NOTES ON THE JACOBI EQUATION

ALEXANDER LYTCHAK

Abstract. We discuss some properties of Jacobi fields that do not involve assumptions on the curvature endomorphism. We compare indices of different spaces of Jacobi fields and give some applications to Riemannian geometry.

1. Introduction

This note is essentially a collection of results about conjugate points of Jacobi fields for which we could not find an appropriate reference in the literature, while we were working on conjugate points in quotients of Riemannian manifolds. We discuss here some basic results about indices of spaces of Jacobi fields that do not involve Morse theory nor comparison results. For the latter the reader can consult [EH90] or any text book on Riemannian geometry, for instance [Sak96].

Let V be an m-dimensional Euclidean vector space and let $R(t), t \in I$ be a smooth family of symmetric endomorphisms defined on an interval $I \subset \mathbb{R}$. The equation

$$Y''(t) + R(t)Y(t) = 0$$

is called the *Jacobi equation* defined by $R(t)$. Solutions of the Jacobi equations are called *Jacobi fields*. By Jac we will denote the vector space of all Jacobi fields. For any $t \in I$ we have an identification $I^t : \mathrm{Jac} \to V \times V$, given by $J \to (J(t), J'(t))$. On Jac the symplectic form $\omega(J_1, J_2) = \langle J_1(t), J_2'(t)\rangle - \langle J_1'(t), J_2(t)\rangle$ is independent of t, due to the symmetry of $R(t)$. For any t, this symplectic form corresponds to the canonical symplectic form on $V \times V$ via the identification I^t.

For a subspace $W \subset \mathrm{Jac}$ we denote by $W^\perp$ the orthogonal complement of W with respect to ω. The subspace W is called *isotropic* if $W \subset W^\perp$; and it is called *Lagrangian* if $W = W^\perp$.

Let W be an isotropic subspace of Jac. For $t \in I$ we set $W(t) = \{J(t) | J \in W\}$ and $W^t = \{J \in W | J(t) = 0\}$. We say that t is W-*focal*

2000 *Mathematics Subject Classification.* 53C20, 34C10.

Key words and phrases. Jacobi fields, conjugate points, focal points, index.

The author was supported in part by the SFB 611 *Singuläre Phänomene und Skalierung in mathematischen Modellen.*

if $\dim(W^t) = \dim(W) - \dim(W(t)) > 0$ and call this number the W-*focal index* of t. This number will be denoted by $f^W(t)$. For an interval $I_0 \subset I$, we define the W-*index* of I_0 to be $\mathrm{ind}_W(I_0) = \Sigma_{t \in I_0} f^W(t)$.

In Riemannian geometry one mostly considers the indices of special Lagrangians defined by some submanifolds (see [Sak96] and Subsection 2.3 below). Here we emphasize a more abstract point of view that involve all Lagrangian subspaces and also more general isotropic subspaces of the space of all Jacobian fields. For the natural appearance of such situations one should look at [Wil04], the paper in which the important tool of transversal Jacobi equation was invented.

Now we can state our results, that seem to be known to the experts in many special cases.

Theorem 1.1. *Let $\Lambda_1, \Lambda_2 \subset Jac$ be any Lagrangians. Then for any interval $I_0 \subset I$ we have $ind_{\Lambda_1}(I_0) - ind_{\Lambda_2}(I_0) \leq dim(V)$.*

See Proposition 4.1, for a slightly more general statement. As a consequence of Theorem 1.1 we deduce:

Corollary 1.2. *Let M be a Riemannian manifold without conjugate points. Then for any submanifold N of M and any geodesic γ orthogonal to N there are at most $dim(N)$ focal points of N along γ (counted with multiplicity).*

Another direct consequence is a non-geometric proof of the following well known differential geometric result (see Subsection 4.1 for the definition of conjugate points):

Corollary 1.3. *Let V, R, Jac be as above. If for some $a < b \in I$ the points a and b are conjugate, then for each $\bar{a} \leq a$ there is some $\bar{b} \in [a, b]$ that is conjugate to $\bar{a}$.*

Another important issue for that we could not find a reference is the following semi-continuity and continuity statement:

Proposition 1.4. *Let $R_n(t)$ be a sequence of families of symmetric endomorphisms converging in the $\mathcal{C}^0$ topology to $R(t)$. Let W_n be isotropic subspaces of R_n-Jacobi fields that converge to an isotropic subspace W of R-Jacobi fields. Let $I_0 = [a, b] \subset I$ be a compact interval and assume that $f^{W_n}(a) = f^W(a)$ and $f^{W_n}(b) = f^W(b)$, for all n large enough. Then $ind_W(I_0) \geq ind_{W_n}(I_0)$, for all n large enough. If all W_n are Lagrangians then this inequality is in fact an equality.*

We prove Theorem 1.1 and its Corollaries by using the continuity principle above and by reducing the claim to the 1-dimensional situation with the help of Wilkings transversal Jacobi equation. The proof involves the decomposition of the index in the sum of the vertical and the horizontal indices with respect to an isotropic subspace (Lemma 3.1). See Subsection 3.2 and [LT07a] for a geometric interpretation of these notions. We would like to mention that the non-continuity of indices for isotropic non-Lagrangian spaces and the decomposition formula of Lemma 3.1 cause strange non-continuous behavior of indices in quotients of Riemannian manifolds ([LT07a] and Remark 3.1).

In Section 2 we discuss basic facts about Jacobi fields, prove the semi-continuity part of Proposition 1.4 and recall the arguments of [Dui76] that relate the index to Lagrangian intersections and imply the continuity part of Proposition 1.4. In Section 3 we recall the construction of the transversal Jacobi equation, due to Wilking. In Section 4 we prove the remaining results.

2. Semi-continuity and continuity of indices

2.1. **Semi-continuity.** We start with the only simple comparison result that will be used in the sequel.

Lemma 2.1. *Let V, R, Jac be as in the introduction. Assume that $||R(t)||$ is bounded above by $C^2 \in \mathbb{R}$. Let $J \in Jac$ be a Jacobi field with $J(t^-) = 0$. Then for all $t^+ \in I$ with $|t^+ - t^-| < \frac{1}{2C}$ we have $||J(t^+) - (t^+ - t^-) \cdot J'(t^+)|| \leq C \cdot ||J'(t^+)|| \cdot (t^+ - t^-)^2$.*

Proof. We may assume $t^+ > t^-$. From Rauch's comparison theorem ([Sak96],p.149) we deduce

$$||J(t)|| \leq \frac{1}{2C} e^{C|t-t^-|} \cdot ||J'(t^-)|| \leq \frac{1}{C} ||J'(t^-)||$$

for all $t \in [t^-, t^+]$. Thus $J''(t) \leq C||J'(t^-)||$, for all $t \in [t^-, t^+]$. Hence

$$J'(t^+) \geq ||J'(t^-)|| - C \cdot ||J'(t^-)|| \cdot |t^+ - t^-| \geq \frac{1}{2} ||J'(t^-)||$$

Due to the Taylor formula, we find some $t \in [t^-, t^+]$ with

$$||J(t^+) - (t^+ - t^-) \cdot J'(t^+)|| \leq \frac{1}{2} ||J''(t)|| \cdot (t^+ - t^-)^2$$

The desired estimate now follows from

$$\frac{1}{2} ||J''(t)|| \leq \frac{1}{2} C \cdot ||J'(t^-)|| \leq C \cdot ||J'(t^+)||$$

□

Let again $C^2 \in \mathbb{R}$ be an upper bound for $||R(t)||$. Let W be an isotropic subspace of Jac and let $t^+ > t^-$ be W-focal points, with $|t^+ - t^-| < \frac{1}{2C}$. Choose any $J_+ \in W^{t^+}$ and $J_- \in W^{t^-}$. Since W is isotropic, we have $\langle J_-(t^+), J'_+(t^+)\rangle = 0$. From the last lemma we obtain now

$$\langle J'_-(t^+), J'_+(t^+)\rangle \leq C \cdot |t^+ - t^-| \cdot ||J'_-(t^-)|| \cdot ||J'_+(t^+)||$$

Thus J^+ and J^- are almost orthogonal with respect to the scalar product s_{t+} on Jac defined by

$$s_{t^+}(J_1, J_2) := \langle J_1(t^+), J_2(t^+)\rangle + \langle J'_1(t^+), J'_2(t^+)\rangle$$

This has the following consequences (cf. [Sak96],p.61,p.101):

Lemma 2.2. *Let W be an isotropic subspace of Jac. Then the W-focal points are discrete in I. Moreover, there is some number ϵ, that depends only on an upper bound of $||R(t)||$, such that for an interval I_0 of length $\leq \epsilon$ the inequality $ind_W(I_0) \leq dim(W)$ holds.*

In the case $\dim(W) = 1$ we get:

Lemma 2.3. *Let J be a non-zero Jacobi field. If $J(t^+) = J(t^-) = 0$, for some $t^+ > t^- \in I$, then $|t^+ - t^-| > \epsilon$, where ϵ depends only on the upper bound on $||R(t)||$.*

Finally we get:

Lemma 2.4. *Let $R_n(t)$ be a sequence of families of symmetric endomorphisms converging in the $\mathcal{C}^1$ topology to $R(t)$. Let W_n be isotropic spaces of R_n-Jacobi fields that converge to an isotropic space W of R-Jacobi fields. Let $I_0 = [a, b] \subset I$ be a compact interval. Then $ind_W(I_0) \geq ind_{W_n}(I_0)$, for all n large enough.*

Proof. It is enough to observe that for $t_n \to t \in I_0$ the limit of W^{t_n} is contained in W^t, and that for sequences $t_n^+ > t_n^-$ converging to the same $t \in I_0$, the limits of $W^{t_n^+}$ and $W^{t_n^-}$ are orthogonal with respect to the scalar product s_t. $\square$

2.2. **Continuity.** If the isotropic subspaces W_n and W in Lemma 2.4 are Lagrangian then the inequality turns out to be an equality. To see this one has either to interpret the W-index as the index of some bilinear form (as it is done in the most important geometric situations, cf. [Sak96],p.99), or to interpret the index as the Maslov index of Lagrangian intersections, as in [Dui76],p.180-186. We are going to sketch the last approach for the convenience of the reader.

Namely, the map $J \to (J, J')$ identifies Jacobi fields with flow lines of the time dependent vector field $X'(t) = A(t)X(t)$ on the vector space $T = V \times V$, where $A(t)$ is given by $A(t)(v_1, v_2) = (v_2, -R(t)v_1)$. This flow preserves the canonical symplectic form ω on T, given by $\omega((v_1, v_2), (w_1, w_2)) = \langle v_1, w_2\rangle - \langle v_2, w_1\rangle$. Thus, for each Lagrangian subspace $\Lambda \subset T$ the family $\Lambda(t) := \{X(t) | X \in \Lambda\}$ is a curve in the space Lagr of all Lagrangians of T.

Consider the fixed Lagrangian subspace $\Lambda_0 = \{0\} \times V$ of T. By definition, for each Lagrangian subspace Λ of Jac, the focal index $f^{\Lambda}(t)$ is given by $f^{\Lambda(t)} = \dim(\Lambda(t) \cup \Lambda_0)$.

The space $Lagr^0$ of all Lagrangians transversal to Λ_0 is a contractible space. Thus each curve $\gamma(t)$ in Lagr whose endpoints are in Lagr^0 can be (uniquely up to homotopy), completed to a closed curve $\bar{\gamma}$ by connecting the endpoints of γ inside of Lagr^0. Hence, such γ gives us a well defined element in $\pi_1(\mathrm{Lagr}) = \mathbb{Z}$. The image of such a curve γ in $\mathbb{Z}$ is called the Maslov-Arnold index of γ and is denoted by $[\gamma]$.

The Maslov-Arnold index $[\gamma]$ is equal to the intersection number of $\bar{\gamma}$ and the cycle given by $\mathrm{Lagr} \setminus \mathrm{Lagr}^0$ and can be computed as follows. For each time t with non-zero intersection $F_t = \gamma(t) \cap \Lambda_0$, one computes the restriction to F_t of the symmetric bilinear form $B \in \mathrm{Sym}(\Lambda_0)$, given by $\gamma'(t) \in T_{\Lambda_0} \mathrm{Lagr} = \mathrm{Sym}(\Lambda_0)$. If this bilinear form B on F_t is non-degenerate, its signature is the contribution of the point $\gamma(t)$ to the Maslov-Arnold index. In our case, $\gamma(t) = \Lambda(t) = \{X(t) | X \in \Lambda, X'(t) = A(t)X(t)\}$, the bilinear form $B = \Lambda'(t)$ is defined by $B(x, y) = \omega(A(t)x, y)$, for $x, y \in \Lambda(t)$. By the definition of $A(t)$, we have $B(x, x) = ||x||^2$ for each $x \in \{0\} \times V = \Lambda_0$. Thus B is positive definite on each intersection space F_t and the contribution of the Λ-focal point t to $[\gamma]$ is precisely the focal index $f^{\Lambda}(t)$. We conclude that the Maslov-Arnold index of the flow line $\Lambda : [a, b] \to \mathrm{Lagr}$ coincides with the Λ-index $\mathrm{ind}^{\Lambda}([a, b])$ if the endpoints a and b are not Λ-focal. Since the Maslov-Arnold index $[\gamma]$ is a topological notion, it is stable under small perturbations and we get the same conclusion for $\mathrm{ind}^{\Lambda}([a, b])$. Now we can finish the

Proof of Proposition 1.4. The semi-continuity in the general case was shown in Lemma 2.4. Thus let us assume that W and W_n are Lagrangian. Due to the semi-continuity of indices, we find some $\epsilon > 0$ such that W and W_n have no focal points in $[a - \epsilon, a)$ and $(b, b + \epsilon]$. Thus the W-indices of $[a, b]$ and of $[a - \epsilon, b + \epsilon]$ coincides and the same statement is true for the W_n-indices. Now the W_n- and W-indices of $[a - \epsilon, b + \epsilon]$ are equal to the corresponding Maslov-Arnold indices and the last ones are stable under small perturbations. $\square$

2.3. **Geometric interpretation.** Let I be an interval and let $T \to I$ be a Riemannian vector bundle with a Riemannian connection and a family of symmetric endomorphisms $R : T \to T$. The connection is flat and defines an isomorphism of T and the canonical bundle $I \times V \to I$. Thus all results discussed above apply to this situation. The most prominent example is the case of the normal bundle $\mathcal{N}$ along a geodesic γ in a Riemannian manifold M, where the endomorphisms R are the curvature endomorphisms $R(X) = \mathcal{R}(X, \gamma')\gamma'$. Most prominent examples of Lagrangian subspaces of the spaces of Jacobi fields are spaces Λ^N of all normal N-Jacobi fields, where N is a submanifold of M orthogonal to γ. A special and most important case is that of a 0-dimensional submanifold $N = \{\gamma(a)\}$ that defines the Lagrangian Λ^a of all Jacobi fields J with $J(a) = 0$. In this case the Λ-index can be interpreted as the index of a symmetric bilinear form on a Hilbert space or as the index of a Morse function on a space of curves. In this case the results discussed above are contained in any book on Riemannian geometry.

3. Transversal Jacobi equation

3.1. **The construction.** Let $T \to I$ be a Riemannian vector bundle over an interval I with a Riemannian connection ∇ and a field $R : T \to T$ of symmetric endomorphisms. Let Jac be the space of Jacobi fields and let $W \subset$ Jac be an isotropic subspace. We are going to describe Wilking's construction of the transversal Jacobi equation ([Wil04],p.3).

Wilking observed that the family $\tilde{W}(t) := W(t) \oplus \{J'(t) | J \in W^t\}$ is a smooth subbundle of T. Notice that $W(t) = \tilde{W}(t)$ for all non-focal values of t. Denote by $\mathcal{H}$ the orthogonal complement of $\tilde{W}$ and by $P : T \to \mathcal{H}$ the orthogonal projection. Then P defines an identification between $\mathcal{H}$ and $T/\tilde{W}$. The mapping $A(J(t)) = P(J'(t))$ extends to a smooth field of homomorphisms $A : \tilde{W} \to H$, by setting $A(J'(t)) = 0$, for all $J \in W^t$.

Consider the field $R^{\mathcal{H}} : \mathcal{H} \to \mathcal{H}$ of symmetric endomorphisms defined by $R^{\mathcal{H}}(Y) = P(R(Y)) + 3AA^*(Y)$. Denote by $\nabla^{\mathcal{H}}$ the induced covariant derivative on $\mathcal{H}$, that is defined by $\nabla^{\mathcal{H}}(Y) = P(\nabla Y)$. Wilking proved ([Wil04],p.5) that for each Jacobi field $J \in W^{\perp} \subset$ Jac, the projection $Y = P(J)$ is an $R^{\mathcal{H}}$-Jacobi field, i.e. we have

$$\nabla^{\mathcal{H}}(\nabla^{\mathcal{H}}(Y)) + R^{\mathcal{H}}(Y) = 0$$

Two R-Jacobi fields $J_1, J_2 \in W^{\perp}$ have the same projection to $\mathcal{H}$ if and only if $J_1 - J_2 \in W$. Thus the induced map $I : W^{\perp}/W \to \mathrm{Jac}^{R^{\mathcal{H}}}$ is injective and by dimensional reasons it is an isomorphisms. Thus

$R^{\mathcal{H}}$-Jacobi fields are precisely the projections of Jacobi fields in $W^{\perp}$; and Lagrangians in $\mathrm{Jac}^{R^{\mathcal{H}}}$ are projections of Lagrangian in Jac that contain W.

Finally, we have the following equality of indices:

Lemma 3.1. *In the notations above, for each Lagrangian subspace $\Lambda \subset Jac$ that contains W we have the equality $ind_W(I) + ind_{\Lambda/W}(I) = ind_{\Lambda}(I)$.*

Proof. Let $t \in I$ be given. For each $J_1 \in \Lambda$ and $J_2 \in W^t$, we have $\langle J_1(t), J_2'(t)\rangle = 0$. Thus for each $J \in \Lambda$ the inclusions $J(t) \in W(t)$ and $J(t) \in \tilde{W}(t)$ are equivalent. Hence $\Lambda(t) \cap \tilde{W}(t) = \Lambda(t) \cap W(t)$ and we deduce $f^t(W) + f^t(\Lambda/W) = f^t(\Lambda)$. Summing up the focal indices gives us the result. □

Remark 3.1. The index formula above imply the following explosion of indices in quotients, see the next subsection for a geometric interpretation. In the notations of Proposition 1.4, let $W_n \subset \Lambda_n$ be pairs of R_n-isotropic and including Lagrangian subspaces converge to the pair $W \subset \Lambda$. We get smooth transversal Jacobi equation with symmetric endomorphisms $R^{\mathcal{H}_n}$ and $R^{\mathcal{H}}$. Note that at all non-focal points of W the transversal endomorphisms $R^{\mathcal{H}_n}$ converge to $R^{\mathcal{H}}$ and Λ_n/W_n converge to Λ/W on the complement of the set of W-focal points. However, if $\mathrm{ind}_{W_n}(I_0) < \mathrm{ind}_W(I_0)$ for all n, a situation that happens very often, then we deduce $\mathrm{ind}_{\Lambda/W}(I_0) < \mathrm{ind}_{\Lambda_n/W_n}(I_0)$. Thus at some W-focal points the transversal endomorphisms $R^{\mathcal{H}_n}$ are forced to have a very steep bump producing focal points. It seems that if indices of W_n are stable, the fields $R^{\mathcal{H}_n}$ should converge to $R^{\mathcal{H}}$ in the $\mathcal{C}^0$ topology, but we have checked this statement only in the special situation of [LT07a].

3.2. **Geometric interpretation.** In the situation of [Wil04], the meaning of Λ/W is not easy to describe. However, the origin of the tensor $R^{\mathcal{H}}$ defined above is the curvature endomorphism in the base of a Riemannian submersion, a situation that we will shortly describe now (cf. [LT07b] for a more detailed exposition). Thus let $f : M \to B$ be a Riemannian submersion, let γ be a geodesic in M and let $\bar{\gamma} = f(\gamma)$ be its image in B. Let $R, \bar{R}$ be the curvature endomorphisms along γ and $\bar{\gamma}$ respectively. Consider the space W of all Jacobi fields along γ that arise as variational fields of geodesic variations γ_s such that $f(\gamma_s) = \bar{\gamma}$, for all s. Then W is an isotropic subspace, since it is contained the space Λ^N of normal N-Jacobi fields, where $N = f^{-1}(f(\gamma(a)))$ for any a. In this case the additional term AA^* is just the O'Neill tensor ([O'N66],p.465) and the field $R^{\mathcal{H}}$ coincides with $\bar{R}$. In this case $W^{\perp}$ consists of all variation fields of variations through horizontal geodesics, as one deduces by counting of dimensions. The "horizontal" index $\mathrm{ind}_{\Lambda/W}(\gamma)$ describes the index of the geodesic $\bar{\gamma}$ in the quotient space. The "vertical" index $\mathrm{ind}_W(\gamma)$ is 0 in this case, but in the similar and much more general situation of a singular Riemannian foliation (cf. [LT07a]) it counts the intersections of γ with singular leaves. Then the formula of Lemma 3.1 describes a natural decomposition of the Λ-index in a horizontal part seen in the quotient below and a vertical part counting the singular leaves.

4. Applications

4.1. **Conjugate points.** Let $V, I \subset \mathbb{R}, R(t)$, Jac be as in the introduction. Points $a < b \in I$ are called *conjugate* if there is some $J \in \mathrm{Jac}$ with $J(a) = J(b) = 0$. Equivalently, one can say that b is Λ^a-focal. Here and below we use the notation $\Lambda^a = \{J \in \mathrm{Jac} \,|\, J(a) = 0\}$. Before proving Theorem 1.1 we are going to derive its consequences Corollary 1.2 and Corollary 1.3.

Proof of Corollary 1.3. Assume the contrary. Then the Lagrangian $\Lambda^{\bar{a}}$ has index 0 on the interval $I_0 = [a, b]$ and $\mathrm{ind}_{\Lambda^a}(I_0) \geq dim(V) + 1$. This contradicts Theorem 1.1. □

Proof of Corollary 1.2. Consider the normal bundle $\mathcal{N}$ along γ with induced connection ∇ and the curvature endomorphism R. Let Λ^N denote the Lagrangian of all normal N-Jacobi fields along γ. Then the number of N-focal points along γ counted with multiplicity is precisely $\mathrm{ind}_{\Lambda^N}(I) - f^{\Lambda^N}(0) = \mathrm{ind}_{\Lambda^N}(I) - ((n-1) - \dim(N))$, where I is the interval of definition of γ.

Thus it is enough to prove $\mathrm{ind}_{\Lambda^N}(I) \leq n - 1$. Due to Theorem 1.1, it is enough to find a Lagrangian Λ without focal points on I. By assumption, for each $a \in \mathbb{R}$ the space Λ^a has no focal points with exception of a. Let the time a go to the a boundary of the interval I and choose a convergent subsequence of the Lagrangian subspaces Λ^a. Then limiting Lagrangian subspace Λ^∞ ("the space of parallel Jacobi fields") has no focal points in I, due to Proposition 1.4. □

4.2. **The main theorem.** Now we are going to prove a slightly more general version of Theorem 1.1.

Proposition 4.1. *Let $V, I \subset \mathbb{R}, R$, Jac be as usual. Then for any Lagrangians $\Lambda_1, \Lambda_2 \subset$ Jac and any interval I_0 we have $ind_{\Lambda_1}(I_0) - ind_{\Lambda_2}(I_0) \leq dim(V) - dim(\Lambda_1 \cap \Lambda_2)$.*

Proof. We proceed by induction on $\dim(V)$ and start with the case $\dim(V) = 1$. Then $\dim(\Lambda_i) = 1$ and we may assume $\Lambda_1 \neq \Lambda_2$. Assume that $\mathrm{ind}_{\Lambda_1}(I_0) - \mathrm{ind}_{\Lambda_2}(I_0) \geq 2$.

Note that for any $c \in [a, b]$ the space Λ^c is 1-dimensional, thus all focal points have multiplicity one. Therefore we find an interval $I_1 \subset I_0$ with $\mathrm{ind}_{\Lambda_1}(I_1) = 2$ and $\mathrm{ind}_{\Lambda_2}(I_1) = 0$. We may assume $I_1 = [a, b]$ and $\Lambda_1 = \Lambda^a = \Lambda^b$. Since $\dim(V) = 1$, the space Lagr of all Lagrangians is homeomorphic to $\mathbb{R}P^1 = S^1$. Consider the continuous map $F : I_1 \to$ Lagr given by $F(c) = \Lambda^c$. Due to Lemma 2.3, the map is locally injective, thus $F((a, b))$ is an open connected subset of S^1. Since $F(a) = F(b) = \Lambda_1$, the image $F(I_1)$ is a compact subset of S^1 with at most one boundary point Λ^1. But no compact subset of S^1 has precisely one boundary point. Thus $F(I_1) =$ Lagr. Therefore, there is some $c \in I_1$ with $\Lambda_2 = \Lambda^c$. This contradicts $\mathrm{ind}_{\Lambda_2}(I_1) = 0$ and finishes the proof in the case $\dim(V) = 1$.

Let us now assume $\dim(V) = m > 1$ and let the result be true in all dimensions smaller than m. Consider the isotropic subspace $W = \Lambda_1 \cap \Lambda_2$ and assume that $W \neq 0$. Then $\mathrm{ind}_{\Lambda_i}(I_0) = \mathrm{ind}_W(I_0) + \mathrm{ind}_{\Lambda_i/W}(I_0)$, for $i = 1, 2$. Thus replacing V, R by the W-transversal Jacobi equation and using Lemma 3.1 and our inductive assumption we get

$$|ind_{\Lambda_1}(I_0) - ind_{\Lambda_2}(I_0)| = |ind_{\Lambda_1/W}(I_0) - ind_{\Lambda_2/W}(I_0)| \leq \dim(V) - \dim(W)$$

This proves the statement in the case $W \neq 0$. In the case $W = 0$ one finds a Lagrangian Λ_3 with $\dim(\Lambda_3 \cap \Lambda_1) = m - 1$ and $\dim(\Lambda_3 \cap \Lambda_2) = 1$ (To find such Λ_3, take any $(m-1)$-dimensional subspace W of Λ_1, find a non-zero vector J in the intersection $\Lambda_2 \cap W^{\perp}$ and set $\Lambda_3 := W \oplus \{J\}$). Using the result for Lagrangians with non-zero intersection we get:

$$|ind_{\Lambda_1}(I_0) - ind_{\Lambda_2}(I_0)| \leq |ind_{\Lambda_1}(I_0) - ind_{\Lambda_3}(I_0)| + |ind_{\Lambda_1}(I_0) - ind_{\Lambda_2}(I_0)| \leq m$$

□

References

[Dui76] J. Duistermaat. On the Morse index in variational calculus. *Advances in Math.*, 21:173–195, 1976.

[EH90] J.-H. Eschenburg and E. Heintze. Comparison theory for Riccati equations. *Manuscripta Math.*, 68:209–214, 1990.

[LT07a] A. Lytchak and G. Thorbergsson. Curvature explosion in quotients and applications. Preprint, 2007.

[LT07b] A. Lytchak and G. Thorbergsson. Variationally complete actions on nonnegatively curved manifolds. *Illinois J. Math.*, 2007.

[O'N66] B. O'Neill. The fundamental equation of a submersion. *Michigan J. Math.*, 13:459–469, 1966.

[Sak96] T. Sakai. *Riemannian geometry.* Translations of Mathematical Monographs, 149. American Mathematical Society, Providence, RI, 1996.

[Wil04] B. Wilking. A duality theorem for Riemannian foliations in non-negative curvature. To appear in Geom. Funct. Anal., 2004.

A. Lytchak, Mathematisches Institut, Universität Bonn, Beringstr. 1, 53115 Bonn, Germany
E-mail address: lytchak@math.uni-bonn.de