_n, r)$ in Y_0 is at least $\frac{\pi}{4}r^2$. But all these balls are disjoint. This contradicts the finiteness of the total area of Y_0 and finishes the proof that Y is compact. From Lemma 12.2 we deduce that Y is homeomorphic to $\bar{D}$.

Since $I(Y_0)$ is dense in Z and Y compact, we obtain $I(Y) = Z$. Since $I : Y_0 \to Z \setminus \partial Z$ is a homeomorphism, we infer that $I^{-1}(\partial Z) = \partial Y$. An open subset U of $\bar{D}$ is contractible if and only if $U \cap D$ is contractible. Since $I : Y \setminus \partial Y \to Z \setminus \partial Z$ is a homeomorphism, we deduce that preimages of open contractible sets are contractible. Therefore, the preimage of any point $z \in Z$ is a cell-like set (cf. [HNV04, p.97] or [LW18, Section 7]). For $z \in \partial Z$, the preimage is a cell-like subset of the circle ∂Y, hence either a point or an arc. Therefore, the restriction $I : \partial Y \to \partial Z$ is weakly monotone.

In particular, the preimage η of the arc $c \subset \partial Z$ is an arc in ∂Y. For any point $z \in Z \setminus c$ there is a small ball O around z such that the metric on $O \setminus \partial Z$ is a length metric by Subsection 13.3. Thus, $I : I^{-1}(O) \to O$ is an isometry.

Assume now that I is arclength preserving on η. Then I is bijective and hence a homeomorphism. Moreover, for any geodesic γ in Z, the map $I : I^{-1}(\gamma) \to \gamma$ preserves the $\mathcal{H}^1$-measure. Thus, $I^{-1}(\gamma)$ has the same length as γ and the map $I^{-1} : Z \to Y$ is 1-Lipschitz. Therefore, I is an isometry. □

In the sequel we will construct Jordan curves $T \subset Y$ whose intersection with η is an arc η_0. In this situation the image $I \circ T$ is a Jordan curve in Z whose Jordan domain is the (locally isometric) image of the Jordan domain of T. The complementary part $k = T \setminus \eta_0$ is mapped by I in an arclength preserving way. Since I is 1-Lipschitz and the image of η_0 is a geodesic, we get

$$\ell_Z(I \circ T) = \ell_Y(k) + \ell_Z(I \circ \eta_0) \leq 2\ell_Y(k) .$$

Moreover, the number $\ell_Y(T) - \ell_Z(I \circ T) = \ell_Y(\eta_0) - \ell_Z(I \circ \eta_0)$ measures the deviation of I from being an isometry.

15. Rectifiable parts

15.1. **Formulation.** We continue to work under the standing assumptions of Lemma 14.1 and use the notations of Proposition 14.3. The aim of this section is to prove:

Proposition 15.1. *The map $I : Y \to Z$ preserves the length of any rectifiable subcurve of η.*

15.2. **Euclidean domains of almost isoperimetric equality.** We begin with a short Euclidean computation. For any sufficiently small $r > 0$, let $T = T_r$ be a Jordan curve in $\mathbb{R}^2$ which consists of an arc of length $2\pi - 2r$ on S^1 and a chord of length $2 \sin r$. Then $\ell(T) < 2\pi$ and the Jordan domain J of T has area

$$\mathcal{H}^2(J) = \pi - r + \frac{\sin(2r)}{2} > \pi - r^3 .$$

Therefore, we deduce:

$$\ell(T) - \sqrt{4\pi\mathcal{H}^2(J)} < 2r^3 .$$

We note that the curve T is contained in a hinge of angle $\pi - r$ enclosed between the chord and the tangent to S^1 at one of the endpoints of the chord. Rescaling the curve T suitably, we obtain:

Lemma 15.2. *For any $\epsilon > 0$ there exist some $L, \delta > 0$ with the following property. Let $\gamma : [0, \infty) \to \mathbb{R}^2$ be one of two rays bounding a* *hinge H of angle $\geq \pi - \delta$. For any $s > 0$ one can find a Jordan curve $T_s \subset H$ with Jordan domain J_s such that the following holds true.*

(1) *The curve T_s contains the initial part of γ of length s.*
(2) $\ell(T_s) \leq L \cdot s$.
(3) $\ell(T_s) - \sqrt{4\pi \mathcal{H}^2(J_s)} < \epsilon \cdot s$.

15.3. **Curved domains of almost isoperimetric equality.** We apply the approximation of hinges by flat cones, provided by Lemma 11.2, and directly deduce from Lemma 15.2:

Lemma 15.3. *For any $\epsilon > 0$ there exist some $L, \delta > 0$ with the following property. Let M be a metric space of non-positive curvature homeomorphic to a surface with boundary. Let H be a hinge in M of angle $\geq \pi - \delta$ and let γ be one of its bounding geodesics. Then for all sufficiently small $s > 0$ there exists a Jordan curve $T_s \subset H \subset M$ with Jordan domain J_s such that the conclusions (1)-(3) of Lemma 15.2 hold true.*

The bound $s_0 > 0$, such that the conclusion of Lemma 15.3 holds true for all $0 < s < s_0$, depends on ϵ, the space M, and the hinge H.

15.4. **Differentials.** In order to approach general rectifiable curves we will use a Rademacher-type theorem. Let $\gamma : [p, q] \to X$ be a rectifiable curve parametrized by arclength in a CAT(0) space X. We say that γ is *differentiable* at the point $t \in (p, q)$ if the in- and outgoing directions of γ are "almost defined by almost opposite geodesic directions". More precisely, we require the following conditions to hold true for the curves $\gamma^{\pm}(s) := \gamma(t \pm s)$. The angle between $\gamma^{\pm}$ is well-defined (cf. [BBI01, 3.6.26]) and equal to π, and, moreover, there are geodesics $\eta_n^{\pm}$ starting at $\gamma(t)$, such that the angles between $\eta_n^{\pm}$ and $\gamma^{\pm}$ are well-defined and converge to 0, as n converges to ∞.

In other words, the angle between η_n^+ and η_n^- converges to π and for any $\delta > 0$ and all sufficiently large n, there exists some $s_0 > 0$ such that for all $0 < s < s_0$ we have $d(\gamma(t \pm s), \eta_n^{\pm}(s)) < \delta s$. The metric differentiability theorem implies ([Lyt04, Theorem 1.6]):

Lemma 15.4. *Let X be a* CAT(0) *space and let $\gamma : [p, q] \to X$ be a rectifiable curve parametrized by arclength. Then for almost all $t \in [p, q]$ the curve γ is differentiable at t.*

Now we are able to deduce that many small parts of any rectifiable curve can be complemented to Jordan curves almost violating the Euclidean isoperimetric inequality. In order to avoid minor difficulties we restrict ourselves to boundary curves, the only case we will need.

Proposition 15.5. *For any $\epsilon > 0$ there exists some $L > 0$ with the following property. Let M be a metric space of non-positive curvature homeomorphic to a surface with boundary and let $\gamma : [p,q] \to \partial M$ be a part of the boundary of M parametrized by arclength. Then for a set S of full measure in $[p,q]$ and any $t \in S$ there exists some $r_0 = r_0(t) > 0$ such that the following holds true. For any $s < r_0$ there exists a Jordan curve T_s in M with Jordan domain J_s such that*

(1) *The intersection of T_s with γ is an arc which contains $\gamma|_{[t,t+s]}$.*
(2) *$\ell(T_s) \leq L \cdot s$.*
(3) *$\ell(T_s) - \sqrt{4\pi\mathcal{H}^2(J_s)} < \epsilon \cdot s$.*

Proof. Choose some $L = L(\frac{\epsilon}{4})$ provided by Lemma 15.3. Let $S \subset (p,q)$ be the set of points in which γ is differentiable. For any $t \in S$, and any $\delta > 0$ we find geodesics $\gamma^{\pm}$ starting in $\gamma(t)$ at an angle not less than $\pi - \delta$, such that $d(\gamma(t \pm s), \gamma^{\pm}(s)) < \delta \cdot s$ for all sufficiently small s.

Let H denote a small hinge in M enclosed by the geodesics $\gamma^{\pm}$. If δ and s are small enough, we apply Lemma 15.3 and find a Jordan curve $T_s' \subset H \subset M$ with Jordan domain J_s' and the following properties. The curve T_s' has length at most $2Ls$ and contains the initial part of γ^+ of length $2s$. Moreover,

$$\ell(T_s') - \sqrt{4\pi\mathcal{H}^2(J_s')} < \frac{\epsilon}{2} \cdot s \ .$$

Now we connect the point $\gamma^+(2s)$ with a nearest point $\gamma(t+\hat{s})$ on γ by a geodesic c_s. By the choice of γ^+, the length of c_s is at most $2\delta s$. Moreover, $|\hat{s} - 2s| = d(\gamma^+(\hat{s}), \gamma^+(2s)) \leq 2\delta s + \delta\hat{s}$ by the triangle inequality. Therefore, for $\delta < \frac{1}{2}$, we deduce $\hat{s} - 2s < 4\delta\hat{s} \leq 8\delta s$.

Since γ^+ is a geodesic, c_s does not intersect T_s' outside of γ^+. Throwing away the common part of c_s and γ^+ we may assume that c_s intersects T_s' only at the initial point of c_s.

Let now the curve T_s arise from T_s' by replacing the initial γ^+-part of T_s' by c_s and the corresponding arc of γ between $\gamma(t)$ and $\gamma(t+\hat{s})$. By construction, the Jordan domain J_s of T_s contains the Jordan domain J_s' of T_s', hence $\mathcal{H}^2(J_s') \leq \mathcal{H}^2(J_s)$. Moreover, the length of $\ell(T_s)$ is at most $\ell(T_s') + 2\delta s + 8\delta s$.

Once δ and s have been chosen small enough, we see that the curve T_s satisfies all requirements of the lemma. □

15.5. **Length preservation.** Now we can easily provide:

Proof of Proposition 15.1. Let $\gamma : [p,q] \to \partial Y$ be a rectifiable subcurve contained in η. We may assume γ to be parametrized by arclength. Since the map I is 1-Lipschitz we have $\ell_Z(I \circ \gamma) \leq \ell_Y(\gamma)$. If the inequality is strict then we find some $\epsilon > 0$ and a set Q of positive measure in $[p,q]$ such that the following holds true. For any $r \in Q$, there is some $\delta = \delta(r) > 0$ such that, for all $h < \delta$, one has

$$\ell_Z(I \circ \gamma|_{[r,r+h]}) \leq h \cdot (1 - 2\epsilon).$$

Applying Proposition 15.5 we find some $r \in Q$ and, for all sufficiently small $h > 0$, we find a Jordan curve T_h as in Proposition 15.5 containing $\gamma_{[r,r+h]}$. Let J_h denote the Jordan domain of T_h. Then $I(J_h)$ is the Jordan domain of the Jordan curve $I(T_h)$. Since on Y_0 the map I is a local isometry, we have $\mathcal{H}^2(J_h) = \mathcal{H}^2(I(J_h))$. By assumption and the 1-Lipschitz property of I, we see that $\ell_Z(I \circ T_h) \leq \ell_Y(T_h) - 2\epsilon \cdot h$.

We deduce $\ell_Z(I \circ T_h) - \sqrt{4\pi\mathcal{H}^2(I(J_h))} < -\epsilon \cdot h$, which contradicts the isoperimetric assumption of Theorem 1.3, (3). This finishes the proof. □

16. Final steps

16.1. **Formulation.** We continue to use the notations from Proposition 14.3. The rest of the section is devoted to the proof of

Proposition 16.1. *The map $I : Y \to Z$ is an isometry.*

Assume the contrary. Due to Proposition 14.3 and Proposition 15.1, the arc η is not rectifiable. We fix a parametrization of η as a simple curve $\eta : [p,q] \to Y$.

For the convenience of the reader, we first outline the main steps of the proof. In Lemma 16.3 we will deduce from the non-rectifiability of η the existence of points on η at which the "differential" of I is arbitrary small. We will then fix such a point y and search for a contradiction to the isoperimetric inequality in a small neighborhood of this point. In Lemma 16.4, we show that the area of small balls around y is almost Euclidean and obtain in Corollary 16.5 a bound on the length of any curve surrounding such a ball. In Lemma 16.7 we connect y by a geodesic γ with a nearby point on η and prove that γ and η are sufficiently close to each other, more precisely, they include (in a rather weak sense) an angle whose tangent is at most $\frac{1}{2}$. In the final subsection, we take two geodesics connecting y with nearby points on η, lying on different sides of y. If the angle enclosed between these geodesics is at least π then we obtain a contradiction in the same way as at the end of the proof of Proposition 15.1 above. If the angle is smaller than π then we consider two points on these geodesics with small distance r from y. We connect these points by a "circular arc" inside the hinge (using Lemma 11.2) and we further connect these points to some points on η using Lemma 16.7. The arising curve is relatively short but nevertheless surrounds a sufficiently large ball around y, leading to a contradiction with Corollary 16.5.

16.2. **Bounding diameter by endpoints on η.** We claim:

Lemma 16.2. *For any $p \leq t < r \leq q$ the diameter of $\eta|_{[t,r]}$ is at most $10 \cdot d(\eta(t), \eta(r))$.*

Proof. Assume the contrary and set $l = d(\eta(t), \eta(r))$. Since $Y_0 = Y \setminus \partial Y$ is a length space, we find a simple curve k in Y of length less than $2l$ connecting $\eta(t)$ and $\eta(r)$ and building a Jordan curve T together with $\eta|_{[t,r]}$. The Jordan domain J of T has area at most $\frac{4}{\pi}l^2$, since the length of the image of T in Z is at most $2 \cdot \ell_Y(k) = 4l$.

Since $\bar{J}$ (which contains $\eta|_{[t,r]}$) has diameter larger than $10l$, we find a point $x \in \bar{J}$ with distance at least $4l$ from the curve k. Thus, the ball $B_Y(x, 4l)$ is contained in $\bar{J}$. From Lemma 14.2 we see that $\mathcal{H}^2(J) \geq \frac{\pi}{4}(2l)^2 = \pi l^2$. This contradicts $\mathcal{H}^2(J) \leq \frac{4}{\pi}l^2$ and finishes the proof. □

16.3. **A consequence of non-rectifiability.** Recall that $I \circ \eta$ is a weakly monotone parametrization of the geodesic $c \subset Z$. From now on, we will consider the curve c with this weakly monotone parametrization $c = I \circ \eta : [p, q] \to Z$, despite the fact that in the rest of the paper all geodesics are parametrized by arclength. We are going to find points at which the "differential" of I is arbitrary small, by using Lemma 16.2 and the fact that η is non-rectifiable.

Lemma 16.3. *For any $\lambda > 0$ there exists some $t \in [p, q]$ and $\epsilon > 0$ such that for all $s \in [p, q]$ with $|s - t| < 2\epsilon$ we have*

$$d(\eta(t), \eta(s)) \geq \lambda \cdot d(c(t), c(s)).$$

Proof. We assume the contrary and take some $\lambda > 0$ for which the claim is wrong. We are going to prove that η is rectifiable, in contradiction to our assumptions.

Consider an arbitrary $\epsilon > 0$. For any $t \in [p, q]$, we find some $t^+ \neq t$ with $|t^+ - t| < 2\epsilon$ and $d(\eta(t), \eta(t^+)) < \lambda \cdot d(c(t), c(t^+))$. If t is one of the endpoints p or q, we set $t^- = t$. If not then, by continuity of η and c, we find t^- arbitrarily close to t on the other side of t from t^+ such that $d(\eta(t^+), \eta(t^-)) < \lambda \cdot d(c(t^+), c(t^-))$.

Denote by I_t the closed interval between t^- and t^+, which by our choice has length smaller than 2ϵ. Changing the order if needed we may assume $t^- < t^+$ for any t. We find a finite covering of $[p, q]$ by some of these intervals $I_{t_1}, ..., I_{t_k}$, such that the intersection number of the covering is at most 2. We reorder the intervals and have $t_i^- \leq t_{i+1}^- \leq t_i^+$, for all i. Thus all endpoints of all the intervals I_{t_i} define a 2ϵ-fine subdivision $p = s_1 < s_1 \le \cdots \le s_{2k} = q$ of $[p, q]$. Each of the intervals $[s_i, s_{i+1}]$ is contained in exactly one or two of the intervals $[t_j^-, t_j^+]$. Due to Lemma 16.2, we have $d(\eta(s_i), \eta(s_{i+1})) \le 10 \cdot d(\eta(t_j^-), \eta(t_j^+))$ in this case. Summing up we deduce

$$\sum_{i=1}^{2k} d(\eta(s_i), \eta(s_{i+1})) \le 20 \cdot \left(\sum_{j=1}^{k} d(\eta(t_j^+), \eta(t_j^-)) \right)$$

$$< 20\lambda \cdot \left(\sum_{j=1}^{k} d(c(t_j^-), c(t_j^+)) \right) \le 40\lambda \cdot \ell(c)\,.$$

Since ϵ was arbitrary, we see that $40\lambda\ell(c)$ provides an upper bound for the length of η, in contradiction to the non-rectifiability of η. □

16.4. **Setting.** We choose some large $\lambda > 0$, to be determined later. We find $t \in [p, q]$ and some $\epsilon > 0$ provided by Lemma 16.3. Since Z is non-positively curved in neighborhoods of the boundary points $c(a_0) = I(\eta(p))$ and $c(b_0) = I(\eta(q))$ (cf. Lemma 14.1), the map I is a local isometry in neighborhoods of $\eta(p)$ and $\eta(q)$. Hence, $t \in (p, q)$. In order to simplify notations we may and will assume $t = 0$ and $[-\epsilon, \epsilon] \subset (p, q)$. Set $y = \eta(0)$ and note that $I(y)$ is not an endpoint of the geodesic c in Z. We choose some $r_0 > 0$ such that $B(y, 2r_0) \cap \partial Y \subset \eta|_{[-\epsilon,\epsilon]}$.

We can now show that balls around y have almost Euclidean area. We emphasize, that the point y and the radius r_0 depend on the choice of the constant λ.

Lemma 16.4. *For any $\alpha_0 > 0$ the following holds true. If λ has been chosen large enough then for any $r < r_0$ the area of $O(y, r) = B(y, r) \cap Y_0$ can be estimated by*

$$\mathcal{H}^2(O(y, r)) \ge (\pi - \alpha_0) r^2\ .$$

Proof. Approximating y by points in Y_0 we obtain from Lemma 14.2 the inequality $\mathcal{H}^2(O(y, r)) \ge \frac{\pi}{4} r^2$ for all $r < r_0$. In order to improve the bound, we argue as in the proof of Lemma 14.2. We consider the distance function $f : Y_0 \to \mathbb{R}$ defined by $f(z) = d(y, z)$. Then $O(y, t)$ is the sublevel set $f^{-1}((0, t))$. Denote by $L_t \subset Y_0$ the level set $f^{-1}(t)$ and by $w(t)$ its length $\mathcal{H}^1(L_t)$. Set $v(t) = \mathcal{H}^2(O(y, t))$. By the co-area inequality, $w \in L^1([0, r_0])$ and $\int_0^t w(s)\, ds \le v(t)$ for almost all $t \in (0, r_0)$. As in the proof of Lemma 14.2, for almost all $t \in (0, r_0)$, the set L_t contains an arc L_t' which connects two points on η and separates $O(y, t)$ from some fixed point $x \in Y_0$ at large distance from y.

Denote by $w_1(t) \leq w(t)$ the length $\mathcal{H}^1(L'_t)$. The statement of the lemma follows by integration, once we have verified for almost all $t \in (0, r_0)$ the inequality

$$v(t) \leq \frac{1}{4\pi} w_1^2(t) \cdot (1 + g_\lambda) \ , \tag{16.1}$$

where the constant g_λ goes to 0 as λ goes to ∞.

We already know $v(t) \geq \frac{\pi}{4}t^2$, and we have seen in the proof of Lemma 14.2 that $\frac{1}{\pi}w_1^2(t) \geq v(t)$. Therefore, $w_1(t) \geq \frac{\pi}{2}t$.

The endpoints $e^\pm$ of L'_t must be contained in $\eta|_{[-\epsilon,\epsilon]}$ and lie on different sides of y. Moreover, by definition, $d(e^\pm, y) = t$. As in Lemma 14.2, we consider the Jordan curve T_t built by L'_t and the part of η between $e^\pm$. From the choice of λ and ϵ we deduce that the length of $I \circ (T_t \cap \eta)$ is at most $\frac{2}{\lambda}t$. Therefore,

$$\ell_Z(I \circ T_t) \leq w_1(t) + \frac{2}{\lambda}t \leq (1 + \frac{4}{\pi\lambda}) \cdot w_1(t) \ .$$

Now, as in the proof of Lemma 14.2, the isoperimetric inequality in Z provides (16.1) with $(1 + g_\lambda) = (1 + \frac{4}{\pi\lambda})^2$. This finishes the proof. □

As a consequence we get:

Corollary 16.5. *If λ is large enough then for any $r < \frac{1}{3}r_0$ the following holds true. Any curve k in $Y \setminus B_Y(y, r)$ which connects two points in $\eta([-\epsilon, \epsilon])$ on different sides of y satisfies the inequality $\ell_Y(k) > (\pi+3)r$.*

Proof. Assume the contrary. Fix a sufficiently small $\alpha_0 > 0$, choose $\lambda > 0$ such that Lemma 16.4 holds true. Consider an arbitary $r < \frac{1}{3}r_0$ and a curve k violating the conclusion of the corollary. Then we find a simple subcurve of k which still connects two points $e^\pm$ in $\eta([-\epsilon, \epsilon])$ on different sides of y and does not intersect η between $e^\pm$. We replace k by this subcurve and consider the Jordan curve T built by k and the part of η between $e^\pm$. The Jordan domain of T contains $O(y, r) = B(y, r) \cap Y_0$. Therefore,

$$\frac{1}{4\pi}\ell_Z^2(I \circ T) \geq (\pi - \alpha_0)r^2 \ , \tag{16.2}$$

due to Lemma 16.4 and the isoperimetric inequality in Z.

We have $\ell_Y(k) \geq \frac{1}{2}\ell_Z(I \circ T)$. If k does not intersect the $3r$-ball around y in Y, then we may replace the term r^2 by $(3r)^2$ on the right hand side of (16.2). In this case we arrive at a contradiction, once α_0 is small enough ($\alpha_0 = \frac{5}{9}\pi$ is sufficient here).

If k intersects the $3r$-ball around y in Y, then $e^\pm$ have distance at most $10r$ to y, since k has length at most $(\pi+3)r$. Therefore, $I \circ (T \cap \eta)$

has length at most $\frac{20}{\lambda}r$. Hence,

$$\frac{1}{4\pi}\left(\ell_Y(k)+\frac{20}{\lambda}r\right)^2 \geq (\pi-\alpha_0)r^2 \,.$$

If λ has been chosen sufficiently large and α_0 sufficiently small, this contradicts the assumption $\ell_Y(k) \leq (\pi+3)r$. $\square$

16.5. **The boundary is close to a geodesic.** Consider the geodesic $\gamma : [0, t_0] \to Y$ starting at y and ending at $\eta(\epsilon)$. Let $P : Y \to \gamma$ be the nearest point projection, which is well-defined and 1-Lipschitz since Y is CAT(0). For any point $x \in Y$, denote by β_x the shortest geodesic from x to $P(x)$. Then $P(\beta_x) = P(x)$. In particular, for $x, w \in Y$, the geodesics β_x and β_w are either disjoint or are sent by P to the same point, their common endpoint. Since any geodesic β_x encloses an angle of at least $\frac{\pi}{2}$ with (the initial part of) γ at $P(x)$, we infer $d(y, \beta_x) = d(y, P(x))$. In other words, any point on β_x has at least the same distance from $y = \gamma(0)$ as $P(x)$. For topological reasons we have:

Lemma 16.6. *The composition $P \circ \eta : [0, \epsilon] \to \gamma$ is a weakly monotone parametrization of γ.*

Denote by Q the union of all geodesics β_x, where x runs over all points on $\eta|_{[0,\epsilon]}$. By definition, Q contains $\eta|_{[0,\epsilon]}$ and γ. Denote by Q_0 the intersection $Q \cap Y_0$ and consider the 1-Lipschitz continuous function $f : Q_0 \to [0, t_0]$ which sends $x \in Q_0$ to the γ-parameter of $P(x)$, thus $f(x) = d(P(x), y)$. By definition, $f^{-1}(t)$ is exactly the (intersection with Y_0 of the) union of all geodesics β_w which start on $\eta|_{[0,\epsilon]}$ and end in $\gamma(t)$.

For $t \in [0, t_0]$ we let $h(t) \geq 0$ be the infimum of lengths of all geodesics β_x which start at some point $x \in \eta|_{[0,\epsilon]}$ and end at $\gamma(t) = P(x)$. By definition, $h(t)$ equals the minimum of the distance function to the geodesic γ on the compact set $P^{-1}(\gamma(t)) \cap \eta|_{[0,\epsilon]}$. By continuity, $P(\eta|_{[0,\epsilon]}) = \gamma$, thus h is well-defined. By compactness, for any $t \in [0, t_0]$ we find a geodesic $\beta^t = \beta_x$ of length $h(t)$ which starts on $\eta|_{[0,\epsilon]}$ and ends at $\gamma(t)$. By minimality, the geodesic β^t intersects η only at the starting point. Again by compactness, the function $h(t)$ is lower semi-continuous.

We set $g(t) = \mathcal{H}^1(f^{-1}(t))$ for $t \in [0, t_0]$. By construction, we have $h(t) \leq g(t)$ for all t. By the co-area inequality, g is integrable and for any $0 \leq t < t' \leq t_0$ we have

$$\mathcal{H}^2(f^{-1}((t,t'))) \geq \int_t^{t'} g(s)\, ds \,. \tag{16.3}$$

With these notations and preparations at hand we can now show that γ and η do not diverge too fast from each other.

Lemma 16.7. *For all $t \in [0, t_0]$ there is some $t \le t' \le 2t$ with $h(t') \le \frac{t'}{2}$. Thus, there exists a geodesic $\beta^{t'}$ of length at most $\frac{t'}{2}$ starting at $\gamma(t')$ orthogonally to γ and ending on $\eta|_{[0,\epsilon]}$.*

Proof. Let $\mathcal{T}$ be the set of all $t \in [0, t_0]$ for which the claim is true. By definition $h(t_0) = 0$, therefore $[\frac{t_0}{2}, t_0] \subset \mathcal{T}$. By the semi-continuity of h, $\mathcal{T}$ is closed. Assume that $\mathcal{T} \neq [0, t_0]$ and let $t_3 \in (0, \frac{t_0}{2}]$ be the smallest number such $[t_3, t_0] \subset \mathcal{T}$.

Consider $t_2 := 2t_3$. From the minimality of t_3 we infer that $h(t_2) \le \frac{1}{2}t_2$ and that for any $t \in [t_3, t_2)$ the inequality $h(t) > \frac{1}{2}t$ holds true. Due to the semi-continuity of h and since $h(0) = 0$, there exists a largest $t_1 \in [0, t_3)$ with $h(t_1) \le \frac{1}{2}t_1$. Summarizing, we have

$$h(t_1) \le \frac{1}{2}t_1;\ h(t_2) \le \frac{1}{2}t_2;\ t_2 > 2t_1 \text{ and } h(t) > \frac{1}{2}t \ \text{ for } t \in (t_1, t_2)\ .$$

We are going to derive a contradiction to the isoperimetric inequality. Consider the geodesic β^{t_1} and β^{t_2}. By construction, these geodesics do not intersect η outside their endpoints. Moreover, $\gamma|_{(t_1,t_2)}$ does not intersect η, since otherwise h were equal to 0 at the intersection point. Thus $\beta^{t_1}, \beta^{t_2}, \gamma|_{(t_1,t_2)}$ and the part of η between the endpoints of β^{t_1} and β^{t_2} constitute a Jordan curve T. Due to Lemma 16.6, the preimage $f^{-1}((t_1, t_2))$ is contained in the Jordan domain J of T. Since $g(t) \ge h(t) > \frac{1}{2}t$ for all $t \in (t_1, t_2)$, we deduce from (16.3) that

$$\mathcal{H}^2(J) > \int_{t_1}^{t_2} \frac{1}{2}s\, ds = \frac{1}{4}(t_2^2 - t_1^2)\,. \tag{16.4}$$

We now estimate the length of $I \circ T$ in Z as follows. By assumption, the lengths of $\beta^{t_1}, \beta^{t_2}, \gamma|_{(t_1,t_2)}$ sum up to at most $\frac{1}{2}(t_1 + t_2) + (t_2 - t_1)$. Moreover, the distance of the starting point of β^{t_2} on η from y is at most $t_2 + \frac{1}{2}t_2$. Thus, the η-part of $I \circ T$ is mapped to a part of the geodesic $c \subset Z$ which has length at most $\frac{1}{\lambda} \cdot \frac{3}{2}t_2$.

We set $q = \frac{t_1}{t_2} < \frac{1}{2}$. The isoperimetric inequality in Z gives us $\mathcal{H}^2(J) \le \frac{1}{4\pi}\ell_Z^2(I \circ\ T)$. Inserting the above estimates we infer:

$$\frac{1}{4}(1 - q^2) \le \frac{1}{4\pi}\left(\frac{3}{2} - \frac{1}{2}q + \frac{3}{2\lambda}\right)^2\,. \tag{16.5}$$

The left hand side is at least $\frac{3}{16}$ since $q < \frac{1}{2}$. The right hand side is at most $\frac{9}{16\pi}(1 + \frac{1}{\lambda})^2$. Since $\pi > 3$ we obtain a contradiction if λ is large enough. This finishes the proof of Lemma 16.7. □

16.6. **Final conclusions.** We look at the other side of y and connect y with $\eta(-\epsilon)$ by a geodesic γ_1. We apply the same considerations to γ_1 which we applied to γ above. We deduce that for all sufficiently small

t there is some $t \leq t' \leq 2t$ and a geodesic $\alpha^{t'}$ from $\gamma_1(t')$ to a point on $\eta|_{[-\epsilon,0]}$ such that $\ell(\alpha^{t'}) \leq \frac{t'}{2} \leq t$ and $d(\alpha^{t'}, y) = t' \geq t$.

The contradiction is now achieved in two steps.

Lemma 16.8. *The angle between γ and γ_1 must be at least π.*

Proof. Assume the contrary. We first claim that for all sufficiently small t there exists a curve k_t between $\gamma(t)$ and $\gamma_1(t)$ such that any point on k_t has distance at least t from y and such that $\ell(k_t) \leq (\pi + \frac{1}{2}) \cdot t$.

Indeed, if the angle between γ and γ_1 is not 0 we apply Lemma 11.2 to the hinge between γ and γ_1. Thus, for any fixed $\delta > 0$ and all sufficiently small t, we find a curve k'_t of length at most $(1+\delta)\pi t$ which connects points $\gamma((1+\delta) \cdot t)$ and $\gamma_1((1+\delta) \cdot t)$ as the image of the corresponding circular arc in the flat hinge under the almost isometric map E provided by Lemma 11.2. Moreover, the distance of any point on k'_t to y is at most t. In order to obtain the required curve k_t, we just need to connect the endpoints of k'_t with $\gamma(t)$ and $\gamma_1(t)$ along γ and γ_1, respectively. On the other hand, if the angle between γ and γ_1 is 0 (or just sufficiently small), we can obtain the required curve k_t for all sufficiently small t as follows: connect $\gamma(2t)$ with $\gamma_1(2t)$ by a geodesic and then connect $\gamma(t)$ with $\gamma(2t)$ and $\gamma_1(2t)$ with $\gamma_1(t)$ along γ and γ_1, respectively.

Now, we consider a sufficiently small t such that the curve β^t has length at most $\frac{t}{2}$. Such t exists by Lemma 16.7. Moreover, we apply Lemma 16.7 to the curve γ_1 instead of γ and find some $t \leq t' \leq 2t$ and a geodesic $\alpha^{t'}$ with the properties provided by Lemma 16.7 and discussed prior to the present lemma.

Let the curve k be the concatenation of $\beta^t, k_t, \gamma_1|_{[t,t']}$ and $\alpha^{t'}$. By construction, the curve k lies completely outside the ball $B(y,t)$, it connects two points on η which lie on different sides of y and the length of k is at most

$$\ell(k) \leq \frac{t}{2} + (\pi + \frac{1}{2})t + t + t \leq (\pi + 3)t \ .$$

This contradicts Corollary 16.5 and finishes the proof. □

The final lemma is proven similarly to the final step in the rectifiable case, Proposition 15.1:

Lemma 16.9. *The angle between γ and γ_1 is strictly smaller than π.*

Proof. We assume the contrary and apply Lemma 15.3 to the hinge H between γ and γ_1. Thus, for all sufficiently small s, we find a Jordan curve T_s in the hinge H which contains the initial part of γ of length

s, and such that for the Jordan domain J_s of T_s we have

$$\ell(T_s) - \sqrt{4\pi\mathcal{H}^2(J_s)} < \frac{1}{3}s \, .$$

We now choose s to be such that β^s has length at most $\frac{s}{2}$ and replace $\gamma|_{[0,s]} \subset T_s$ by the concatenation of β^s and the part of η between the starting point of β^s and y. The arising Jordan curve T_s' contains J_s in its Jordan domain. The image Jordan curve $I \circ T_s'$ has length at most

$$\ell_Z(I \circ T_s') \leq \ell_Y(T_s) - s + \frac{s}{2} + \frac{3s}{2\lambda} = \ell_Y(T) - \frac{1}{2}s + \frac{3s}{2\lambda} \, .$$

If we have chosen $\lambda > 9$, then the curve $I \circ T_s'$ does not satisfy the isoperimetric inequality in Z. □

The contradiction between Lemma 16.9 and Lemma 16.8 finishes the proof of Proposition 16.1 and therefore the proof of Theorem 1.3.

Appendix.

17. Generalization to non-zero curvature bounds

We sketch the proof of Theorem 1.4, which generalizes Theorem 1.1 to the case of non-zero curvature bounds. We refer to [Bal04] and [AKP16] for basics on CAT(κ) spaces and recall that Reshetnyak's majorization theorem holds for all κ. Thus, any closed curve Γ of length smaller than R_κ in any $CAT(\kappa)$ space is majorized by a convex subset in M^2_κ. Now, the proof of Lemma 3.2 shows the "only if part" of Theorem 1.4.

Starting the proof of the "if part", assume that X satisfies the conditions of Theorem 1.4. Due to $\lim_{r\to 0} \frac{\delta_\kappa(r)}{r^2} = \frac{1}{4\pi}$, the arguments from Section 5 remain valid and prove that X satisfies property (ET). Arguing as in Subsection 6.3 we reduce the proof of the "if part" to the following claim: every intrinsic minimal disc Z in X corresponding to a solution of the Plateau problem $u \in \Lambda(\Gamma, X)$ is a CAT(κ) space. Here Γ is any Jordan curve in X of length smaller than R_κ.

The isoperimeric property of X implies the same isoperimetric property for all Jordan curves in Z. Thus, as in Section 7, we deduce that the conformal factor f of u satisfies the integral inequality (for all $z \in D$ and almost all $0 < r < 1 - |z|$):

$$\int_{B(z,r)} f^2 \leq \delta_\kappa \left(\int_{\partial B(z,r)} f \right) . \tag{17.1}$$

Arguing as in Sections 7 and 8 and using [Res61] instead of [BR33] we obtain a metric of curvature $\leq \kappa$ on the disc D which is defined on D by the canonical semi-continuous representative of the conformal factor f. As in Section 9 we reduce the proof to the following analogue of Theorem 1.3.

Theorem 17.1. *Let Z be a geodesic metric space homeomorphic to $\bar{D}$. Assume that for any Jordan curve Γ in Z the Jordan domain J enclosed by Γ satisfies $\mathcal{H}^2(J) \leq \delta_\kappa(\ell(\Gamma))$. Assume further that $Z \setminus \partial Z$ has curvature $\leq \kappa$. Then Z is $\mathrm{CAT}(\kappa)$ and $Z \setminus \partial Z$ is a length space.*

To prove Theorem 17.1 we closely follow the second part of this paper. The approximation by flat cones Lemma 11.2 is valid without changes for all $\kappa \neq 0$. As in Subsection 13.3, the $\mathrm{CAT}(\kappa)$ property of Z implies that $Z \setminus \partial Z$ is a length space. In order to prove that Z is $\mathrm{CAT}(\kappa)$, we need the following additional lemma which can be used instead of the theorem of Cartan-Hadamard.

Lemma 17.2. *If Z has curvature $\leq \kappa$ then Z is $\mathrm{CAT}(\kappa)$.*

Assuming that the lemma is wrong we obtain an isometric embedding into Z of a circle Γ of length $2i < R_\kappa$, where i is the injectivity radius of Z, cf. [Bal04, Section 6]. Then one can either obtain a contradiction by directly estimating the area of the Jordan domain J of Γ which contains a rather large metric ball, or apply the fact that a round hemisphere is a minimal filling of a circle (cf. [Iva11]) to deduce that $\mathcal{H}^2(J) \geq \frac{1}{2\pi}\ell^2(\Gamma)$ which contradicts the isoperimetric inequality.

Lemma 17.2 shows that the closed Jordan domain of any Jordan polygon in $Z \setminus \partial Z$ is $\mathrm{CAT}(\kappa)$ in its intrinsic metric. Thus as in Section 12, we obtain that the completion Y of the space $Y_0 = Z \setminus \partial Z$, equipped with the induced length metric, must be a $\mathrm{CAT}(\kappa)$ space. From here the rest of the proof goes without changes, we only need to restrict the attention to sufficiently small distances, where δ_κ almost coincides with δ_0.

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