ON ORBIT SPACES OF
REPRESENTATIONS OF COMPACT LIE GROUPS

CLAUDIO GORODSKI AND ALEXANDER LYTCHAK

ABSTRACT. We investigate classes of orthogonal representations of compact Lie groups that have isometric orbit spaces. We find invariants and general structural results of such classes. Our main technical result provides a geometric description of classes which contain representations with very different algebraic properties. Such classes contain most representations arising naturally in geometric problems, for instance, representations of low copolarity and most representations of low cohomogeneity.

1. INTRODUCTION

1.1. **Motivation.** For an orthogonal representation $\rho : G \to \mathbf{O}(V)$ of a compact Lie group G, the quotient metric space V/G is the most important invariant of the action, at least from the metric point of view. We are going to call two representations $\rho_i : G_i \to \mathbf{O}(V_i)$, $i = 1, 2$, *quotient-equivalent* if V_1/G_1 and V_2/G_2 are isometric. If $\dim(G_2) < \dim(G_1)$, the representation ρ_2 will be called a *reduction* of ρ_1. If ρ_2 does not admit a reduction, it will be called a *minimal reduction* of ρ_1 and the dimension of G_2 will be called the *abstract copolarity* of ρ_1 (for reasons that will become clear in the sequel). A representation that does not admit a reduction will be called *reduced*; in this case, we will refer to the dimension of its underlying group as its abstract copolarity.

Beyond the natural question of finding a minimal element in a quotient-equivalence class, our interest in reductions is two-fold. On one hand, the existence of a reduction implies some, probably severe, restrictions on the original representation. The reduction simplifies the action and bounds its "complexity" and that of the singular quotient space. On the other hand, while most representations do not admit reductions (see below), many geometrically interesting ones do. For instance, it is the case for many representations of low cohomogeneity and for the important class of *polar* representations. These are representations admitting reductions to discrete groups; in the terms above, a representation is polar if and only if its abstract copolarity is 0 (see Subsection 2.3). Another class is built by representations with non-trivial principal isotropy groups, and, generalizing both previous examples, by representations with *non-trivial copolarity* [GOT04]. [1] These are representations, for which the isometry $I : V_2/G_2 \to V_1/G_1$ of a reduction can be realized via an isometric embedding $V_2 \to V_1$ and a homomorphism $G_2 \to G_1$ (up to orbit-equivalence, cf. Subsection 2.3) i.e., when one can find a reduction "inside" the original representation. The minimal dimension of all reductions of this kind is called the *copolarity* of the representation (see Subsection 2.3); in particular, the abstract copolarity is bounded above by the copolarity.

It seems interesting to understand non-reduced actions and to describe their minimal reductions. In the case of (abstract) copolarity 0, i.e., in the case of polar actions, the

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[1]Strictly speaking, representations of *discrete* groups are always polar and have trivial copolarity.

cohomogeneity can be arbitrarily large and all such actions (with connected group) have been classified [Dad85, EH99, Ber01]. In the case of higher copolarity, it seems to be extremely involved and, at the moment, not very promising to describe all reducible representations. For irreducible ones there is more hope. In [GOT04] it has been proved that an irreducible representation of a connected group G has copolarity 1 if and only if it has cohomogeneity 3. Those have been classified ([HL71]; see also [Uch80, Yas86, Str96]) and appear in many geometric problems (cf. [GT03]).

1.2. **Results.** While methods of submanifold geometry have been strongly used in [GOT04], here we only apply some elementary observations concerning the geometry of quotients to describe the minimal reductions of irreducible representations of connected groups of low copolarity. We prove:

Theorem 1.1. *Let $\rho : H \to \mathbf{O}(W)$ be a non-reduced non-polar irreducible representation of a connected compact Lie group H. If the copolarity k of ρ is at most 6, then the cohomogeneity of ρ is exactly $k + 2$.*

As mentioned above, we prove Theorem 1.1 by only using the fact that the action and its reduction are quotient equivalent, and by studying the invariants of their quotient equivalence class. Some geometric invariants of quotient equivalence classes like cohomogeneity or the property of being polar (Subsection 2.3) are easy to find. It seems to be considerably more difficult to derive helpful algebraic invariants from the geometric properties of the quotient. The bulk of our paper consists in studying on quotient equivalence classes a variant of the most simple algebraic property: reducibility of the representation. While reducibility or irreducibility of a representation is an invariant of the quotient equivalence class (Lemma 5.1), the restriction of the representations to the respective identity components may behave differently for different representations in one class. That it indeed happens in many important cases can be seen at the example of irreducible representations of connected groups whose minimal reductions have Abelian identity components (for instance, if the abstract copolarity is at most 2).

Our main result describes all quotient equivalence classes in which reducibility of the restricted representation of the identity component is not (!) an invariant of the class:

Theorem 1.2. *Let $\rho : H \to \mathbf{O}(W)$ and $\rho' : H' \to \mathbf{O}(W')$ be quotient equivalent representations. Assume that the action of the identity component H^0 on W is irreducible and that of $(H')^0$ on W' is reducible. Then there is precisely one effective representation $\tau : G \to \mathbf{O}(V)$ in the quotient class of ρ and ρ' which has trivial copolarity. If this quotient equivalence class is non-polar, then the identity component of G is a torus T^k and its action on V can be identified with that of a maximal torus of $\mathbf{SU}(k+1)$ on $\mathbf{C}^{k+1}$.*

As the most important special case we obtain:

Corollary 1.3. *Let $\rho : H \to \mathbf{O}(W)$ be a non-reduced non-polar irreducible representation of a connected compact Lie group H. Let $\tau : G \to \mathbf{O}(V)$ be a minimal reduction of ρ, which we assume to be effective. Let G^0 be the identity component of G. If G^0 acts reducibly on V, then G^0 is a torus T^k and its action on V can be identified with that of a maximal torus of $\mathbf{SU}(k+1)$ on $\mathbf{C}^{k+1}$.*

Further remarks are in order. First of all, there are infinite series of representations satisfying the assumptions of Corollary 1.3, see [GOT04]. All examples appearing therein

have simple relations to some slightly larger polar representations. In general, it seems that all representations satisfying the assumptions of Corollary 1.3 have interesting geometric and topological properties and are closely related to isoparametric submanifolds. We expect to address geometric properties and characterizations of such representations in a forthcoming work.

Note that Theorem 1.1 follows directly from Theorem 1.2 if the copolarity is 1, 2 or 5. The remaining cases as well as the proof of Theorem 1.2 are based on the observation that a representation can admit a reduction only if its quotient has a non-empty boundary, see Proposition 5.2. On the other hand, it is quite difficult for a representation to have non-empty boundary in the quotient. While it is possible to analyze the existence of the boundary in each concrete case, it seems quite difficult to describe all such representations. In the case of simple groups, such classification has been announced by Kollross and Wilking. If such a classification could be carried out for non-simple groups as well (which should be much more difficult), one probably could describe all representations admitting reductions.

A few cases that we could not exclude by direct geometric arguments are ruled out in the second part of the paper, in which we provide a classification of all irreducible representations of connected groups acting with cohomogeneities 4 or 5. These classifications are of independent interest and should be useful for other purposes as well. We would like to mention that, unlike the case of cohomogeneity 3, (and, probably, the case of all cohomogeneities larger than 5), in cohomogeneities 4 and 5 there are only a few representations and no families. As a by-product of the classification, there is an example of cohomogeneity 5 and copolarity 7, which shows that Theorem 1.1 is not valid for $k = 7$. The classification is summarized as follows:

Theorem 1.4. *The non-polar irreducible representations of connected compact Lie groups of cohomogeneities 4 or 5 as well as their copolarities and presence or not of boundary in the orbit space are listed in the following two tables.*

G	ρ	Copolarity	Boundary
$\mathbf{SO}(3)$	$\mathbf{R}^7$	*trivial*	*no*
$\mathbf{U}(2)$	$\mathbf{C}^4$	*trivial*	*no*
$\mathbf{SO}(3) \times \mathbf{G}_2$	$\mathbf{R}^3 \otimes_{\mathbf{R}} \mathbf{R}^7$	2	*yes*
$\mathbf{SU}(3)$	$S^2\mathbf{C}^3$	2	*yes*
$\mathbf{SU}(6)$	$\Lambda^2\mathbf{C}^6$	2	*yes*
$\mathbf{SU}(3) \times \mathbf{SU}(3)$	$\mathbf{C}^3 \otimes_{\mathbf{C}} \mathbf{C}^3$	2	*yes*
$\mathbf{E}_6$	$\mathbf{C}^{27}$	2	*yes*

TABLE 1: COHOMOGENEITY 4

G	ρ	Copolarity	Boundary
$\mathbf{SU}(2)$	$\mathbf{C}^4$	*trivial*	*no*
$\mathbf{SO}(3) \times \mathbf{U}(2)$	$\mathbf{R}^3 \otimes_{\mathbf{R}} \mathbf{R}^4$	*trivial*	*yes*
$\mathbf{SU}(4)$	$S^2\mathbf{C}^4$	3	*yes*
$\mathbf{SU}(8)$	$\Lambda^2\mathbf{C}^8$	3	*yes*
$\mathbf{SU}(4) \times \mathbf{SU}(4)$	$\mathbf{C}^4 \otimes_{\mathbf{C}} \mathbf{C}^4$	3	*yes*
$\mathbf{SO}(4) \times \mathbf{Spin}(7)$	$\mathbf{R}^4 \otimes_{\mathbf{R}} \mathbf{R}^8$	3	*yes*
$\mathbf{U}(3) \times \mathbf{Sp}(2)$	$\mathbf{C}^3 \otimes_{\mathbf{C}} \mathbf{C}^4$	7	*yes*

TABLE 2: COHOMOGENEITY 5

1.3. **Explanation.** We would like to explain the general strategy of this paper by sketching the proof of Theorem 1.1 in the case of $k = 1$. Thus let us assume that $\tau : G \to \mathbf{O}(V)$ is the minimal reduction of an irreducible representation $\rho : H \to \mathbf{O}(W)$ of a connected group H. Assume further that $\dim(G) = 1$, i.e., that G^0 is a circle $\mathbf{U}(1)$. Since ρ is irreducible, so must be τ (Lemma 5.1). Hence the action of G must act transitively on the set of isotypical components of V with respect to the action of G^0. Since G^0 is a circle and the action is effective, we deduce that there is only one G^0-isotypical component and it is given by the complex multiplication of $\mathbf{U}(1)$ on a complex vector space $V = \mathbf{C}^l$. For $l = 1$, the action is polar, thus it is not the minimal reduction. Hence $l \geq 2$. From Proposition 5.2 we deduce that $W/H = V/G = (V/G^0)/\Gamma$ has non-empty boundary, where Γ is the discrete group G/G^0. Hence the quotient of the unit sphere $S(V)/G = (S(V)/G^0)/\Gamma$ has non-empty boundary as well. But $S(V)/G^0$ is the complex projective space $\mathbf{C}P^{l-1}$. The finite quotient $\mathbf{C}P^{l-1}/\Gamma$ can have non-empty boundary if and only if some element of Γ acts as a reflection at a totally geodesic hypersurface. But such hypersurfaces exist in $\mathbf{C}P^{l-1}$ only for $l = 2$. Thus $V = \mathbf{C}^2$.

1.4. **Questions.** We would like to finish the introduction by formulating some basic questions about quotient-equivalence classes, closely related to our results.

Question 1.1. Assume that $\rho_i : G_i \to \mathbf{O}(V_i)$, where $i = 1$, 2, are effective, reduced and quotient-equivalent. Is it true that the ρ_i must be equivalent as representations?

Question 1.2. Let the irreducible representation of G on V be reduced. What is the isometry group of the quotient space V/G? Can it be much larger than the $N(G)/G$, where $N(G)$ is the normalizer of G in $\mathbf{O}(V)$?

Question 1.3. For a representation $\rho : G \to \mathbf{O}(V)$ of a group G consider the following four conditions.
  (C1) There is an orbit-equivalent action of some group G' which has non-trivial principal isotropy groups.
  (C2) The representation has non-trivial copolarity.
  (C3) The action has a non-trivial reduction.
  (C4) The quotient V/G has non-empty boundary.
  We have implications (C1) $\Rightarrow$ (C2) $\Rightarrow$ (C3) $\Rightarrow$ (C4). We do not know a single representation which satisfies (C3) but not (C1) and only very few representations of connected groups satisfying (C4) but not (C1). Are there some reverse implications?

For all representations appearing in our main results the conditions (C1), (C2) and (C3) are equivalent. We formulate a special case of this observation that follows directly from the proofs of the main theorems:

Corollary 1.5. *Let ρ be an irreducible representation of a connected compact Lie group. If the cohomogeneity is at most 5 or the abstract copolarity is at most 6, then the copolarity of the representation coincides with its abstract copolarity.*

1.5. **Structure.** The paper is divided into two parts. In the first part we study the geometry of reductions and prove Theorems 1.1 and 1.2. After a section on preliminaries, we investigate orbifold parts of quotient spaces in Section 3. We use orbifold fundamental groups to reduce the investigation of quotient equivalent actions of G_i on M_i, $i = 1$, 2, to the case where G_i/G_i^0 acts on the subquotient M_i/G_i^0 as a *reflection group* (Proposition 3.2). Moreover, we

find an easy criterion when one can replace one of the groups G_i by its identity component (Proposition 3.1). In Section 4 we study the restrictions imposed by the triviality of the principal isotropy groups on the strata of low codimensions. In Section 5, we specialize to representations and prove some basic structural results: the invariance of irreducibility (Lemma 5.1) and the necessity of boundaries for the existence of reductions (Proposition 5.2). The results from these preparatory sections may be useful for other related problems as well.

In Section 6, the technical heart of the paper, we study irreducible representations whose restricted representations to the identity component are reducible and prove Proposition 6.1, where our special representations of maximal tori T^k of $\mathbf{SU}(k+1)$ turn up. In Section 7 we apply results of Sections 3, 4 and 5, to study the geometry of $\mathbf{C}^{k+1}/T^k$. In Section 8, we collect the harvest and prove Theorem 1.2. In Section 9, we start with the proof of Theorem 1.1, which is finished in the two subsequent sections, dealing with the cases of connected, respectively, disconnected minimal reduction of our original presentation.

In the final part of the proof of Theorem 1.2 as well as of Theorem 1.1, a few cases that we cannot resolve by direct geometric arguments show up. To exclude these few possible reductions, we rely on the classification of irreducible representations of low cohomogeneity (Theorem 1.4). This theorem is proved independently in the second part of the paper, by invoking classifications of representations of simple groups with low cohomogeneity, followed by a straightforward but a bit tedious analysis of tensor products of representations. In the last section we discuss copolarities of the found representations.

Part 1. Geometry of reductions

2. Preliminaries

In this section we collect a few basic results about actions of a compact group of isometries G on a connected complete Riemannian manifold M. We assume that the actions are effective. In the later sections, M will always be either an Euclidean space or an Euclidean sphere. Let X be the quotient space M/G with the induced quotient metric.

2.1. **Stratification.** For a point $p \in M$ we denote by G_p its isotropy group, and we denote by $St(p)$ the stratum of p, i.e., the *connected component* through p of the set of points $q \in M$ whose isotropy groups G_q are conjugate to G_p.

Denote by x the image point $x = G \cdot p \in X$. The stratum $St(p)$ is a manifold and projects to a Riemannian totally geodesic submanifold $St_X(x)$ of the space X, called a stratum of X.

The dimension of the stratum $St(p)$ coincides with the topological dimension of $\dim(G \cdot F)$, where F is the connected component through p of the set of fixed points of G_p. We have $\dim(G \cdot F) = f_p + g - n_p$, where f_p is the dimension of F, where g is the dimension of G and n_p is the dimension of the normalizer $N(G_p)$ of G_p in G.

Locally at a point $p \in M$, the orbit decomposition of M is completely determined by the *slice representation* of the isotropy group G_p on the normal space $\mathcal{H}_p := N_p(G \cdot p)$ to the orbit $G \cdot p$. According to our convention $St(p)$ is connected, therefore the equivalence class of the slice representation along $St(p)$ is constant. The set of fixed vectors of G_p in $\mathcal{H}_p$ is tangent to the stratum $St(p)$, and the action of G_p on its orthogonal complement $\mathcal{H}_p^\dagger$ in $\mathcal{H}_p$ has cohomogeneity $\dim(\mathcal{H}_p^\dagger/G_p)$, which equals to the codimension of the stratum $St_X(x)$ in X, where $x = G \cdot p$.

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A point $p \in M$ is called *regular* if $\mathcal{H}_p^{\dagger}$ is trivial. It is called *exceptional* if it is not regular and the action of G_p on $\mathcal{H}_p^{\dagger}$ has discrete orbits. If it is neither regular nor exceptional, it is called *singular*. The set M_{reg} of all regular points in M is open and dense, and $X_{reg} = M_{reg}/G$ is connected. X_{reg} is the stratum corresponding to the unique conjugacy class of minimal appearing isotropy groups. These isotropy groups are called the *principal isotropy groups*.

2.2. Boundary.

The set X_{reg} of regular points of X is exactly the set of points that have neighborhoods isometric to Riemannian manifolds. It is the unique maximally dimensional stratum in X. By definition, the *boundary* of X is the closure of union of all strata that have codimension 1 in X. It is denoted by ∂X. A point $p \in M$ is mapped to a stratum of codimension 1 in X if and only if the isotropy group G_p acts on $H_p^{\dagger}$ with cohomogeneity 1. Since boundary points will play a special role in subsequent considerations, we are going to call a point $p \in M$ that projects to a point on a stratum of codimension 1 in X an *important point*, or, if the action needs to be specified, a *G-important point*.

Let now G' be a normal subgroup of G, with finite quotient $\Gamma = G/G'$. Then Γ acts by isometries on $X' := M/G'$, and $X = X'/\Gamma$. Since X and X' have the same dimension, and since strata of X' are mapped to unions of strata of X, we have $\pi(\partial X') \subset \partial X$. Here, $\pi : X' \to X$ is the canonical projection. On the other hand, any G-important point p that is not G'-important must be G'-regular. In this case, we deduce that G_p/G_p' must act on the normal space $\mathcal{H}_p$ as a single reflection at a hyperplane in $\mathcal{H}_p$.

2.3. Copolarity and polar actions.

We refer to [GOT04] for a detailed discussion of the following notions. A *generalized section* of the action of G on M is a connected complete totally geodesic submanifold Σ whose intersection $\Sigma \cap M_{reg}$ with the set of regular points is not empty and satisfies $\mathcal{H}_p \subset T_p\Sigma$ for any $p \in \Sigma \cap M_{reg}$. For any minimal generalized section Σ, the group $G_\Sigma = \{g \in G | g\Sigma = \Sigma\}$ acts on Σ and the canonical map $\Sigma/G_\Sigma \to M/G$ is an isometry. The minimal dimension of such G_Σ is called the *copolarity* of the action. In addition, we say that the copolarity of the action of G on M is non-trivial if $\Sigma \neq M$, i.e., if $G_\Sigma \neq G$.

If the action of G on M has non-trivial principal isotropy groups then a connected component containing regular points of the set of fixed points of any principal isotropy group is a generalized section. Thus such an action has non-trivial copolarity.

An action of another compact Lie group G' by isometries on another connected complete Riemannian manifold M' is called *orbit-equivalent* to the action of G on M if both actions have the same orbits, up to isometry. More precisely, they are orbit-equivalent if there exists an isometry $F : M \to M'$ such that $F(G(p)) = G'(F(p))$ for every $p \in M$. In this case, we can identify M and M' via F and view G and G' as subgroups of the isometry group of M. If G' does not coincide with G inside the isometry group of M then the group generated by G and G' is also orbit equivalent to both actions and has non-trivial principal isotropy groups. Thus this group and therefore the actions of G and G' have non-trivial copolarity in this case.

An action is called *polar* if it has copolarity 0, i.e., if and only if it admits a generalized section Σ with $\dim(\Sigma) = \dim M/G$ (in which case Σ is called a *section* of the action).

If M is a Euclidean space V, then by the above, any non-trivial generalized section defines a reduction. Thus the abstract copolarity is bounded above by the copolarity.

It is known that an orthogonal representation G on V is polar if and only if it is orbit equivalent to the isotropy representation of a symmetric space [Dad85]. Another equivalent

formulation is that the set of regular points X_{reg} of X is flat [HLO06, Ale06]. Therefore, if a representation has a reduction to a discrete group, it must be polar. Thus a representation has abstract copolarity 0 if and only if it has copolarity 0.

The classification result of [Str94] implies, in terms of copolarity, that any representation of cohomogeneity at most 3 has copolarity at most 1.

3. REFLECTIONS IN QUOTIENTS

3.1. **Formulation.** Given two isometric quotients M_1/G_1 and M_2/G_2 of Riemannian manifolds modulo compact groups of isometries, one would like to replace the groups by some smaller subgroups of finite index (for instance, by the identity components) preserving the property of having isometric quotients. In this section we are going to prove two useful criterions for such reductions. The first one is quite general:

Proposition 3.1. *Let M_1, M_2 be simply connected Riemannian manifolds. Let G_i be a compact group of isometries of M_i, for $i = 1, 2$. Assume that M_1/G_1 and M_2/G_2 are isometric. If the quotient M_1/G_1^0 has no boundary, then for any subgroup G_2' of finite index in G_2 there is a subgroup G_1' of finite index in G_1 such that $M_2/G_2' = M_1/G_1'$.*

To formulate our criterion in the presence of boundaries we need a definition. A *reflection* on a Riemannian manifold is an isometry whose set of fixed points has codimension 1. Let $X = M/G$ be a quotient space. An isometry of X is called a *reflection* on X if its restriction to the regular part X_{reg} is a reflection. A discrete group of isometries of X that is generated by reflections is called a *reflection group* on X. We refer to [DAM07] for more about reflection groups on manifolds.

The second result of this section is the following:

Proposition 3.2. *Let G_i be a compact group of isometries of a simply connected Riemannian manifold M_i for $i = 1, 2$. Assume that M_1/G_1 and M_2/G_2 are isometric. Then there are normal subgroups of finite index G_i^+ in G_i such that the group G_i^+/G_i^0 acts on the quotient M_i/G_i^0 as a reflection group, $i = 1, 2$, and M_1/G_1^+ and M_2/G_2^+ are isometric. In particular if, say, G_1 is connected, we can take $G_i^+ = G_i$ for $i = 1, 2$ and then G_2/G_2^0 acts on M_2/G_2^0 as a reflection group.*

To prove the results we are going to study more closely a slightly larger part of the quotient X than X_{reg}, its *orbifold part* X_{orb}.

3.2. **Riemannian orbifolds and their fundamental groups.** We are going to use a bit about orbifolds and orbifold fundamental groups. We refer the reader to the notes by Thurston [Thu80] and to those by Davis [Dav10].

A *Riemannian orbifold* is a metric space C where each point has a neighborhood isometric to a finite quotient of a smooth Riemannian manifold. Since it is locally represented as a quotient in a unique manner, it comes along with a natural stratification and a unique underlying structure of a smooth orbifold.

Let C be a connected Riemannian orbifold. Let $\pi_1^{orb}(C)$ denote the *orbifold fundamental group* of C (cf. [Dav10]). This group acts as a group of discrete isometries on a connected Riemannian orbifold $\tilde{C}$, called the *universal orbifold covering* of C, such that $\tilde{C}/\pi_1^{orb}(C) = C$. For any other presentation of the orbifold C as a quotient $C = B/\Gamma$ of a connected Riemannian orbifold B modulo a discrete group of isometries Γ, there is an (essentially unique) normal subgroup Γ' of $\pi_1^{orb}(C)$, such that $\tilde{C}/\Gamma' = B$ and such that $\Gamma = \pi_1^{orb}(C)/\Gamma'$.

A Riemannian orbifold C has a trivial orbifold fundamental group if and only if it coincides with its universal orbifold covering $\tilde{C}$. In this case C has no boundary and it is simply connected as a topological space. In general, one can write a presentation of the orbifold fundamental group of C in terms of its usual fundamental group and its strata of codimension 1 and 2 [Dav10]. Looking at this presentation of the orbifold fundamental group one notes that the embedding $C \setminus \partial C \to C$ induces an injection $\pi_1^{orb}(C \setminus \partial C) \to \pi_1^{orb}(C)$.

If $\pi_1^{orb}(C \setminus \partial C) = 1$ (essentially, the only case that will be of importance in the sequel, see Lemma 3.5) then the presentation of $\pi_1^{orb}(C)$ has the following simple form. For any stratum Σ of codimension one in C one takes a generator w_Σ that has by definition order 2. Whenever codimension one strata Σ_1, Σ_2 meet at a stratum P of codimension 2, (i.e. $P \subset \bar{\Sigma}_1 \cap \bar{\Sigma}_2$) one adds a relation $(w_{\Sigma_1} \cdot w_{\Sigma_2})^{m(P)} = 1$, where $m(P)$ is the natural number defined such Σ_1 and Σ_2 meet at P at the angle $\pi/m(P)$. In particular, $\pi_1^{orb}(C)$ is a Coxeter group. If all occuring angles $\pi/m(P)$ are equal to $\pi/2$, then the group $\pi_1^{orb}(C)$ must be Abelian or infinite.

3.3. Reflections in orbifolds.
Let C be a connected Riemannian orbifold. For a discrete group Γ of isometries of C we denote by Γ_{refl} the subgroup of Γ that is generated by reflections of C that are contained in Γ. Since a conjugate of a reflection is a reflection, Γ_{refl} is a normal subgroup of Γ.

The following lemma will provide us with sufficiently many reflections:

Lemma 3.3. *Let B be a Riemannian orbifold. Let Γ be a discrete group of isometries of B. Let C be the orbifold $C = B/\Gamma$. Set $C_0 = C \setminus \partial C$. If the orbifold fundamental group of C_0 is trivial, then Γ is a reflection group on B.*

Proof. Denote by Γ' the quotient group $\Gamma' = \Gamma/\Gamma_{refl}$. Then Γ' acts by isometries on $B' := B/\Gamma_{refl}$ such that $B'/\Gamma' = C$. If an element w in Γ' acts as a reflection on B', then we can find a manifold point p in B that is projected to a manifold point in B', but whose projection to C lies on a stratum of codimension 1 in C. Thus this point p must be fixed by a reflection in Γ that is not in Γ_{refl}, providing a contradiction.

Hence Γ' does not contain reflections of B'. Therefore, the projection $B' \to C$ has the property that the preimage of a boundary point in C is a boundary point in B'. Hence the preimage of C_0 is exactly $B'_0 = B' \setminus \partial(B')$, i.e., a connected orbifold. Hence $C_0 = B'_0/\Gamma'$. Since $\pi_1^{orb}(C_0) = 1$, the group Γ' acts trivially on B'. Thus $\Gamma = \Gamma_{refl}$. $\qquad\square$

Consider now the action of $\pi_1^{orb}(C)$ on the universal covering $\tilde{C}$ of C. We obtain the reflection group $\pi_1^{orb}(C)_{refl} =: \pi_{1,refl}^{orb}(C)$ of isometries of $\tilde{C}$. The quotient $\tilde{C}/\pi_{1,refl}^{orb}(C)$ is a Riemannian orbifold, which will be denoted by C_{refl}. The quotient group $\pi_{1,nonrefl}^{orb}(C) := \pi_1^{orb}(C)/\pi_{1,refl}^{orb}(C)$ acts on C_{refl} with $C_{refl}/\pi_{1,nonrefl}^{orb}(C) = C$. By construction, C_{refl} is the unique minimal orbifold covering of C with property that its orbifold fundamental group is generated by reflections.

The following observation is probably well known to the experts. We include it here with a sketchy proof for the sake of completeness:

Proposition 3.4. *In the notations above we have $\pi_{1,nonrefl}^{orb}(C) = \pi_1^{orb}(C \setminus \partial C)$. Thus for any Riemannian orbifold there is a split exact sequence:*

$$1 \to \pi_{1,refl}^{orb}(C) \to \pi_1^{orb}(C) \to \pi_1^{orb}(C \setminus \partial C) \to 1$$

Proof. The proof of Lemma 3.3 shows that $\pi^{orb}_{1,nonrefl}(C)$ does not contain reflections of C_{refl}. Moreover, for the projection from C_{refl} to C, the preimage of $C \setminus \partial C$ is the connected Riemannian orbifold $C_{refl} \setminus \partial C_{refl}$.

Now, by definition, the orbifold fundamental group of C_{refl} is generated by reflections. Looking at the presentation of the orbifold fundamental group in terms of the strata [Dav10], one deduces that $C_{refl} \setminus \partial C_{refl}$ has trivial orbifold fundamental group. Hence $\pi^{orb}_{1,nonrefl}(C)$ acts on the simply connected orbifold $C_{refl} \setminus \partial C_{refl}$ with quotient orbifold identified with $C \setminus \partial C$. This implies the result. $\qquad\square$

3.4. **Orbifold points in quotients.** We call a point x in our quotient $X = M/G$ an orbifold point if it has a neighborhood isometric to a Riemannian orbifold. It has been shown in [LT10] that a point $p \in M$ is projected to an orbifold point in X if and only if the slice representation of G_p on $\mathcal{H}_p$ is polar. The orbifold X_{orb} of all orbifold points in X is open, connected and it is a union of strata. It contains all strata that have codimension at most 2 in X, in particular, all G-important orbits. The set X_{orb} has a non-empty boundary if and only if X has non-empty boundary.

The following result, probably folklore, can be found in [Lyt10]:

Lemma 3.5. *Let M be a simply connected complete Riemannian manifold. Let G be a connected compact group of isometries of M. Let X be the quotient M/G. Let X_{orb} be the set of orbifold points in X and set $X_0 = X_{orb} \setminus \partial X_{orb}$. Then X_0 is exactly the set of non-singular G-orbits. Moreover, X_0 has trivial orbifold fundamental group.*

In particular, the above lemma says, that if M is simply connected and G connected, no G-important point may lie on an exceptional orbit.

The open subsets X_{reg} and X_{orb} are dense and convex in the quotient space X. Thus for a pair of quotients $X = M/G$ and $Y = N/H$ any isometry between X_{orb} and Y_{orb} (or between X_{reg} and Y_{reg}) extends uniquely to an isometry between their completions X and Y.

Consider again the quotient $X = M/G$. If G' is a subgroup of finite index in G then $\pi(X'_{orb}) = X_{orb}$, where $X' = M/G'$ and $\pi : X' \to X$ is the canonical projection.

Let X^0 be the subquotient M/G^0. Consider the orbifold parts X_{orb} and X^0_{orb}. The group G acts on X^0_{orb} by isometries and we obtain a homomorphism j from G/G^0 onto a finite group D of isometries of X^0_{orb}, such that $X_{orb} = X^0_{orb}/D$. Thus for the orbifold fundamental groups $\Gamma = \pi^{orb}_1(X_{orb})$ and $\Gamma_0 = \pi^{orb}_1(X^0_{orb})$, the group Γ_0 is a normal subgroup of Γ with $D = \Gamma/\Gamma_0$.

Any subgroup G' of finite index in G has a subquotient $X' = M/G'$ such that the orbifold fundamental group of X'_{orb} is contained in $\pi^{orb}_1(X_{orb})$ and contains $\pi^{orb}_1(X^0_{orb})$. On the other hand, any group Γ' with $\Gamma_0 \subset \Gamma' \subset \Gamma$ projects to a subgroup of D. Taking the preimage of this subgroup under j in G we obtain a subgroup G' of finite index in G. Then the orbifold part X'_{orb} of the quotient $X' = M/G'$ has Γ' as its orbifold fundamental group.

Now we are in position to prove the main results of this section.

Proof of Proposition 3.1. Set $X = M_1/G_1 = M_2/G_2$. By assumption, $X^0 = M_1/G^0_1$ has no boundary. Hence X^0_{orb} has trivial orbifold fundamental group by Lemma 3.5. Consider the orbifold covering $X'_{orb} = (M_2/G'_2)_{orb}$ of X_{orb} defined by the subgroup G'_2 of G_2. Since $\pi^{orb}_1(X^0_{orb}) = 1$, using considerations preceding the proof, we find a subgroup G'_1 of G_1 such that $(M_1/G'_1)_{orb} = X'_{orb}$. $\qquad\square$

Proof of Proposition 3.2. Let $X = M_1/G_1 = M_2/G_2$ and consider the orbifolds $(M_i/G_i^0)_{orb}$ for $i = 1, 2$. We deduce from Lemma 3.5 and Lemma 3.3 that $\Gamma^i = \pi_1^{orb}((M_i/G_i^0)_{orb})$ is generated by reflections. Consider now the subgroup Γ_{refl} of $\Gamma = \pi_1^{orb}(X_{orb})$ that is generated by all reflections in Γ. Then Γ_{refl} contains Γ^i, hence due to the considerations above, we find subgroups of finite index G_i^+ in G_i such that $(M_i/G_i^+)_{orb}$ is isometric to $(X_{orb})_{refl}$, i.e., to the quotient of the universal orbifold covering of X_{orb} modulo Γ_{refl}. Hence M_i/G_i^+ for $i = 1, 2$ are isometric.

Since Γ_{refl} is generated by reflections on the universal orbifold covering of X_{orb}, the group Γ_{refl}/Γ^i acts as a reflection group on $(M_i/G_i^0)_{orb}$. $\qquad\square$

4. Triviality of the principal isotropy group

4.1. Boundary points.
Let the compact group G act (effectively) on the simply connected manifold M with trivial principal isotropy groups. Then the principal isotropy group of any slice representation is trivial as well. If $p \in M$ is a G-important point then the isotropy group G_p must act transitively on the unit sphere S^a in $\mathcal{H}_p^\dagger$, the normal space to the stratum $St(p)$. Since the action must have trivial principal isotropy as well, G_p must be diffeomorphic to S^a. But this can happen only for $a = 0$, 1 or 3. Note that if $a = 1$ or 3, then G_p is contained in the identity component G^0 of G and p is G^0-important. On the other hand, if p is a G^0-important point, then it cannot lie on an G^0-exceptional orbit, due to Lemma 3.5. Thus G_p must be non-discrete and $a \neq 0$ in this case.

Hence p is G-important and not G^0-important if and only if G_p is a group with only one non-trivial element w. This element w is an involution in $G \setminus G^0$ which normalizes G^0. Moreover, it acts as a reflection on M/G^0.

Summarizing and using Subsection 2.1 we arrive at:

Lemma 4.1. *Let the compact group G with identity component G^0 act effectively on a simply connected manifold M. Assume that the principal isotropy group of G is trivial. If a point p is G-important, then $G_p = S^a$, for a equal to 0, 1 or 3. We have $a = 0$ if and only if p is not G^0-important. Moreover the stratum $St(p)$ through p has dimension $\dim St(p) = \dim(M) - a - 1 = f_p + g - n_p$, where f_p denotes the dimension of the connected component of the set of fixed points of G_p, g is the dimension of G and n_p is the dimension of the normalizer of G_p in G.*

4.2. Codimension two strata.
Let a *connected* compact group G act on a simply connected manifold M with trivial principal isotropy group. Let X be the quotient M/G, and let x be a point in a stratum of codimension two in X. Then $x \in X_{orb}$. If x is not contained in the boundary ∂X it must be an exceptional orbit of the G-action (Lemma 3.5). Otherwise, $x \in \partial X$. Take a point p in the G-orbit corresponding to x. Then the tangent cone $C_x X$ at x to X is isometric to the quotient $\mathcal{H}_p/G_p$ of the slice representation. The assumption that x lies in a stratum of codimension two implies that G_p fixes a subspace $\mathbf{R}^{k-2}$ of $\mathcal{H}_p$ and acts on the orthogonal complement $\mathcal{H}_p^\dagger$ with codimension two.

The representation of $(G_p)^0$ on $\mathcal{H}_p^\dagger$ is of cohomogeneity two and the quotient $\mathcal{H}_p^\dagger/(G_p)^0$ is isometric to $\mathbf{R}^2/D_m$, where D_m is the dihedral group of order $2m$ that is generated by two reflections at lines enclosing the angle π/m. The classification of representations of cohomogeneity two [HL71] implies that m can be only 2, 3, 4 or 6 (much more sophisticated topological argument is given in [Mün81]). Moreover, the classification shows that the action of $(G_p)^0$ on $\mathcal{H}_p^\dagger$ has non-trivial principal isotropy groups, possibly unless if $m = 2$. If the

action of $(G_p)^0$ has non-trivial principal isotropy, so does the action of G. Thus $\mathcal{H}_p/(G_p)^0$ is isometric to $\mathbf{R}^{k-2} \times \mathbf{R}^2/D_2$.

We claim that the action of G_p on $\mathcal{H}_p$ is orbit-equivalent to the action of $(G_p)^0$. Otherwise, $G_p/(G_p)^0$ acts as a non-trivial group of isometries on $\mathcal{H}_p^\dagger/(G_p)^0$. However, the only non-trivial isometry of the quadrant $\mathbf{R}^2/D_2$ is the reflection at the midline of the quadrant. Any non-zero small vector $v \in \mathcal{H}_p$ in the preimage of this line is G_p-important and $(G_p)^0$-regular. Exponentiating v, we find a point p' in M close to p that is G-important and lies on an exceptional orbit. This contradicts Lemma 3.5.

Thus, we have shown that the tangent cone $C_x X$ is isometric to $\mathbf{R}^{k-2} \times \mathbf{R}^2/D_2$. But this means that the two strata of codimension 1, whose closures contain the point x, meet at the point x at a right angle.

Lemma 4.2. *Let a connected compact Lie group G act isometrically on a simply connected manifold M. Assume that the principal isotropy groups of the action are trivial. Then codimension 1 strata of $X = M/G$ can only meet at a right angle, i.e., for any point x in a stratum of codimension two in X which is contained in the boundary, the tangent cone $C_x X$ at x is isometric to $\mathbf{R}^{k-2} \times \mathbf{R}^2/D_2$. Here, D_2 is the dihedral group of order 4 generated by reflections at two orthogonal lines.*

4.3. Nice involutions. Assume now that a (possibly disconnected) group G acts on Riemannian manifold M with trivial principal isotropy group. Assume moreover that $\Gamma = G/G^0$ acts on M/G^0 as a reflection group. Since the action has trivial principal isotropy groups, the action of Γ on M/G^0 is effective. By definition, the group Γ is generated by elements $w' \in G$ that act on the quotient M/G^0 as reflections. Given any $w' \in G$ that acts on M/G^0 as a reflection, we take a regular point $x \in M/G^0$ that is fixed by w'. Let p be a preimage of x in M. Then p is G^0-regular and G-important. Thus G_p contains only one non-trivial element w, equal to $w'g$, for some $g \in G^0$. This element w is an involution and the connected component F through p of its set of fixed points satisfies $\dim(G \cdot F) = \dim(F) + \dim(G) - \dim(C) = \dim(M) - 1$, where C is the centralizer of w (hence the normalizer of G_p) in G. Since w and w' are equivalent modulo G^0, involutions of the kind of w generate Γ; we will call them *nice involutions*.

5. Basic observations

In this section we collect some basic observations about quotient-equivalences. For a Euclidean space U, we will denote by $S(U)$ the unit sphere of U.

5.1. Fixed points and origins. Let U be a Euclidean space and let K be a nontrivial closed subgroup of $O(U)$. Let F be the set of fixed points of K, and let $F^\perp$ be its orthogonal complement. Then $X = U/K$ splits as $F \times (F^\perp/K)$. Moreover, for any unit vector v in $U \setminus F$ there is no unit vector v' in U with $d(K \cdot v, v') = 2$. Thus there is no geodesic in U/K that starts in $K \cdot v$ and has the origin $K \cdot 0 = 0_U$ as its midpoint. Thus F contains the set of all lines through the origin. In particular, it is the unique maximal Euclidean factor of U/K.

On the other hand, in the factor $F^\perp/K$ the origin 0_U is the only point that is not the midpoint of some geodesic (all other points lie on the ray that starts at 0_U).

Therefore, for any other representation of a group K' on an Euclidean space U', any isometry $I : U/K \to U'/K'$ must be given as a product of isometries $I_1 : F \to F'$ and

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$I_2 : F^\perp/K \to (F')^\perp/K'$. Moreover, the second isometry I_2 must send the origin to the origin.

Changing our isometry I by an isometry of the Euclidean space F, if needed, we may therefore assume that I sends the origin 0_U to the origin $0_{U'}$. From now on we will make this assumption.

Since the quotient of the unit spheres $S(U)/K$ is just the unit distance sphere in the quotient, our isometry I induces an isometry between the spherical quotients $I : S(U)/K \to S(U')/K'$.

If the set of fixed points F is non-trivial, then $S(U)/K$ has diameter π. On the other hand, assume that the diameter of $S(U)/K$ is larger than $\pi/2$. Then, for some orbit $K \cdot v$ of a unit vector v, the set of points in the unit sphere with distance $\geq \pi/2 + \epsilon$ to this orbit is non-empty, for some positive ϵ. But this set is compact, convex, K-invariant and does not contain great circles. Hence it has a unique center which must be fixed by K.

Thus we see that the action has non-zero fixed points if and only if the diameter of $S(U)/K$ is larger than $\pi/2$ (in which case it is equal to π).

5.2. Invariance of reducibility.
We want to prove that invariant subspaces can be recognized metrically and thus are invariants of the quotient-equivalence classes.

Thus let K act on U as above. Consider the restricted action on the sphere $S(U)$ with quotient $X = S(U)/K$. Assume that there are no fixed points of K in $S(U)$.

We claim that a closed subset $Z \subset X$ has the form $S(V)/K$ for some K-invariant subspace V if and only if there is some $Z' \subset X$ such that Z is the set of all points $z \in X$ with $d(z, z') = \pi/2$, for all $z' \in Z'$.

Namely, if $Z = S(V)/K$ where V is K-invariant, one can consider its orthogonal complement $V^\perp$ and set $Z' := S(V^\perp)/K$. On the other hand, if $Z \subset X$ is given in terms of Z' as above, then the preimage of Z in $S(U)$ is a compact convex K-invariant subset of $S(U)$. Since K does not have fixed points in U, this subset must be a great subsphere of $S(U)$. Thus it is the unit sphere of a K-invariant subspace.

This provides a metric description of projections of invariant subspaces under the assumption of the absence of fixed points. Combining it with the previous subsection we arrive at:

Lemma 5.1. *Let $\rho : K \to \mathbf{O}(U)$ and $\rho' : K' \to \mathbf{O}(U')$ be quotient-equivalent representations, with projections $\pi : U \to U/K$ and $\pi' : U' \to U'/K'$. Then ρ is irreducible if and only if ρ' is. More precisely, if $I : U/K \to U'/K'$ is an (origin preserving) isometry then for any K-invariant subspace V of U the subset $\pi'^{-1}(I(\pi(V)))$ is a K'-invariant subspace of U'.*

5.3. Existence of boundary points.
Given a representation of $\rho : G \to \mathbf{O}(V)$ of a group G, the copolarity of G and of its identity component G^0 coincide [GOT04]. On the other hand, the corresponding statement about the abstract copolarity is not clear to us. The following lemma, which is most basic to our results, is formulated in terms of the identity component. In view of this lemma, it seems that the abstract copolarity of the representation may always coincide with that of the identity component.

Proposition 5.2. *Let $\rho : G \to \mathbf{O}(V)$ be an effective representation and let G^0 be the identity component of G. If $X_0 = V/G^0$ has an empty boundary then the representation ρ is reduced.*

Proof. Assume that the boundary of X_0 is empty and assume that $X = V/G$ is isometric to W/H, for a representation of a group H on W with $\dim(W) < \dim(V)$.

The "unit sphere" Y_0 in X_0, i.e., the distance sphere to the orbit of the origin, is isometric to $S(V)/G^0$. Since X_0 is the Euclidean cone over Y_0, the quotient Y_0 does not have boundary as well. The unit sphere Y of X is isometric to $S(V)/G$ and to $S(W)/H$. By construction, Y is a finite quotient $Y = Y_0/\Gamma$, for $\Gamma = G/G^0$.

Since Y_0 has no strata of codimension 1, we find an infinite geodesic in Y_0 that is contained in the set of G^0-regular orbits and starts at a point $y \in Y_0$, such that y is projected to a regular point in Y. Take a part γ of this geodesic that has length π. Let m be the index of this geodesic (i.e., the number of conjugate points along γ, counted with multiplicites). For the Riemannian submersion $S(V)_{reg} \to (Y_0)_{reg}$, consider any horizontal lift η of γ. Then the index m of γ is equal to the L-index of η (i.e. the number of L-focal points along η), where L is the G^0-orbit corresponding to the point y, and where η is considered as an L-geodesic. But in the round sphere $S(V)$, the L-index of any L-geodesic η of length π is exactly the dimension of L. Thus $m = \dim(L)$ in this case.

Consider now the image γ' of γ in Y. It is contained in the orbifold part of Y and is (by definition) an *orbifold geodesic* that starts at a regular point. Consider a lift η' of γ' to an H-horizontal geodesic in $S(W)$ that starts on a regular H-orbit L'. It has been shown in [LT10] that the L'-index of the geodesic η' is equal to the sum of the index of the orbifold-geodesic γ' and a "vertical index", a non-negative number that counts the number of intersection of η' with H-singular orbits. In particular, it is not smaller than m, the index of the orbifold-geodesic γ'. Using again that the L'-index of η' is given by the dimension of L', we get $\dim(L') \geq m$.

But this contradicts $\dim(V) = \dim(X) + \dim(L) > \dim(W) = \dim(X) + \dim(L')$. $\qquad\square$

Remark 5.1. Applying Wilking's transversal Jacobi equation [Wil07, Cor. 10], one can deduce in a similar way the following related statement. Let a compact group G act on a compact positively curved manifold M. If the quotient M/G^0 has no boundary then the action has trivial copolarity.

6. MAIN ARGUMENT

In this section we are going to prove the following result:

Proposition 6.1. *Let a non-discrete group G act with trivial copolarity on a Euclidean vector space V. Assume that G/G^0 acts on V/G^0 as a reflection group. Assume also that G acts irreducibly but that the action of G^0 is reducible. Then either the action of G^0 on V can be identified with the action of the maximal torus of $\mathbf{SU}(n)$ on $\mathbf{C}^n$, or G^0 is one of the groups $\mathbf{U}(2)$ or $\mathbf{U}(1) \cdot \mathbf{Sp}(2)$ and V is the double of the vector representation (on $\mathbf{C}^2$ or $\mathbf{H}^2$, respectively).*

6.1. **Basic lemma.** Since the representation has trivial copolarity, it has trivial principal isotropy group. Thus G/G_0 is generated by nice involutions in terms of Subsection 4.3. Since G^0 is normal in G, any element of G sends a G^0-irreducible subspace to another G^0-irreducible subspace. Thus we obtain an action of the finite group $\Gamma := G/G^0$ on the set of G^0-isotypical components and on the set of G^0-irreducible subspaces.

Recall that, by assumption, our group G has positive dimension. Since its representation on V has trivial copolarity, it is not polar.

The basic step is the following observation:

Lemma 6.2. *Under our general assumptions, let $U_{\pm 1}$ be G^0-invariant subspaces with $U_1 \cap U_{-1} = \{0\}$. Let $w \in G$ be a nice involution that satisfies $w(U_{-1}) = U_1$. Then the action of G^0 on $U_{\pm 1}$ is of cohomogeneity 1.*

Proof. Denote by F the subspace of fixed points of w in V. By Subsection 4.3 the subset $G \cdot F$ must be of codimension 1 in V. In particular, the subset $G^0 \cdot F_0$ must be of codimension 1 in $U_1 \oplus U_{-1}$, where F_0 is the space of all $(u + w(u))$ for $u \in U_1$. However, $G^0 \cdot F_0$ is an algebraic set that is contained in the subspace Δ of all $u + v \in U_1 \oplus U_{-1}$ with $|u| = |v|$. Hence, $G^0 \cdot F_0$ contains an open subset of Δ.

Thus, for some unit $u \in U_1$ and all v in an open subset of the unit sphere of U_{-1} there is some $u' \in U_1$ and $h \in G^0$ with $hu' = u$ and $hwu' = v$. In particular, we have $hwh^{-1}u = v$, thus $(whw)h^{-1}u = wv$. Hence, the orbit $G^0 u$ contains an open subset of the unit sphere in U_1. If $\dim U_1 \geq 2$, then G^0 acts transitively on the unit sphere in U_1. If $\dim(U_1) = 1$, the statement is clear anyway. $\qquad\square$

6.2. Isotypical components.
Recall that the action of G on V is irreducible. Hence the action of $\Gamma = G/G^0$ on the set of G^0-isotypical components is transitive. From this and Lemma 6.2 we derive:

Lemma 6.3. *Under the assumptions above, let $V = V_1 \oplus \cdots \oplus V_l$ be the decomposition of V into G^0-isotypical components, and assume that $l > 1$. Then the action of G^0 on each V_i is of cohomogeneity 1; in particular, each V_i is G^0-irreducible. The group Γ acts on the set $S = \{V_1, \ldots, V_l\}$ of G^0-isotypical components as the full permutation group.*

Proof. If there is a trivial G^0-isotypical component, then from the transitivity of the action of Γ, we deduce that all isotypical components are trivial. Since the action of G is effective, we would get that G is discrete, in contradiction to our assumption.

The group Γ is generated by nice involutions. Since the action of Γ on S is transitive, for any V_i, we find a nice involution $w \in G$ which moves V_i to some $V_j \neq V_i$. We use Lemma 6.2 to see that the action of G on V_i has cohomogeneity 1.

We claim that such w leaves all other V_k, $k \neq i$, j invariant. Otherwise, we could set $U_{-1} := V_i \oplus V_k$ in Lemma 6.2 and obtain that G^0 acts with cohomogeneity 1 on $V_i \oplus V_k$, which is impossible. Thus the action of the group Γ on the finite set S is generated by transpositions. Since it is also transitive, the image of Γ must be the full group of permutations of S. $\qquad\square$

In the same way we are going to deduce:

Lemma 6.4. *Assume that there is only one G^0-isotypical component. Then the action of G^0 on each G^0-irreducible subspace of V is of cohomogeneity 1. Moreover, there is a nice involution $w \in G$ and a decomposition of $V = V_1 \oplus \cdots \oplus V_l$ into G^0-irreducible subspaces V_i such that $w(V_1) = V_2$ and $w(V_i) = V_i$, for all $i \geq 3$.*

Proof. Take a G^0-irreducible subspace V_1. Since V_1 is not G-invariant, some element of Γ moves V_1 to another G^0-irreducible subspace. We find a nice involution $w \in G$ that does not leave V_1 invariant. We may apply Lemma 6.2 and deduce that the action of G^0 on V_1 and hence on any G^0-irreducible subspace has cohomogeneity 1. Set $V_2 = w(V_1)$ and choose pairwise orthogonal G^0-invariant subspaces $V_3, \ldots, V_l$ that are orthogonal to V_1, V_2 and satisfy $V = V_1 \oplus \cdots \oplus V_l$. By the same argument as in the proof of Lemma 6.3, we deduce that w must leave the V_i, $i \geq 3$, invariant. $\qquad\square$

6.3. **Hopf action and its brothers.** In this subsection we are going to analyze the case of one isotypical component. We start with the following simple observation:

Lemma 6.5. *Let a connected group K act effectively with cohomogeneity 1 on a vector space U. If the doubling representation of K on $U \oplus U$ has trivial copolarity then K is one of the classical groups $\mathbf{U}(1)$, $\mathbf{SU}(2) = \mathbf{Sp}(1)$, $\mathbf{U}(2)$, $\mathbf{Sp}(2)$, $\mathbf{U}(1) \cdot \mathbf{Sp}(2)$ with its vector representation on $U = \mathbf{C}^1$, $\mathbf{C}^2 = \mathbf{H}$, $\mathbf{C}^2$, $\mathbf{H}^2$, $\mathbf{H}^2$, respectively.*

Proof. The representations of cohomogeneity 1 are listed in Subsection 12.7. Going through the list, one observes that if K acts on $U \oplus U$ with trivial principal isotropy group then K is either one of the groups above, or $K = \mathbf{SO}(3)$, or $K = \mathbf{SU}(3)$.

However, the action of $\mathbf{SO}(3)$ (resp. $\mathbf{SU}(3)$) on $\mathbf{R}^3 \oplus \mathbf{R}^3$ (resp. $\mathbf{C}^3 \oplus \mathbf{C}^3$) is orbit equivalent to the action of $\mathbf{O}(3)$ (resp. $\mathbf{U}(3)$). Thus it has non-trivial copolarity. $\qquad\square$

Proposition 6.6. *Under the general assumptions of this section, assume that there is only one G^0-isotypical component in V. Then G^0 is one of the groups $\mathbf{U}(1)$, $\mathbf{U}(2)$, $\mathbf{U}(1) \cdot \mathbf{Sp}(2)$, $V = V_1 \oplus V_2$, and V_1, V_2 are the vector representations of G^0, i.e., $\mathbf{C}$, $\mathbf{C}^2$, $\mathbf{H}^2$, respectively.*

Proof. Consider an involution w in G and a decomposition $V = V_1 \oplus \cdots \oplus V_l$ into G^0-irreducible subspaces, as in Lemma 6.4. Since G acts effectively, the assumption that there is only one isotypical component implies that G^0 acts effectively on V_1. We identify G^0 with its image in $\mathbf{SO}(V_1)$ and recall that it acts on V_1 with cohomogeneity 1. Since there is only one isotypical component, we fix (G^0-equivariant) identifications of V_i with V_1 for all i.

We write elements in V as $(v_1, \ldots, v_l)$, for $v_i \in V_i = V_1$. Then the action of G^0 on V is given in these coordinates by $g \cdot (v_1, \ldots, v_l) = (g \cdot v_1, \ldots, g \cdot v_l)$. Moreover, by assumption, there are isometries $p_1, \ldots, p_l : V_1 \to V_1$ such that the action of w is given by

$$w(v_1, \ldots, v_l) = (p_2(v_2), p_1(v_1), p_3(v_3), \ldots, p_l(v_l)).$$

Since w is an involution, we have $p_2 = p_1^{-1}$ and $p_i^2 = 1$, for $i \geq 3$. We set $p = p_1$. For an element $g \in G^0$, the conjugation $g^w = w^{-1}gw$ acts as

$$g^w(v_1, \ldots, v_l) = (g^p(v_1), g^{p^{-1}}(v_2), g^{p_3}(v_3), \ldots, g^{p_l}(v_l)).$$

By assumption, w normalizes G^0. Thus we must have $g^p = g^{p^{-1}} = g^{p_i}$, for all $i \geq 3$. And this element g^p is contained in G^0. Denoting by N and C the normalizer and the centralizer of G^0 in $\mathbf{O}(V_1)$, we deduce: $p, p_i \in N$ and $p^2, pp_i^{-1} \in C$ for $i \geq 3$.

Let F be the subspace of fixed points of w in V. It consists of all elements of the form

$$(v_1, p(v_1), f_3, \ldots, f_l),$$

where $v_1 \in V_1$ is arbitrary and $f_i \in V_1$ is fixed by p_i. Since w is a nice involution, the set $\mathcal{S} = G^0 \cdot F$ is of codimension 1 in V.

First, assume $l \geq 3$. Then there is some $c \in C$ with $p = c \cdot p_3$. Thus any element $(u_1, \ldots, u_l) \in \mathcal{S}$ has the first three coordinates given by $(u_1, u_2, u_3) = (g(v), g(c \cdot p_3(v)), g(f_3))$, for some $g \in G^0$, some $v \in V_1$ and some $f_3 \in V_1$ fixed by the involution p_3.

Hence we have $|u_1| = |u_2|$ and $(u_2 - c(u_1)) \perp c(u_3)$. The assumption $\dim \mathcal{S} = \dim V - 1$ implies $u_2 = c(u_1)$, and, therefore, $\dim(V_1) = 1$. Then G is discrete in contradiction to our assumption. We deduce $l \leq 2$, so $l = 2$.

Using Lemma 6.5 and our assumption that G acts with trivial copolarity, we deduce that if the lemma does not hold, then $G^0 = \mathbf{Sp}(m)$ for $m = 1$ or $m = 2$ and $V_1 = \mathbf{H}^m$. Thus we only need to exclude these two cases.

Under the assumption $l = 2$, we have
$$\mathcal{S} = G^0 \cdot F = \{(v, p^g(v)) \mid g \in G^0,\ v \in V_1\}.$$
We have
$$\langle p^g(v), v \rangle = \langle (p^g)^2(v), p^g(v) \rangle = \langle (p^2)^g(v), p^g(v) \rangle = \langle p^g(v), p^2(v) \rangle,$$
since $p^2 \in C$. Thus all elements (u_1, u_2) in $\mathcal{S}$ satisfy $|u_1| = |u_2|$ and $u_2 \perp (u_1 - p^2(u_1))$. Since $\mathcal{S}$ has codimension 1, we deduce that p^2 must be the identity.

Let now c be an arbitrary element in C. Then $cp \in N$ and $(cp)^2 = c(pcp^{-1})p^2 \in C$. The same calculation as above reveals, $(cp)^g(v) \perp (v - (cp)^2(v))$, hence $p^g(v) \perp c^{-1}(v - (cp)^2(v))$. Again the assumption that $\mathcal{S}$ has codimension 1 implies that $(cp)^2$ is the identity. Thus we have shown that for all $c \in C$ the equality $(cp)^2 = 1$ must hold.

If now $G^0 = \mathbf{Sp}(m)$ then its normalizer is $N = \mathbf{Sp}(m) \cdot \mathbf{Sp}(1)$ and its centralizer is $C = \mathbf{Sp}(1)$. If the involution $p \in N$ is given by $\bar{A} \cdot \bar{p}$, for $\bar{A} \in \mathbf{Sp}(m)$, $\bar{p} \in \mathbf{Sp}(1)$, then we have $\bar{A}^2 = \pm 1$. From above we deduce that for all $c \in \mathbf{Sp}(1)$ we must have $(c \cdot \bar{p})^2 = \pm 1$. But this is impossible. $\qquad\square$

Remark 6.1. Unfortunately, by this type of arguments it is impossible to exclude the remaining cases $G^0 = \mathbf{U}(2)$, $G^0 = \mathbf{U}(1) \cdot \mathbf{Sp}(2)$. In fact, one can find an extension of the action of such G^0 on $V_1 \oplus V_1$ by an involution w such that w acts as a reflection on $(V_1 \oplus V_1)/G^0$.

We finish this subsection by noting that the Hopf action of $\mathbf{U}(1)$ on $\mathbf{C}^2$ is equivalent to the action of the unit torus of $\mathbf{SU}(2)$ on $\mathbf{C}^2$. Thus we have finished the proof of Proposition 6.1, under the additional assumption that the action of G^0 has only one isotypical component.

6.4. **Generalized toric actions I.** Now we proceed with the examination of the case of several isotypical components. In this and in the next subsection we will work under the general assumptions of the present section, and assume, in addition, that the action of G^0 on V has several isotypical components. We will denote by $V = V_1 \oplus \cdots \oplus V_l$ the unique decomposition into isotypical components. Due to Lemma 6.3, the group G^0 acts on all V_i irreducibly and with cohomogeneity 1. Moreover, the group $\Gamma = G/G^0$ acts on $S = \{V_1, \ldots, V_l\}$ as the full group of permutations.

Proposition 6.7. *Assume, in addition, that G^0 is a k-dimensional torus T. Then the action of G^0 on V can be identified with the action of the maximal torus of the special unitary group $\mathbf{SU}(k+1)$ on $\mathbf{C}^{k+1}$.*

Proof. Since all irreducible non-trivial representations of T have dimension 2, all V_i are two-dimensional. Thus each V_i is given by a character $\phi_i : T \to S^1$ which is defined up to conjugation, i.e., it is given by a (integral) linear functional $d\phi_i$ on the Lie algebra $\mathfrak{t}$ of T, which is defined up to sign. Since each irreducible summand of V is an isotypical component, each $d\phi_i$ occurs exactly once, and $\Gamma = G/G^0$ acts as the full permutation group on the set of all $d\phi_i$.

We fix a Γ-invariant flat metric on T. Then the $d\phi_i$ have all the same length and we can identify each $d\phi_i$ with the line t_i which is orthogonal to the kernel of $d\phi_i$ in $\mathfrak{t}$. The image Q of Γ in the orthogonal group $\mathbf{O}(k) = \mathbf{O}(\mathfrak{t})$ permutes the l elements t_i of the projective space $\mathbf{R}P^{k-1} = P(\mathfrak{t})$.

Since the action of T is effective, the lines t_i span $\mathfrak{t}$, thus $l \geq k$. If $l = k$, then T is a maximal torus of $\mathbf{O}(V)$ and the action of T on V is polar (hence of non-trivial copolarity). Thus $l \geq k + 1$. As we have seen, Q permutes the finite set S of all t_i and acts as the

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full permutation group of this set S. Thus the distance between each pair of different t_i, t_j in the projective space does not depend on i and j. Therefore the lines t_i are *equiangular* in the Euclidean space $\mathfrak{t}$. In general, the sets of equiangular lines can be quite large and difficult to understand (cf. [LS66]). However, in the presence of the large group Q, they can be described quite easily:

Lemma 6.8. *Let $\mathfrak{t}$ be a k-dimensional Euclidean vector space. Let S be a set of lines (1-dimensional subspaces) in $\mathfrak{t}$ that consists of $l > k$ elements and generates $\mathfrak{t}$. Let Q be a finite subgroup of $\mathbf{O}(\mathfrak{t})$ that leaves S invariant and acts on S as the full group of permutations of S. Then $l = k + 1$ and the lines of S are generated by the vertices of a regular simplex centered at the origin in $\mathfrak{t}$.*

Proof. We proceed by induction on k. For $k = 2$ the claim is easily verified (for $k = 1$, such sets of lines do not exist). Assume the statement is already shown for the dimension $k - 1$, for some $k \geq 3$.

Let $\alpha \leq \pi/2$ be the angle between any pair of lines from S, which by assumption does not depend on the pair. If $\alpha = \pi/2$, all lines are pairwise orthogonal, thus the number of lines is bounded from above by k, in contradiction to our assumption. If 3 lines t_1, t_2, $t_3 \in S$ lie in a plane, we must have $\alpha = \pi/3$. Choose another line t_4. We can choose unit vectors X_i on t_i such that the spherical distances (i.e., the Euclidean angles) satisfy $d(X_1, X_2) = d(X_2, X_3) = d(X_2, X_4) = \pi/3$. Since X_4 does not coincide with $\pm X_1$ or $\pm X_3$, its distances to X_1 and X_3 are less than $2\pi/3$. Thus they must be equal to $\pi/3$. But such quadruple of points does not exist in the unit sphere S^2.

Thus $\alpha < \pi/2$ and any three lines in S generate a 3-dimensional space. Consider the subgroup Q_1 of Q that leaves t_1 invariant. Then it acts on $\mathfrak{t}_1$, the orthogonal complement of t_1 and it preserves the set S_1 of lines in $\mathfrak{t}_1$ that are projections of lines in S different from t_1. Since no three lines lie in a plane, projections of different lines are different, thus S_1 has $k - 1$ elements. By assumption, Q_1 acts as the full permutation group of S_1. By our inductive assumption, $l - 1 = (k - 1) + 1$. Hence $l = k + 1$. Moreover, by induction, the lines in S_1 are given by the vertices $Y_2, \ldots, Y_l$ of a regular simplex in $\mathfrak{t}_1$.

Fix a unit vector X_1 on t_1. Choose the unit vector $\bar{X}_i$ on $t_i, i = 2, \ldots, k + 1$, with $d(X_1, \bar{X}_i) = \alpha$. Then $\bar{X}_i$ lies on the spherical geodesic from X_1 to Y_i. Moreover, by our assumption, for all $i \neq j$, we have either $d(\bar{X}_i, \bar{X}_j) = \alpha$ or $d(\bar{X}_i, \bar{X}_j) = \pi - \alpha$. However, the angle between X_1Y_i and X_1Y_j is larger than $\pi/2$. Thus the triangle $X_1\bar{X}_i\bar{X}_j$ cannot be equilateral. Hence $d(\bar{X}_i, \bar{X}_j) = \pi - \alpha$ for all $i \neq j$. Therefore if we set $X_i = -\bar{X}_i$ for $i \geq 2$, the $(k + 1)$ unit vectors X_i have the same pairwise distances, given by $\pi - \alpha$. Thus X_i are the vertices of a regular simplex. $\square$

Now we can finish the proof of Proposition 6.7. We deduce form the last lemma that $l = k + 1$ and that, up to an eventual change of ϕ_i by its conjugate (that does not effect the representation of the torus T), the homomorphisms $\phi_i : T \to S^1$ have differentials $d\phi_i$ that are the vertices of a regular simplex. Hence these differentials satisfy $d\phi_1 + d\phi_2 + \ldots + d\phi_{k+1} = 0$. But this is exactly the defining equation of the maximal torus of $\mathbf{SU}(k + 1)$. $\square$

6.5. **Generalized toric actions II.** In this subsection we are going to finish the analysis of the situation where the action of G^0 on V has several isotypical components $V = V_1 \oplus \cdots \oplus V_l$. Here we assume that G^0 is not commutative. Since all isotypical components are permuted by $\Gamma = G/G^0$, the images K_i of G^0 in the isometry group of the isotypical components V_i are congruent inside $\mathbf{O}(V)$.

Denote by K' the image of G^0 in $\mathbf{O}(V_1)$. Since G^0 is contained in a product of groups isomorphic to K', the group K' is not commutative. Going through the list of groups K' acting with cohomogeneity one on a vector space, we get three cases that we will analyze separately: K' is covered by a simple group K; K' is covered by a group K with two different factors; K' is covered by $K = \mathbf{Sp}(1) \times \mathbf{Sp}(1)$.

Case 1. Let us assume that K is a simple group. Then G^0 is a finite quotient of a product K^m. Moreover, for any component V_i, exactly one of the m factors is mapped not to the identity in $\mathbf{SO}(V_i)$. If $K \neq \mathbf{Spin}(8)$ then there is exactly one irreducible representation of K with cohomogeneity 1 on a vector space of the fixed dimension $\dim V_1 = \dim V_2 = \cdots = \dim V_l$. Thus, in this case, $m = l$ and for each $j = 1, \ldots, m$ there is exactly one V_i such that the j-th factor K is mapped non-trivially into $\mathbf{SO}(V_i)$. We deduce that the action of G^0 on V is an l-fold product of the actions of K on V_1. However, this action is polar. This contradicts our assumption.

If $K = \mathbf{Spin}(8)$ then K has 3 representations ρ_1, ρ_2, ρ_3 of cohomogeneity 1 on $\mathbf{R}^8$ that are permuted by the triality automorphism. Then the only part of the representation of G^0 on which a fixed factor K may act non-trivially is a sum of different ρ_j, $j = 1$, 2, 3, i.e., a vector space of dimension at most 24. However, $\dim \mathbf{Spin}(8) = 28$. Thus the action of K on this subspace has non-trivial principal isotropy groups. Then the action of G^0 on V has non-trivial principal isotropy groups, as well. Thus it has non-trivial copolarity in contradiction to our assumption.

Case 2. Now, we assume that the image K' is covered by a product K of two different factors (i.e., $\mathbf{U}(1) \times \mathbf{SU}(n)$, $\mathbf{U}(1) \times \mathbf{Sp}(n)$, $\mathbf{Sp}(1) \times \mathbf{Sp}(n)$, for $n \geq 2$). In all cases, the action of one factor $\hat{K}$ of K is still of cohomogeneity 1. Consider the connected normal subgroup N of G^0 whose Lie algebra is the sum of all factors isomorphic to $\hat{\mathfrak{k}}$, the corresponding Lie algebra. Since the factors of K are different, N is normalized by G.

Consider the decomposition $V = W_1 \oplus \cdots \oplus W_r$ into N-isotypical components. Since G^0 acts on N via inner automorphisms, any element of G^0 preserves each summand W_i. Any element $g \in G$ permutes the N-isotypical components. If $r > 1$ then, arguing as in Lemma 6.3, we deduce that the action of G^0 on each W_i has cohomogeneity 1, hence each W_i coincides with some V_j. As in the previous case, the action of N on V is a direct product of cohomogeneity 1 actions of N on V_i. We deduce that the action of G^0 is orbit equivalent to the action of N on V and that both actions are polar. Contradiction.

Therefore we may assume that $r = 1$. Then N has only one factor, i.e., N is locally isomorphic to $\hat{K}$, which is either $\mathbf{SU}(n)$ or $\mathbf{Sp}(n)$. The representation of N on V is given by the l-fold sum of the vector representation V_1 of $\hat{K}$. Any nice involution $w \in G$ normalizes N. Arguing as in Subsection 6.3, we get a presentation of w as $w(v_1, \ldots, v_l) = (p^{-1}(v_2), p(v_1), p_3(v_3), \ldots, p_l(v_l))$, for some involutions $p_i \in \mathbf{O}(V_1)$ and some isometry $p \in \mathbf{O}(V_1)$ such that they all normalize $\hat{K}$, and such that p^2, pp_i^{-1} lie in the centralizer of $\hat{K}$ inside $\mathbf{O}(V_1)$, for $i \geq 3$.

Assume $l \geq 3$. The set F of fixed points of w is given by all $(v, p(v), f_3, \ldots, f_l)$ with $v \in V_1$ and the f_i fixed by p_i. The extension of the action of N to the action of G^0 is given by some (complex, respectively, quaternionic) scalar multiplications in each component. Note that p and p_i induce the same action on the projective spaces $\mathbf{C}P^{n-1}$ and $\mathbf{H}P^{n-1}$, respectively. Thus the set $\mathcal{S} = G^0 \cdot F$ is contained in the set of all points with the first three coordinates (u_1, u_2, u_3) given by $(g(v)\lambda_1, g(p_3(v))\lambda_2, g(f_3)\lambda_3)$ for some scalars λ_i, some element $g \in \hat{K}$, some $v \in V_1$ and some f_3 in V_1 fixed by p_3.

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We have $|u_1| = |u_2|$. Moreover, the "lines" $[u_1]$, $[u_2]$, $[u_3]$ (i.e. the elements of the corresponding projective space) are given by $g([v])$, $g(p_3[v])$, $g([f_3])$, where $[f_3]$ is fixed by the involution p_3 in the projective space. Thus $d([u_1], [u_3]) = d([u_2], [u_3])$ in the projective space $P(V_1)$ (over $\mathbf{C}$ and $\mathbf{H}$, respectively). Hence the set $\mathcal{S}$ has codimension at least 2 in V, and w cannot act as a reflection. This provides a contradiction in the case $l > 2$.

We deduce $l = 2$. Since the action of G^0 has trivial copolarity, the action of G^0 and therefore of N on $V = V_1 \oplus V_2$ has trivial principal isotropy groups. From the classification of cohomogeneity 1 actions, we deduce that $\hat{K} = \mathbf{SU}(n)$, with $n = 2$ or 3, or $\hat{K} = \mathbf{Sp}(2)$.

If $\dim(G^0/N) \geq 3$ then $\dim(V) - \dim(G^0) \leq 3$ in all cases. Thus, due to [Str94], the action of G^0 cannot have trivial copolarity. Hence $\dim(G^0/N) \leq 2$. If $\dim(G^0/N) = 1$, then, using the fact that the representations V_1 and V_2 of G^0 are congruent inside G, we see that both representations are equivalent. Hence there is only one G^0-isotypical component, in contradiction to our assumption. Therefore G^0/N is the two-dimensional torus T^2. If $\hat{K} = \mathbf{SU}(n)$ we get $\dim(V) - \dim(G^0) \leq 3$ again, in contradiction to the trivial copolarity assumption. Thus G^0 must be finitely covered by $\mathbf{Sp}(2) \times \mathbf{U}(1)^2$. Now $V_1 = V_2 = \mathbf{H}^2$ and we use the canonical real basis $\{1, i, j, k\}$ of $\mathbf{H}$. Up to orbit-equivalence, we may assume that each $\mathbf{U}(1)$-factor acts on the corresponding summand $\mathbf{H}^2$ by right multiplication by matrices of the form $e^{i\theta}$ id. Now $\sigma(v_1, v_2) = (-jv_1 j, -jv_2 j)$, for $(v_1, v_2) \in V_1 \oplus V_2$, is an involution that preserves each G^0-orbit. It follows that the action of G^0 on V has non-trivial copolarity in contradiction to our assumption.

Case 3. Now assume that the image is covered by a product K of equal factors. Then, K is $\mathbf{Sp}(1) \times \mathbf{Sp}(1)$, $\dim V_i = 4$ for all i, and G^0 is mapped onto $\mathbf{SO}(4)$ in this case. Thus G^0 is covered by a product $\mathbf{Sp}(1)^m$ for some m.

Denote as above by Γ the group G/G^0. This group permutes the m factors of (the universal covering of) G^0, hence is mapped onto a subgroup Γ' of the symmetric group S_m. Note that on each V_i exactly 2 factors of G^0 act non-trivially. Thus the isotypical components can be indexed by a subset A of the set of unordered pairs of different factors. We will denote such unordered pairs by $i \otimes j$, $1 \leq i, j \leq m$. The action of Γ on the set of G^0-components factors through Γ'. Moreover, the action of Γ' on A is induced by the canonical action of S_m on the set of unordered pairs. Due to Lemma 6.3, the image of Γ' is the whole permutation group of A.

The cardinality of A is ℓ. Since Γ' acts as the full permutation group of A, we have $\ell \leq m$. For any $i = 1, \ldots, m$, we set $A_i = \{j \mid i \otimes j \in A\}$ and denote by $a(i)$ the cardinality of A_i. If $a(i) \geq 2$, for all i, then A has at least m elements, so $\ell = m$. It follows that $\Gamma' = S_m$ and hence A has $m(m-1)/2 = m$ elements. Hence $m = 3$. Then $\dim(V) = 12$ and $\dim(G^0) = 9$. Thus the action of G^0 on V cannot have trivial copolarity by [Str94].

Hence there exists some i with $a(i) = 1$. If $a(i) = 1$, for all i, then the action of G^0 is the direct product of the cohomogeneity 1 representations of G^0 on V_i. Thus the representation is polar in contradiction to our assumption.

Thus, there are some i with $a(i) = 1$ and some j with $a(j) > 1$. Note that the number $a(i)$ is constant on orbits of Γ'. Since Γ' acts transitively on A, for all pairs $i \otimes j \in A$, the unordered pair $a(i) \otimes a(j)$ does not depend on $i \otimes j$. Hence there are exactly two orbits B and C of Γ' in $\{1, \ldots, m\}$, one of them, say B, consisting of all i with $a(i) = 1$, and the other one C consisting of all elements j with $a(j) = a > 1$.

Assume that C has at least two elements j_1, j_2. Choose distinct elements $i_\pm \in B$ such that $j_1 \otimes i_\pm \in A$. Then there is no element in Γ' that leaves $j_1 \otimes i_+$ invariant and moves

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$j_1 \otimes i_-$ to some pair $j_2 \otimes i$. Contradiction to the fact that Γ' acts as the full permutation group on A.

Thus, C has only one element that we may assume to be $\{m\}$. Then B has $m-1$ elements, and the action of the first $(m-1)$ factors on V is a (polar) product action. One sees that the action of G^0 is in this case polar as well (and has non-trivial principal isotropy group).

This finishes the proof in the case the image of G^0 in V_1 is covered by a product of equal factors.

Taking all pieces together, we have completed the proof of Proposition 6.1.

7. An example

Before we finish the proof of Theorem 1.2, we need to collect a few observations about the geometry of the action of a maximal torus T of $\mathbf{SU}(k+1)$ on $V = \mathbf{C}^{k+1}$.

Lemma 7.1. *The quotient V/T has no boundary.*

Proof. No circle inside T fixes a subset F of complex codimension 1. Since T is commutative we derive the result from the dimension formula (Lemma 4.1). $\square$

In particular, the action of any finite extension of T has trivial abstract copolarity, due to Proposition 5.2.

Note that the normalizer $N = N(T)$ of T in $\mathbf{O}(2k+2)$ normalizes the centralizer $\bar{T}$ of T in $\mathbf{O}(2k+2)$. Assume $k \geq 2$. Since $\bar{T}$ is a maximal torus of $\mathbf{U}(k+1)$ which is also a maximal torus of $\mathbf{O}(2k+2)$, we see that $\bar{T}$ is the identity component of the normalizer N. Moreover, N is generated by $\bar{T}$, the complex conjugation c and the symmetric group S_{k+1} of permutations of complex coordinates. On the other hand, the case $k = 1$ is discussed in [Str94, p.14].

Lemma 7.2. *Assume $k \geq 1$. Then the action of the normalizer $N(T) = N$ of T in $\mathbf{O}(2k+2)$ on the quotient V/T induces an isomorphism of N/T with the isometry group $Iso(V/T)$ of V/T.*

Proof. For $k = 1$, the result is contained in [Str94, p.14]. Thus we will assume $k \geq 2$.

If an element of $N \setminus T$ acts trivially on V/T we obtain an action by a group T' larger than T that is orbit equivalent to the action of T. This would imply that the action of T' had non-trivial principal isotropy group. Hence the action of T' (and thus of T) had non-trivial copolarity. Contradiction. Thus the map from N/T to $Iso(V/T)$ is injective.

To prove the surjectivity, take an isometry I of V/T to itself. Restrict it to the unit sphere $X = S(V)/T$. There are $k + 1$ irreducible subspaces of V, each of them defines a unique point p_i in X, for $i = 1, ..., k + 1$. Due to Lemma 5.1, these points are permuted by I. Composing I with an element of N that permutes the complex coordinates backwards, we may assume that I fixes the points p_i.

Let N' denote the subgroup of the isometry group of V/T that fixes all the points p_i. The intersection of N' with the image of N is the group $\mathbf{O}(2)$ generated by the image of $\bar{T}$ and the complex conjugation. We have to prove that $N' = \mathbf{O}(2)$. In order to do so, consider the map $F = (d_{p_1}, d_{p_2}, \ldots, d_{p_{k+1}}) : X \to \mathbf{R}^{k+1}$ whose coordinates are distance functions to the points p_i. By construction, the function F is N'-invariant. We claim that the fibers of F are the orbits of $\bar{T}$. Namely, due to the N'-invariance, the orbits of $\bar{T}$ are contained in the fibers of F. On the other hand, $Y = X/\bar{T} = S(V)/\bar{T}$ is the rectangular spherical simplex

with vertices given by the points p_i. And the function F that descends to Y separates the points of Y.

Thus the regular fibers of F are circles. The group N' acts effectively on X having circles as regular orbits. This implies that N' can be only $\mathbf{U}(1)$ or $\mathbf{O}(2)$. Since its intersection with N is already $\mathbf{O}(2)$, the whole group N' must coincide with this intersection. $\square$

As a consequence we deduce:

Corollary 7.3. *Let T_1, T_2 be finite extensions of T in $\mathbf{O}(2k+2)$. If V/T_1 and V/T_2 are isometric, then T_1 and T_2 are conjugate inside $N(T)$.*

Proof. Consider an isometry $J : V/T_1 \to V/T_2$. Since V/T does not have boundary, $(V/T)_{orb}$ is the universal orbifold covering of $(V/T_i)_{orb}$ (Lemma 3.5). Then the isometry J is induced by an isometry $I : V/T \to V/T$, and we can use the preceding lemma to lift I to an element n of $N(T)$. Now nT_1n^{-1}/T and T_2/T have the same orbits in V/T implying that nT_1n^{-1} and T_2 have the same orbits in V. Since they both act with trivial copolarity it follows that they are equal. $\square$

Lemma 7.4. *Assume $k \geq 2$. Let T^+ be a finite extension of T such that T^+/T acts on V/T as a reflection group and such that T^+ acts on V irreducibly. Then there are two codimension 1 strata in V/T^+ that meet at an angle not equal to $\pi/2$.*

Proof. We have seen in Lemma 6.3 that T^+/T acts as the full permutation group on the components of V. Thus the finite reflection group T^+/T, which is the orbifold fundamental group of $(V/T^+)_{orb}$ surjects onto the non-Abelian symmetric group S_{k+1}. However, if all strata of codimension 1 would only meet at right angles, the orbifold fundamental group of $(V/T^+)_{orb}$ would be Abelian, cf. the end of Subsection 3.2. (More directly, it is not difficult to observe that two reflections w_1, w_2 corresponding to non-commuting transpositions in S_{k+1} define strata of codimension 1 that meet at an angle equal to $\pi/3$.) $\square$

From this we deduce our final piece:

Corollary 7.5. *Assume that $k \geq 1$. Assume that a representation $\rho : H \to \mathbf{O}(W)$ is quotient equivalent to a finite extension T_1 of T in $\mathbf{O}(2k+2)$. If the action of H^0 on W is irreducible, then the principal isotropy group of H is non-trivial.*

Proof. The representation ρ is irreducible. For $k = 1$, it has cohomogeneity 3 and thus it is listed in group III, Table II in [Str94]. Those have non-trivial principal isotropy groups.

Assume $k \geq 2$. Due to Proposition 3.1 and Lemma 7.1, we find a subgroup T^+ of T_1 of finite index in T_1 such that the representation of H^0 on W and of T^+ on V are quotient equivalent. Since H^0 is connected, we see that T^+/T acts on V/T as a reflection group (Lemma 3.5 and Lemma 3.3). Since H^0 acts irreducibly, so does T^+ (Lemma 5.1). Applying the previous lemma and Lemma 4.2, we deduce that H^0 (and therefore H) acts with non-trivial principal isotropy group. $\square$

Remark 7.1. We would like to mention that Lemma 7.4 does not hold for $k = 1$, see [Str94, Table II].

8. Conclusion

Now we are going to finish the proof of Theorem 1.2.

Consider the action of H' on W' and take its reduction to a minimal generalized section V. Let $\tau : G \to \mathbf{O}(V)$ be this reduction. By construction it has trivial copolarity and is in the same quotient equivalence class as $\rho : H \to \mathbf{O}(W)$. Since V is also a generalized section of $(H')^0$, we see that the action of G^0 on V is reducible.

Due to Proposition 3.2, we find subgroups of finite index H^+ in H and G^+ in G such that the corresponding restricted representations are still quotient equivalent and such that G^+/G^0 acts on V/G^0 as a reflection group. Since H^+ contains H^0 it acts irreducibly and so does G^+, due to Lemma 5.1. In particular, G^+/G^0 is a non-trivial group. Now, if G is non-discrete, we are in the situation of Proposition 6.1. We deduce that either the action of G^0 on V is as required, or that G^0 is one of the groups $\mathbf{U}(2)$ or $\mathbf{U}(1) \cdot \mathbf{Sp}(2)$ and V is the double of the vector representation (on $\mathbf{C}^2$ and $\mathbf{H}^2$, respectively).

Assume that $G^0 = \mathbf{U}(2)$. Then the action of G, hence that of H has cohomogeneity 4. We use the explicit classification of all irreducible representations of cohomogeneity 4 of connected groups (Theorem 1.4 that is proven independently in the second part of the paper). If H^0 is $\mathbf{SO}(3)$ or $\mathbf{U}(2)$ (the first two lines in the table), then W/H^0 has no boundary. Due to Proposition 3.1, we find a subgroup H_1 of finite index in H such that W/H_1 and V/G^0 are isometric. But H_1 acts irreducibly and G^0 acts reducibly, in contradiction to Lemma 5.1. On the other hand, if H^0 has non-trivial copolarity, then it has copolarity 2, according to Theorem 1.4. Since the copolarities of H and H^0 coincide, we find a reduction of H to a representation of a group K on U with $K^0 = T^2$. The action of K has trivial copolarity and the representation of K^0 is reducible. Thus applying the arguments we have applied to G (Propositions 3.2 and 6.1), we see that K^0 is the maximal torus of $\mathbf{SU}(3)$ and $U = \mathbf{C}^3$. Since U/K^0 has no boundary (Lemma 7.1) we use Proposition 3.1, to find a finite index subgroup K_1 of K such that U/K_1 and V/G^0 are isometric. But the action of G^0 on V has infinitely many invariant subspaces (it is reducible with one isotypical component) and the action of K_1 on U only finitely many. This contradicts Lemma 5.1. Hence the case $G^0 = \mathbf{U}(2)$ cannot occur.

Assume now that $G^0 = \mathbf{U}(1) \cdot \mathbf{Sp}(2)$. Then the action of G^0 has cohomogeneity 5, hence so does the action of H. We use again the explicit classification of all irreducible representations of cohomogeneity 5 of connected groups (Theorem 1.4 that is proven independently in the second part of the paper). If H^0 is $\mathbf{SU}(2)$, then W/H^0 has no boundary and we get a contradiction as above (Proposition 3.1 and Lemma 5.1). On the other hand, if the copolarity of H^0, hence of H is 3, then by the direct computations in the second part, the reduction to the minimal generalized section has a torus T^3 as identity component. Then arguing as in the case of cohomogeneity 4 we obtain a contradiction. It remains to analyze the cases $H^0 = \mathbf{SO}(3) \times \mathbf{U}(2)$ and $H^0 = \mathbf{U}(3) \times \mathbf{Sp}(2)$. Since the latter action has a reduction to the former one in terms of a generalized section, we may assume that $H^0 = \mathbf{SO}(3) \times \mathbf{U}(2)$.

To exclude this remaining case, we observe that $S(V)/G^0$ is a Riemannian orbifold. In fact it is easy to determine the equivalence classes of the slice representations at singular points. There are only three possibilities, in which the nontrivial components of the slice representation are respectively orbit equivalent to $(\mathbf{U}(1), \mathbf{C}) \oplus (\mathbf{Sp}(1), \mathbf{H})$, $(\mathbf{U}(1), \mathbf{C})$ and $(\mathbf{Sp}(1), \mathbf{H})$. Since these are polar representations, the claim follows from the main result of [LT10].

On the other hand, the slice representation of the identity component of the isotropy group of H^0 at a vector $v_1 \otimes v_2$ is given by T^2 acting on $\mathbf{C} \oplus \mathbf{C} \oplus \mathbf{C}$ with weights $(1,0)$, $(1,1)$, $(1,-1)$, hence it is non-polar. Thus $S(W)/H$ has non-orbifold points [LT10].

This proves that G^0 is the maximal torus of $\mathbf{SU}(k+1)$ and $V = \mathbf{C}^{k+1}$.

Remark 8.1. A more conceptual proof that the two special cases cannot occur can be obtained as follows. One shows that both quotient spaces $S(V)/G^0$ are orbifolds of constant curvature, i.e., finite quotients of some round spheres. Hence the same is true for $S(V)/G = S(W)/H$. One invokes a recent theorem of Wiesendorf [Wie11] saying that in this case all H-orbits are taut submanifolds of W. Now one uses the main result of [GT03] that states that the action of H on W must be of cohomogeneity 3.

Assume now that there is another representation $\rho_1 : G_1 \to \mathbf{O}(V_1)$ in the same quotient equivalence class that has trivial copolarity. Due to Corollary 7.5, the representation of G_1^0 cannot be irreducible. Since it is reducible, we may apply the same arguments we have applied to G, to deduce that the restriction of ρ_1 to G_1^0 is given by the action of a maximal torus of $\mathbf{SU}(k'+1)$ on $\mathbf{C}^{k'+1}$. Since the actions have the same cohomogeneity, $k = k'$ and the representation spaces of G_1^0 and of G^0 can be identified. To prove that ρ and ρ_1 are the same, we only need to invoke Corollary 7.3.

9. NEW SETTING

In the next two sections we are going to prove Theorem 1.1. Thus let here and in the sequel $\rho : H \to \mathbf{O}(W)$ be a non-reduced non-polar irreducible representation of a connected compact Lie group. Let $\tau : G \to \mathbf{O}(V)$ be the minimal reduction. Note that if the copolarity k of ρ as in the assumption of the theorem is at most 6, then the abstract copolarity $\dim(G)$ is not larger than 6 as well. If the action of G^0 is reducible, then we can apply Theorem 1.2 to deduce that the copolarity and the abstract copolarity coincide and that the cohomogeneity is equal to $k + 2$.

Thus Theorem 1.1 is a consequence of the following result that will be proved in the next two sections.

Proposition 9.1. *Assume that $\rho : H \to \mathbf{O}(W)$ and $\tau : G \to \mathbf{O}(V)$ are quotient equivalent. Assume that H is connected, τ has trivial copolarity $k = \dim(G) \leq 6$ and that $\dim(H) > \dim(G)$. Then the representation of G^0 is reducible.*

Form now on assume, to the contrary, that the action of G^0 is irreducible. By assumption, the action of G is effective, hence the Abelian summand of the Lie algebra of G^0 is at most one-dimensional. Since $\dim(G) \leq 6$, the group G^0 is locally isomorphic to $\mathbf{Sp}(1)$, or $\mathbf{U}(2)$, or $\mathbf{Sp}(1) \times \mathbf{Sp}(1)$, respectively.

We will distinguish two cases depending on whether G is connected or not. In the first case $V/G = V/G^0$ has boundary. We will exclude this case in the next section using a bit of representation theory. If $G \neq G^0$ then there is a nice involution $w \in G \backslash G^0$ (Proposition 3.2). Using the dimension formula for the set F of its fixed points, we will obtain a contradiction in all but two cases that will be excluded by the general classification in the second part of the paper.

10. CONNECTED CASE

10.1. **Basics.** In addition to our assumption from the previous section, we assume here that G is connected. Then there is some G-important point p such that G_p is either $\mathbf{U}(1)$ or $\mathbf{SU}(2)$. In all cases ($G = G^0$ covered by $\mathbf{SU}(2)$ or $\mathbf{SU}(2) \times \mathbf{U}(1)$ or $\mathbf{SU}(2) \times \mathbf{SU}(2)$) any $\mathbf{SU}(2)$-subgroup contained in G has a unique involution that is central in G. Since such a

central involution cannot have fixed points (our representation is irreducible!), we cannot have $G_p \cong \mathbf{SU}(2)$.

We fix an important point p whose stabilizer G_p is some $\mathbf{U}(1)$. Let F be the set of fixed points of G_p and let f denote its dimension. Then we have (Lemma 4.1) the dimension formula $f = \dim(V) - \dim(G) + \dim(N(G_p)) - 2$.

10.2. A note on weight spaces.

To obtain an upper bound for f we make the following general observation that allows us to estimate F using the weights of the representation.

Let τ be a representation of a connected compact group K on a complex vector space U. Let L be a subgroup isomorphic to $\mathbf{U}(1)$. Choose a maximal torus T containing L. Let F be the set of fixed points of L. Then F is T-invariant, hence it is a sum of weight spaces. Note that all these weights vanish on L, hence the weight spaces appearing in F are associated to weights contained in a hyperplane in the dual space of the Lie algebra $\mathfrak{t}$ of T.

If U is a real vector space, then one can consider its complexification $U^c = U \oplus iU$, and obtains that the set $F^c \subset U^c$ is a sum of weight spaces whose corresponding weights are contained in a hyperplane of the dual space of $\mathfrak{t}$.

If the rank of K is 1, then the only hyperplane in the dual of the Lie algebra of T is $\{0\}$, hence F (respectively F^c) is contained in the 0-weight space.

Assume now that K has rank 2. Then all hyperplanes are one-dimensional. Thus all weight spaces appearing in F must be linearly dependent. (In fact, they are all multiples of the weight given by $T \to T/L \cong \mathbf{U}(1)$.)

10.3. G is covered by $\mathbf{SU}(2)$.

Here we assume that G has rank 1. Then all irreducible representations of G are given by the quaternionic representations ρ_{2n} on the complex space $\mathbf{C}^{2n}$ of homogeneous polynomials of degree $2n - 1$ in two complex variables and by the real representations ρ_{2n+1} on $\mathbf{R}^{2n+1}$.

From the previous subsection we see that f is bounded from above by the real dimension of the 0-weight space in the quaternionic case and by the complex dimension of the 0-weight space in the real case.

In the quaternionic case, the 0-weight space is trivial (this already implies that there are no G-important points!) and, in the real case, it is 1-dimensional. Thus we get $\dim(V) = \dim(G) - \dim(N(G_p)) + f + 2 \leq 3 - 1 + 1 + 2 = 5$. If follows that the cohomogeneity of the action is at most 2 and hence it is polar. Contradiction.

10.4. G is covered by $\mathbf{U}(1) \times \mathbf{SU}(2)$.

Then all irreducible representations of G are complex. For each n there is a exactly one irreducible representation ρ_n of G on $\mathbf{C}^n$. The restriction of each weight to the central $\mathbf{U}(1)$ of G is independent of the weight, thus there are no linearly dependent non-equal weights. One knows that each weight space is complex 1-dimensional. Thus we deduce $f \leq 2$. Since $\dim(N(G_p)) \geq \dim(T) \geq 2$, we get from the dimension formula $\dim(V) \leq 6$. Again the cohomogeneity of the action is at most 2 and hence the representation must be polar. Contradiction.

10.5. G is covered by $\mathbf{SU}(2) \times \mathbf{SU}(2)$.

In this case the complex irreducible representations of G are given by $\rho_m \otimes \rho_n$ on $\mathbf{C}^m \otimes_{\mathbf{C}} \mathbf{C}^n$. Using that all weights have multiplicities 1 for ρ_n and ρ_m, we see that any number of linearly dependent weights of $\rho_m \otimes \rho_n$ has at most $\min\{m, n\}$ elements. Without loss of generality, we assume $m \geq n$.

If m, n have different parity, then the representation V is quaternionic and we get $\dim(V) = 2mn$, $\dim(F) \leq 2n$. On the other hand we get from the dimension formula: $2mn \leq 6 + 2n$. Hence $n(m-1) \leq 3$, which is impossible as $m \geq n$.

If m, n have the same parity, then the representation V is real, and the dimension formula gives us $mn \leq 6 + n$. Hence, $n(m-1) \leq 6$. Thus either $m = n = 3$ and the corresponding representation of $\mathbf{SO}(3) \times \mathbf{SO}(3)$ on $\mathbf{R}^3 \otimes \mathbf{R}^3$ is polar, or $m = n = 2$ or $m = 4$, $n = 2$ which have cohomogeneity at most 2 and hence are also polar.

11. DISCONNECTED CASE

11.1. General useful observations.
Here we are going to work with a nice involution $w \in G \setminus G^0$. This involution normalizes G^0. We denote the set of the fixed points of w by F and the dimension of F by f. We have the dimension formula: $f = \dim(V) - \dim(G) + \dim(C) - 1$. Here C is the centralizer of w in G.

We will often use the following observation:

Lemma 11.1. *Let $K \subset \mathbf{O}(U)$ be a closed subgroup such that K^0 is locally isomorphic to $\mathbf{Sp}(n)$. Assume that K^0 acts irreducibly on U and there is an involution $w \in K \setminus K^0$. Then either $K^0 = \mathbf{Sp}(n)/\mathbf{Z}_2$ and $w = -w'$, for some involution $w' \in K^0$, or $K^0 = \mathbf{Sp}(n)$ and, for some complex structure on U, the involution w and the group K^0 are contained in the unitary group of U.*

Proof. The involution w acts by conjugation as an automorphism on K^0. Since K^0 does not have outer automorphisms, we find some $j \in K^0$ such that conjugation with j induces the same automorphism as conjugation with w. Then the element $q = j^{-1}w$ commutes with K^0.

Since w is an involution, we have $(jq)^2 = j^2q^2 = 1$. Thus $q^2 \in K^0$. Since q commutes with K^0, the element q^2 must be in the center of K^0. If $K^0 = \mathbf{Sp}(n)/\mathbf{Z}_2$ then q^2 must be the identity, i.e., q is an involution. Since the representation of K^0 is irreducible and w is not in K^0, we must have $q = -1$, hence $w = -j$.

If $K^0 = \mathbf{Sp}(n)$, then the element -1 is contained in K^0 (it is the only non-trivial involution that can commute with an irreducible representation). Hence, the same reasoning as before shows that $q^2 = 1$ would imply that w is contained in K^0. Thus we must have $q^2 = -1$. Then j and w commute with q, and we finish the proof by taking the complex structure defined by q. $\qquad\square$

11.2. G^0 is covered by $\mathbf{SU}(2)$.
Assume that G^0 is covered by $\mathbf{Sp}(1) = \mathbf{SU}(2)$. If $G^0 = \mathbf{SU}(2)$ then dimension of V is even and, due to Lemma 11.1, the involution w must preserve a complex structure. Hence its set F of fixed points has even dimension. The dimension formula $f = \dim(V) - 3 + 1 - 1$ provides a contradiction.

Assume now that $G^0 = \mathbf{SO}(3)$. Then $V = \mathbf{R}^{2n+1}$ for some n. The dimension formula gives us $f = 2n + 1 - 3 + 1 - 1 = 2n - 2$. On the other hand, due to Lemma 11.1, the involution w is given by $-w'$ for some involution $w' \in \mathbf{SO}(3)$. However, the (up to conjugation unique) involution w' in $\mathbf{SO}(3)$ fixes a subspace of dimension n, if n is odd, and of dimension $n + 1$, if n is even. Thus $f = 2n + 1 - n = n + 1$ or $f = 2n + 1 - (n+1) = n$, respectively.

By inserting into the previous equation, we deduce that $n = 3$ or $n = 2$. The case $n = 2$ is polar (thus excluded) and we are left with the case $n = 3$.

Summarizing, we have shown that if G^0 is locally isomorphic to $\mathbf{Sp}(1)$, we must have $G^0 = \mathbf{SO}(3)$ and $V = \mathbf{R}^7$. However, in this case the action of G on V has cohomogeneity

4. The explicit classification given by Theorem 1.4 shows that this case cannot occur as a non-trivial reduction, since there are no representations of copolarity 3 listed in Table 1.

11.3. G^0 is covered by $\mathbf{U}(1) \times \mathbf{SU}(2)$. The representation space V is the complex space $\mathbf{C}^n$. The center of G^0 is a circle $\mathbf{U}(1)$ that acts on $\mathbf{C}^n$ as complex multiplication. Our nice involution w normalizes this circle. Thus w is either complex linear or complex antilinear. We have that f is even in the first case and $f = n$ in the second case.

In the complex linear case, the centralizer of w has dimension 2 and the dimension formula yields $f = 2n - 4 + 2 - 1 = 2n - 3$. This contradicts the fact that f is even.

In the complex antilinear case, the centralizer of w has dimension 3, if w commutes with $\mathbf{SU}(2)$, and dimension 1, if it does not commute. In the first case we deduce $f = 2n - 4 + 3 - 1 = 2n - 2$ and in the second case $f = 2n - 4 + 1 - 1 = 2n - 4$. Using that $f = n$ we derive $n = 2$ in the first case and $n = 4$ in the second case.

The case $n = 2$ is polar. Thus we are left with the case $n = 4$. However, in this case the quotient V/G^0 has again cohomogeneity 4. Thus this case is excluded by the classification result Theorem 1.4.

11.4. G^0 is covered by $\mathbf{SU}(2) \times \mathbf{SU}(2)$. There are three cases to be analyzed:

    (I) $\mathbf{SU}(2) \cdot \mathbf{SU}(2)$ acting on $\mathbf{H}^m \otimes_{\mathbf{H}} \mathbf{H}^n$;
    (II) $\mathbf{SO}(3) \times \mathbf{SO}(3)$ acting on $\mathbf{R}^{2m+1} \otimes_{\mathbf{R}} \mathbf{R}^{2n+1}$; and
    (III) $\mathbf{SO}(3) \times \mathbf{SU}(2)$ acting on $\mathbf{R}^{2m+1} \otimes_{\mathbf{R}} \mathbf{H}^n$.

For our nice involution w we have the formula $f = \dim(V) - 7 + \dim(C)$. The conjugation by w can act on G^0 as an inner or as an outer automorphism. We deal with these two cases separately.

Outer automorphism. We assume first that w acts on G^0 not as an inner automorphism. Then w must interchange both factors of G^0. Thus we must be in cases (I) or (II). Moreover, the equality of dimensions $m = n$ must hold true.

By conjugating w with an element in G^0 we may assume that w acts on G^0 by interchanging the factors, hence by sending (g_1, g_2) to (g_2, g_1).

Consider the involution $i : V \to V$ defined by interchanging the equal factors of V: $i(h_1 \otimes h_2) = h_2 \otimes h_1$. Then i normalizes G^0 and the conjugation with i acts on G^0 by interchanging the factors of G^0. Therefore, $w \circ i$ commutes with G^0. Since the representation of G^0 is of real type, we must have $w = \pm i$. Moreover, in both cases, the centralizer of w is the diagonal of G^0 that has dimension 3.

From the dimension formula we conclude that $\dim(V) - f = 4$. In case (II), the dimensions of the sets of fixed points of i and $-i$ are given by $(2n + 1)(2n + 2)/2$ and $(2n + 1)(2n)/2$, respectively. Since both numbers are larger than 4 for $n \geq 2$ ($n = 1$ is polar), we get $\dim(V) - f > 4$ and a contradiction.

In case (I), the dimensions of the sets of fixed points of i and $-i$ are $n(2n-1)$ and $n(2n+1)$, respectively (note that viewing $\mathbf{H}^n \otimes_{\mathbf{H}} \mathbf{H}^n$ as the real vector space of quaternionic matrices of order n, the involution i corresponds to transpose conjugation $X \mapsto X^*$). Again, for $n > 1$ ($n = 1$ is polar), both numbers are larger than 4. Hence $\dim(V) - f$ is larger than 4 and we derive a contradiction.

Inner automorphism. We assume now that w acts on G^0 by an inner automorphism. Then $w = qj$, for some q that centralizes G^0 and some $j \in G^0$.

In cases (I) and (II) the representations are of real type, hence the centralizer of G^0 consists of ± 1. Since w is not in G^0, q must lie outside G^0. In case (I), the element -1 is contained in G^0, thus we get a contradiction directly.

In case (II) we must have $q = -1$. Thus $w = -j$ and j must be an involution in G^0 that is given by a product $j = j_1 \cdot j_2$, where each j_i is either an involution or the identity in the corresponding factor of G^0.

By the dimension formula, the set of fixed points of j must have dimension equal to $7 - c$, where $c = \dim(C)$. Denoting by f_i the set of fixed points of $-j_i$ and by e_i the set of fixed points of j_i, we get $7 - c = f_1 \cdot f_2 + e_1 \cdot e_2$.

If j_1 is not the identity, then $e_1 = m$ if m is odd and $m + 1$ if m is even. And the corresponding statement is true for j_2.

If $j_1 = 1$ then $c = 4$ and we have $3 = (2m + 1)e_2$. Hence $e_2 = 1$ and $m = n = 1$ and our representation is polar, in contradiction to our assumption. Similarly, $j_2 = 1$ is impossible.

If j_1 and j_2 are both different from the identity, then $c = 2$ and in all cases we get $5 > 2mn$. For $m = n = 1$, we have a polar representation and, for $m = 1$, $n = 2$, the dimension formula reads as $5 = 1 \cdot 3 + 2 \cdot 2$, thus we derive a contradiction.

We are left with the case (III). In this case $q^2 = j^{-2}$ is contained in the center of G^0. Thus either $q^2 = 1$ or $q^2 = -1$. If $q^2 = 1$ then (since q commutes with G^0), it must be that $q = \pm 1$. But -1 is contained in G^0, hence we would get $q \in G^0$, which is impossible.

Hence, we must have $q^2 = -1$. Therefore w is a complex involution with respect to the complex structure defined by q. Therefore, $\dim(V) - f$ is an even number. However, c is either equal to 2 or to 4, hence $7 - c$ is odd. This provides a contradiction.

Part 2. Irreducible representations of cohomogeneity 4 or 5

In this part, we classify the non-polar irreducible representations of cohomogeneity 4 or 5 and prove Theorem 1.4. Throughout, we use the lists of isotropy representations of symmetric spaces [Wol84], additional polar representations [EH99], and representations of cohomogeneity at most 3 [HL71] (see also [Uch80, Yas86, Str96]).

12. PRELIMINARIES

Let $\rho = (G, V)$ be a real irreducible representation of a compact connected Lie group G on a real vector space V. Denote by $c(\rho)$ the cohomogeneity of ρ.

12.1. **Basic dimension bound.** It is obvious that

$$\dim V \leq \dim G + c(\rho).$$

12.2. **Types.** Recall that, by Schur's lemma, the centralizer of $\rho(G)$ in $\mathrm{End}(V)$ is an associative real division algebra, thus, by Frobenius theorem, isomorphic to one of $\mathbf{R}$, $\mathbf{C}$ or $\mathbf{H}$; accordingly, we say that ρ is of real, complex of quaternionic type. There are many alternative characterizations of such types; the following one is often useful. ρ is of real type if and only if its complexification ρ^c remains irreducible; in this case, ρ is a real form of a complex irreducible representation. Otherwise, $\rho^c = \pi \oplus \pi^*$, where π is complex irreducible and ρ is equivalent to the realification π^r; here ρ is of quaternionic (resp. complex) type if π and π^* are equivalent (resp. not equivalent), where π^* denotes the dual representation.

12.3. Quaternionic matrices. For practical computations, it is often useful to work with quaternionic matrices. We view a quaternionic vector space V as a right $\mathbf{H}$-module $V_{\mathbf{H}}$. The set $\mathrm{Hom}(V_{\mathbf{H}}, W_{\mathbf{H}})$ of $\mathbf{H}$-linear maps $V_{\mathbf{H}} \to W_{\mathbf{H}}$ is a real vector space; in the special case $V^* = \mathrm{Hom}(V_{\mathbf{H}}, \mathbf{H}_{\mathbf{H}})$, since $\mathbf{H}$ is a bimodule, we get a left $\mathbf{H}$-module structure $_{\mathbf{H}}V^*$. Now $W_{\mathbf{H}} \otimes_{\mathbf{H}} V^*$ is well defined and a real vector space, and there is the usual canonical isomorphism $W_{\mathbf{H}} \otimes {}_{\mathbf{H}}V^* \cong \mathrm{Hom}(V_{\mathbf{H}}, W_{\mathbf{H}})$; this is G-equivariant in case V, W are G-representations of quaternionic type. It is also clear that $W_{\mathbf{H}} \otimes {}_{\mathbf{H}}V^*$ is a real form of $W \otimes_{\mathbf{C}} V^*$. Having said this, henceforth we write just $W \otimes_{\mathbf{H}} V$ for $W_{\mathbf{H}} \otimes {}_{\mathbf{H}}V^*$.

12.4. Slices and sums. The slice representation at $v \in V$ is the induced representation of the isotropy group G_v on the normal space $N_v(Gv)$. It is known that the cohomogeneity of a slice representation is equal to the cohomogeneity of the original representation. This works as an inductive argument allowing one to compute precisely the cohomogeneity (compare [HH70]). One can also apply this remark to sums. Let $\rho = \rho_1 \oplus \rho_2$ be the representation $(G, V_1 \oplus V_2)$. Let $v_1 \in V_1$ be a regular point for (G, V_1). Then consideration of the slice of ρ at v_1 yields

$$c(\rho) = c(\rho_1) + c(G_{v_1}, V_2).$$

In particular, $c(\rho) \geq c(\rho_1) + c(\rho_2)$ and equality holds if and only if (G_{v_1}, V_2) is orbit equivalent to $\rho_2 = (G, V_2)$.

12.5. Tensor products. It is convenient to introduce the following notation. If A and B are classical groups, we shall denote by $A \otimes B$ the irreducible representation given by the tensor product of the vector representations (the field over which the tensor product is being taking is determined by the types of the factor-representations).

In general, for a tensor product $\rho = (G = G_1 \times G_2, V_1 \otimes_{\mathbf{F}} V_2)$, where $\dim V_i = n_i$ and $\mathbf{F} = \mathbf{R}$, $\mathbf{C}$ or $\mathbf{H}$, we respectively have

$$\rho(G) \subset \mathbf{SO}(n_1) \otimes \mathbf{SO}(n_2); \text{ or}$$
$$\rho(G) \subset \mathbf{U}(n_1) \otimes \mathbf{U}(n_2); \text{ or}$$
$$\rho(G) \subset \mathbf{Sp}(n_1) \otimes \mathbf{Sp}(n_2).$$

It follows that

$$c(\rho) \geq \min\{n_1, n_2\}.$$

The following monotonicity lemma will be used to simplify the estimation of the cohomogeneity of some representations.

Lemma 12.1. *Let $\rho_1 = (G_1, V_1)$ be an arbitrary real representation, and let $\rho_2(n) = (G_2(n), \mathbf{F}^n)$ be the standard representation of $\mathbf{O}(n)$, $\mathbf{U}(n)$, $\mathbf{Sp}(n)$ on $\mathbf{R}^n$, $\mathbf{C}^n$, $\mathbf{H}^n$, respectively. Assume ρ_1 is of $\tilde{\mathbf{F}}$-type for $\tilde{\mathbf{F}} \subset \mathbf{F}$ and consider the tensor product $\rho(n) = (G(n) = G_1 \times G_2(n), V_1 \otimes_{\tilde{\mathbf{F}}} \mathbf{F}^n)$. Then $c(\rho(n)) \leq c(\rho(n+1))$.*

Proof. Since the cohomogeneity is the topological dimension of the orbit space, it is enough to show that the orbit space of $\rho(n)$ injects into that of $\rho(n+1)$. Since V_1 is a real representation, we can identify V_1 with V_1^* and replace the tensor product by the space of linear maps $\mathcal{L}_n = \mathrm{Hom}_{\tilde{\mathbf{F}}}(V_1, \mathbf{F}^n)$. Now $(g_1, g_2) \in G_1 \times G_2(n)$ acts on $A \in \mathcal{L}_n$ by mapping it to $g_2 A g_1^{-1}$. We consider the standard embedding $\mathcal{L}_n \to \mathcal{L}_{n+1}$. To prove the desired injectivity, we just need to show that any two elements A, $B \in \mathcal{L}_n$ that are in the same $G(n+1)$-orbit must be in the same $G(n)$-orbit. It is obvious that we can restrict to the case $G_1 = \{1\}$. In this case A and B are respectively given by m-tuples $(a_1, \ldots, a_m)$, $(b_1, \ldots, b_m)$

of elements in $\mathbf{F}^n \subset \mathbf{F}^{n+1}$ where $m = \dim_{\widetilde{\mathbf{F}}} V_1$. Let $g_2 \in G_2(n+1)$ such that $g_2 A = B$. One restricts g_2 to the subspace of $\mathbf{F}^n$ spanned by $a_1, \ldots, a_m$ and then extends it to an element of $G_2(n) \subset G_2(n+1)$. $\qquad\square$

Corollary 12.2. *We have* $c(G_1 \otimes \mathbf{SU}(n)) \leq c(G_1 \otimes \mathbf{SU}(n+1)) + 1$.

12.6. **Slices of products.** Let ρ_i be a real irreducible representation of a Lie group G_i on V_i, where $i = 1, 2$. Then $\rho = \rho_1 \otimes_{\mathbf{R}} \rho_2$ is a real representation of $G = G_1 \times G_2$ on $V = V_1 \otimes_{\mathbf{R}} V_2$ which is irreducible if at least one of the ρ_i is of real type. Let $v = v_1 \otimes v_2 \in V$ be a pure tensor where $v_i \in V_i$ is regular. Then G_{v_i} is a principal isotropy subgroup of ρ_i and the connected isotropy group $(G_v)^0$ equals $H = H_1 \times H_2$, where $H_i = (G_{v_i})^0$. Since the normal space

$$N_v(Gv) = \mathbf{R}v \oplus [v_1 \otimes (N_{v_2}(G_2 v_2) \ominus \mathbf{R}v_2)] \oplus [(N_{v_1}(G_1 v_1) \ominus \mathbf{R}v_1) \otimes v_2] \oplus \left[v_1^{\perp} \otimes v_2^{\perp}\right],$$

by considering the slice representation at v, we get

$$c(\rho) = c(H, v_1^{\perp} \otimes v_2^{\perp}) + c(\rho_1) + c(\rho_2) - 1.$$

We will mostly use this remark in case $c(\rho_i) = 1$ for $i = 1, 2$; then the decomposition of $v_1^{\perp} \otimes v_2^{\perp}$ into irreducible components is easier.

In the cases of complex and quaternionic tensor products, one uses similar though slightly more complicated arguments that we explicit below in the individual cases.

12.7. **Cohomogeneity** $c(\rho) = 1$. The list of transitive linear actions on spheres is well known (cf. Theorem of Borel-Montgomery-Samelson). We include it for easy reference. Here and below, ρ always denotes a real representation, so e.g. $\rho = (\mathbf{SU}(n), \mathbf{C}^n)$ means the realification of $\mathbf{C}^n$. On the other hand, if W is complex, $[W]_{\mathbf{R}}$ denotes a real form.

G	ρ	Real dim	Remarks
$\mathbf{SO}(n)$	$\mathbf{R}^n$	n	$-$
$\mathbf{SU}(n)$	$\mathbf{C}^n$	$2n$	$-$
$\mathbf{Sp}(n)$	$\mathbf{C}^{2n}$	$4n$	$-$
$\mathbf{G}_2$	$\mathbf{R}^7$	7	$-$
$\mathbf{Spin}(7)$	$\mathbf{R}^8$	8	spin representation
$\mathbf{Spin}(9)$	$\mathbf{R}^{16}$	16	spin representation
$\mathbf{SU}(n) \cdot \mathbf{U}(1)$	$\mathbf{C}^n \otimes_{\mathbf{C}} \mathbf{C}$	$2n$	$-$
$\mathbf{Sp}(n) \cdot \mathbf{U}(1)$	$\mathbf{H}^n \otimes_{\mathbf{C}} \mathbf{C}$	$4n$	$-$
$\mathbf{Sp}(n) \cdot \mathbf{Sp}(1)$	$\mathbf{H}^n \otimes_{\mathbf{H}} \mathbf{H}$	$4n$	$-$

12.8. G **is simple and** $2 \leq c(\rho) \leq 8$. A lemma of Onishchik [Oni62, Lemma 3.1] explains that the dimension of a complex irreducible representation is an increasing function of the highest weight, where one uses a partial order naturally defined on the set of dominant integral weights. Using this lemma, it is a matter of patience to list real irreducible representations of low cohomogeneity of simple groups. In the tables below we go up to cohomogeneity 8 (the tables in [HH70, ch. I, §2] are also helpful). Up to a few cases, our list can also be obtained from Lemma 2.6 in [Kol02].

12.8.1. *Polar representations.*

G	ρ	Conditions	$c(\rho)$	Type	Symmetric space
$\mathbf{SO}(n)$	$S_0^2(\mathbf{R}^n)$	$3 \le n \le 9$	$n-1$	r	$\mathbf{SU}(n)/\mathbf{SO}(n)$
$\mathbf{Sp}(n)$	$[(\Lambda^2\mathbf{C}^{2n} \ominus \mathbf{C}]_\mathbf{R}$	$3 \le n \le 9$	$n-1$	r	$\mathbf{SU}(2n)/\mathbf{Sp}(n)$
$\mathbf{F}_4$	$\mathbf{R}^{26}$	$-$	2	r	$\mathbf{E}_6/\mathbf{F}_4$
$\mathbf{SU}(n)$	$[\mathbf{C}^n \otimes_\mathbf{C} \mathbf{C}^{n*} \ominus \mathbf{C}]_\mathbf{R}$	$3 \le n \le 9$	$n-1$	r	Adjoint
$\mathbf{SO}(n)$	$\Lambda^2\mathbf{R}^n$	$5 \le n \le 17$	$\left[\frac{n}{2}\right]$	r	Adjoint
$\mathbf{Sp}(n)$	$[S^2(\mathbf{C}^{2n})]_\mathbf{R}$	$2 \le n \le 8$	n	r	Adjoint
$\mathbf{F}_4$	$\mathbf{R}^{52}$	$-$	4	r	Adjoint
$\mathbf{G}_2$	$\mathbf{R}^{14}$	$-$	2	r	Adjoint
$\mathbf{E}_6$	$\mathbf{R}^{78}$	$-$	6	r	Adjoint
$\mathbf{E}_7$	$\mathbf{R}^{133}$	$-$	7	r	Adjoint
$\mathbf{E}_8$	$\mathbf{R}^{248}$	$-$	8	r	Adjoint
$\mathbf{SU}(n)$	$\Lambda^2\mathbf{C}^n$	$5 \le n \le 17, n$ odd	$\frac{n-1}{2}$	c	$\mathbf{SO}(2n)/\mathbf{U}(n)$
$\mathbf{Spin}(10)$	$\mathbf{C}^{16}$	$-$	2	c	$\mathbf{E}_6/(\mathbf{U}(1)\mathbf{Spin}(10))$
$\mathbf{Sp}(4)$	$[\Lambda^4\mathbf{C}^8 \ominus \Lambda^2\mathbf{C}^8]_\mathbf{R}$	$-$	6	r	$\mathbf{E}_6/\mathbf{Sp}(4)$
$\mathbf{SU}(8)$	$[\Lambda^4\mathbf{C}^8]_\mathbf{R}$	$-$	7	r	$\mathbf{E}_7/\mathbf{SU}(8)$
$\mathbf{Spin}(16)$	$\mathbf{R}^{128}$	$-$	8	r	$\mathbf{E}_8/\mathbf{Spin}(16)$

12.8.2. *Non-polar representations.*

G	ρ	Conditions	$c(\rho)$	Type
$\mathbf{SO}(3)$	$\mathbf{R}^n$	$n = 7, 9, 11$	$n-3$	r
$\mathbf{SU}(2)$	$\mathbf{C}^4$	$-$	5	q
$\mathbf{SU}(6)$	$\Lambda^3\mathbf{C}^6$	$-$	7	q
$\mathbf{SU}(n)$	$\Lambda^2\mathbf{C}^n$	$6 \le n \le 14, n$ even	$\frac{n}{2}+1$	c
$\mathbf{SU}(n)$	$S^2\mathbf{C}^n$	$3 \le n \le 7$	$n+1$	c
$\mathbf{Sp}(3)$	$\Lambda^3\mathbf{C}^6 \ominus \mathbf{C}^6$	$-$	7	q
$\mathbf{Spin}(12)$	$\mathbf{C}^{32}$	$-$	7	q
$\mathbf{E}_6$	$\mathbf{C}^{27}$	$-$	4	c
$\mathbf{E}_7$	$\mathbf{C}^{56}$	$-$	7	q

13. PRODUCTS OF MIXED TYPE

Having already discussed simple groups, we next turn to the case of representations ρ of non-simple groups G with $c(\rho) = 4$ or 5. We roughly divide the discussion according to the types of the factors and start with the case of mixed type.

Lemma 13.1. *We have $c(\mathbf{SO}(m) \otimes \mathbf{Sp}(n)) \ge 11$ for $m \ge 3$ and $n \ge 2$, and $c(\mathbf{SO}(m) \otimes \mathbf{Sp}(1)) \ge 7$ for $m \ge 4$. Moreover $c(\mathbf{SO}(3) \otimes \mathbf{Sp}(1)) = 6$ and $c(\mathbf{SO}(3) \otimes \mathbf{U}(2)) = 5$.*

Proof. By dimensional reasons, $c(\mathbf{SO}(3) \times \mathbf{Sp}(2)) \ge 11$ and $c(\mathbf{SO}(4) \times \mathbf{Sp}(1)) \ge 7$, so the first two assertions follow from Lemma 12.1. The cohomogeneity of $\mathbf{SO}(3) \otimes \mathbf{U}(2)$ can be directly computed by using slices and noticing that the principal isotropy group is trivial, and then the other cohomogeneity follows. $\square$

Lemma 13.2. *For $m \ge 3$, $n \ge 3$ we have $c(\mathbf{SU}(m) \otimes \mathbf{SO}(n)) \ge 7$ unless $m \ge 4$ and $n = 3$. Moreover $c(\mathbf{U}(m) \otimes \mathbf{SO}(3)) = 6$ for $m \ge 3$.*

Proof. By dimensional reasons, $c(\mathbf{U}(3) \otimes \mathbf{SO}(4)) \geq 9$. Hence $c(\mathbf{SU}(m) \otimes \mathbf{SO}(n)) \geq 9$ if $m \geq 3$ and $n \geq 4$.

The case $n = 3$ is dealt with separately:

$$
\begin{aligned}
c(\mathbf{SU}(m) \otimes \mathbf{SO}(3)) &= 1 + c(\mathbf{SU}(m-1) \times \mathbf{SO}(2), i\mathbf{R}^2 \oplus \mathbf{C}^{m-1} \otimes_{\mathbf{C}} \mathbf{C}^2) \\
&= 2 + c(\mathbf{SU}(m-1), 2\mathbf{C}^{m-1}) \\
&= \begin{cases} 7, & \text{if } m = 3, \\ 6, & \text{if } m > 3. \end{cases}
\end{aligned}
$$

A similar computation shows that $c(\mathbf{U}(m) \otimes \mathbf{SO}(3)) = 6$ for $m \geq 3$. $\qquad\square$

Lemma 13.3. *We have:*

    (a) $c(\mathbf{U}(m) \otimes \mathbf{Sp}(2)) \geq 6$ *if* $m \geq 4$;
    (b) $c(\mathbf{U}(3) \otimes \mathbf{Sp}(2)) = 5$ *and* $c(\mathbf{SU}(3) \otimes \mathbf{Sp}(2)) = 6$;
    (c) $c(\mathbf{SU}(m) \otimes \mathbf{Sp}(n)) \geq 7$ *if* $m \geq 3$ *and* $n \geq 3$.

Proof. By dimensional reasons, $c(\mathbf{U}(4) \otimes \mathbf{Sp}(2)) \geq 6$, $c(\mathbf{U}(4) \otimes \mathbf{Sp}(3)) \geq 11$ and $c(\mathbf{SU}(3) \otimes \mathbf{Sp}(3)) \geq 7$. Hence (a) and (c) follow from Lemma 12.1.

On the other hand, one computes directly that $c(\mathbf{U}(3) \otimes \mathbf{Sp}(2)) = 5$, from which follows $c(\mathbf{SU}(3) \otimes \mathbf{Sp}(2)) = 6$. This proves (b). $\qquad\square$

14. The case $G = G_1 \times G_2$, $V = V_1 \otimes_{\mathbf{R}} V_2$

Here G_1 and G_2 are non-necessarily simple, $\rho_i = (G_i, V_i)$ are real irreducible representations and at least one of them is of real type. Let $m = \dim_{\mathbf{R}} V_1 \leq n = \dim_{\mathbf{R}} V_2$. Recall that $\mathbf{SO}(m) \otimes \mathbf{SO}(n)$ is polar. Since $c(\rho) = 4$ or 5, in view of Subsection 12.5 we may assume that $2 \leq m \leq 4$. Moreover, owing to the next lemma, $c(\rho_1) = c(\rho_2) = 1$.

Lemma 14.1. *Let* $\rho_1 = (G_1, V_1)$ *and* $\rho_2 = (G_2, V_2)$ *be real irreducible non-trivial representations (non-necessarily of real type) and consider their real tensor product* $\rho = (G, V)$*, where* $G = G_1 \times G_2$ *and* $V = V_1 \otimes_{\mathbf{R}} V_2$*. If either* ρ_1 *or* ρ_2 *has cohomogeneity bigger than one, then* $c(\rho) \geq 6$*.*

Proof. Fix $v = v_1 \otimes v_2 \in V$ where $v_i \in V_i$ is G_i-regular. Write $H = (G_v)^0$ and $H_i = ((G_i)_{v_i})^0$. Then $H = H_1 \times H_2$. Without loss of generality, we may assume that $c(\rho_1) = 1$ and $n = c(\rho_2) \geq 2$. Then

$$c(\rho) = n + c(H, v_1^{\perp} \otimes v_2^{\perp}).$$

Put $U = T_{v_2}(G_2 v_2)$. Then $U \neq 0$ and

$$v_1^{\perp} \otimes v_2^{\perp} = (n-1)v_1^{\perp} \otimes \mathbf{R} \oplus v_1^{\perp} \otimes U.$$

If $n \geq 3$, then it follows from the above that $c(\rho) \geq 6$. Suppose $n = 2$. Then ρ_2 is polar and $G_2(v_2)$ is an isoparametric submanifold of V_2. Since ρ_2 is irreducible, $G_2(v_2)$ has at least 3 distinct principal curvatures so U has at least 3 H_2-irreducible components. Thus $c(H, v_1^{\perp} \otimes U) \geq 3$ and it follows that

$$c(H, v_1^{\perp} \otimes v_2^{\perp}) \geq 4$$

and

$$c(\rho) \geq 2 + 4 = 6,$$

as desired. $\qquad\square$

14.1. $m = 2$. Then $\rho_1 = (\mathbf{SO}(2), \mathbf{R}^2)$. Also, $\rho_2 = (\mathbf{G}_2, V_2)$ is of real type. Running through the cases in which $c(\rho_2) = 1$, the only non-polar ρ that we get are $(\mathbf{SO}(2) \times \mathbf{Spin}(9), \mathbf{R}^2 \otimes \mathbf{R}^{16})$, and $(\mathbf{SO}(2) \times \mathbf{Sp}(n)\mathbf{Sp}(1), \mathbf{R}^2 \otimes (\mathbf{H}^n \otimes_{\mathbf{H}} \mathbf{H}))$ where $n \geq 2$, both of which have $c(\rho) = 3$ and copolarity 1 [GT00, Prop. 7.12].

14.2. $m = 3$ **or** 4. Assume first that $\rho_1 = (\mathbf{SO}(m), \mathbf{R}^m)$. Running through the cases in which ρ_2 is of real type and using the dimension bound, we get that ρ is one of $(\mathbf{SO}(3) \times \mathbf{G}_2, \mathbf{R}^3 \otimes \mathbf{R}^7)$, $(\mathbf{SO}(3) \times \mathbf{Spin}(7), \mathbf{R}^3 \otimes \mathbf{R}^8)$ or $(\mathbf{SO}(4) \times \mathbf{Spin}(7), \mathbf{R}^4 \otimes \mathbf{R}^8)$. (Note that $c(\mathbf{SO}(m) \otimes \mathbf{Sp}(n)\mathbf{Sp}(1)) \geq 8$ for $m \geq 3$ and $n \geq 2$ by Lemma 13.1.) The second representation is orbit equivalent to $(\mathbf{SO}(3) \times \mathbf{SO}(8), \mathbf{R}^3 \otimes \mathbf{R}^8)$ and hence polar [EH99]. The first one has $c(\rho) = 4$ and copolarity 2 [GT03, Lemma 6.11]. The last one has $c(\rho) = 5$ and copolarity 3, which can be checked via a similar reasoning as in [GT03, Lemma 6.11] and using Lemma 7.13 *loc. cit.* If ρ_2 is not of real type, we invoke Section 13 to get one more example, namely $\mathbf{SO}(3) \otimes \mathbf{U}(2)$.

Finally, the only other possibilities for ρ_1 are that it equals $(\mathbf{SU}(2), \mathbf{C}^2)$ or $(\mathbf{U}(2), \mathbf{C}^2)$. Then ρ_2 is of real type; again this case is covered by Section 13 and we get no other examples.

15. THE CASE $G = G_1 \times G_2$, $V = V_1 \otimes_{\mathbf{H}} V_2$

In principle, G_1 and G_2 could be non-simple. The $\rho_i = (G_i, V_i)$ are real irreducible representations and both of them are of quaternionic type. The first remark is that we may assume that G_1 and G_2 are simple. Indeed, if, say G_2, is not simple, then V_2 is a tensor product $W_1 \otimes_{\mathbf{R}} W_2$ where W_1 is of real type and W_2 is of quaternionic type; by rearranging the factors in V we see that this case is included in Section 14. Recall that $\mathbf{Sp}(m) \otimes \mathbf{Sp}(n)$ is polar, so this case will be omitted in the sequel.

Assume $G_1 = \mathbf{Sp}(1)$. If $V_1 = \mathbf{H}$ then we must have $c(\rho_2) \leq c(\rho) + \dim \mathbf{Sp}(1) \leq 8$. Since G_2 is simple, running through the list in section 12.8.2 we get five polar representations ρ associated to irreducible quaternionic-Kähler symmetric spaces.

For $m \geq 3$, we have $c(\mathbf{Sp}(n), m \cdot \mathbf{H}^n) > 8$ for $n \geq 1$. Therefore $c(\mathbf{Sp}(1) \times G_2, \mathbf{H}^m \otimes_{\mathbf{H}} V_2) \geq c(\mathbf{Sp}(1) \times \mathbf{Sp}(n), \mathbf{H}^m \otimes_{\mathbf{H}} \mathbf{H}^n) \geq c(\mathbf{Sp}(n), m \cdot \mathbf{H}^n) - 3 > 5$. Moreover, if $c(\rho_2) > 1$ then $c(\rho_2) \geq 5$, due to Subsection 12.8.2. Hence $c(2\rho_2) \geq 10$ and $c(\mathbf{Sp}(1) \times G_2, \mathbf{H}^2 \otimes_{\mathbf{H}} V_2) \geq 7$.

In the case $G_1 = \mathbf{Sp}(1)$, we are thus left with the representations $(\mathbf{Sp}(1) \times \mathbf{Sp}(n), \mathbf{H}^2 \otimes_{\mathbf{H}} \mathbf{H}^n)$, but they have cohomogeneity 3 and copolarity 1 [GT00, Prop. 7.12].

Thus we may assume that both groups G_i are not equal to $\mathbf{Sp}(1)$. Let $m = \dim_{\mathbf{H}} V_1 \leq n = \dim_{\mathbf{H}} V_2$. Since $\mathbf{Sp}(m) \otimes \mathbf{Sp}(n)$ is polar and we are interested in the cases $c(\rho) = 4$ or 5, we may assume that $m \leq 4$. Hence $2 \leq m \leq 4$ and we deduce that $\rho_1(G_1) = \mathbf{Sp}(m)$.

We rule out those cases by showing $c(\rho) \geq 6$ as follows. In view of Lemma 12.1, we may assume $m = 2$. Now if $c(\rho_2) \geq 8$, then the same argument as above, together with $\dim(G_1) = 10$, implies that $c(\rho) \geq 2 \cdot 8 - 10 = 6$. Hence it remains to check for ρ_2 equal to one of the four representations in the table in Subsection 12.8.2. In all cases, we finish by using the basic dimension bound.

16. THE REMAINING CASE: $G = G_1 \times G_2$, $V = V_1 \otimes_{\mathbf{C}} V_2$

For $\rho = (G, V)$ with G non-simple, we may assume that all tensor products in V are over $\mathbf{C}$ due to Sections 14 and 15. It follows that there are no factors of real type and at most one factor of quaternionic type. It is also useful to recall that $\mathbf{SU}(m) \otimes \mathbf{U}(n)$ is always polar and $\mathbf{SU}(m) \otimes \mathbf{SU}(n)$ is polar if and only if $m \neq n$; otherwise, it has cohomogeneity $m + 1$ and copolarity $m - 1$ [GOT04, § 3.3].

Consider first the case $\rho = (\mathbf{U}(1) \times G_2, \mathbf{C} \otimes_{\mathbf{C}} V_2)$ where G_2 is simple, $\rho_2 = (G_2, V_2)$ is real irreducible of complex or quaternionic type and $c(\rho_2) > 1$. A glance at the tables in Section 12.8 yields only one example, namely, $(\mathbf{U}(2), \mathbf{C}^4)$. We will show there is only one further example, with cohomogeneity 5, a circle factor and trivial principal isotropy group. Hence we may assume in the sequel that a circle factor is always present.

Assume the factor of quaternionic type is present. We consider first the case $\rho(G) \subset \mathbf{SU}(2) \otimes \mathbf{SU}(m) \otimes \mathbf{U}(n) = \mathbf{SU}(2) \otimes \mathbf{U}(m) \otimes \mathbf{SU}(n)$ where $3 \le m \le n$. It follows from Lemma 12.1 that $c_{m,n} = c(\mathbf{SU}(2) \otimes \mathbf{SU}(m) \otimes \mathbf{U}(n))$ increases with m and n, but note that $c_{3,3} \ge 16$ by dimensional reasons.

Consider next the case ρ is of the form $(\mathbf{U}(2) \times G_2, \mathbf{C}^2 \otimes_{\mathbf{C}} V_2)$, where $\rho_2 = (G_2, V_2)$ is real irreducible of complex type with G_2 simple. We have $c(\rho) \ge c(2\rho_2) - 4 > 2c(\rho_2) - 4 \ge 6$ for $c(\rho_2) \ge 5$. Otherwise $2 \le c(\rho_2) \le 4$ and we can use the dimension bound to rule out the few possibilities given in the tables in Section 12.8.

Hence if the factor of quaternionic type is present, the discussion in Section 13 yields that ρ is $\mathbf{U}(3) \otimes \mathbf{Sp}(2)$.

Finally we consider the case in which all factors are of complex type. The case $\rho(G) \subset \mathbf{SU}(m) \otimes \mathbf{SU}(n) \otimes \mathbf{U}(p)$ where $3 \le m \le n \le p$ is discarded by means of Lemma 12.1 as above. Hence we may assume that G is the product of two simple groups and a circle factor. Now $\rho(G) \subset \mathbf{U}(m) \otimes \mathbf{SU}(n) = \mathbf{SU}(m) \otimes \mathbf{U}(n)$ and we may assume $m \le n$. Since we are interested in $c(\rho) = 4$ or 5, in view of Subsection 12.5 we may assume $m = 3$ or 4. Again invoking Lemma 12.1, it suffices to exclude the case $\rho = (\mathbf{U}(3) \times G_2, \mathbf{C}^3 \otimes_{\mathbf{C}} V_2)$ where G_2 is simple. In this case $c(\rho) \ge c(3\rho_2) - 9 > 3c(\rho_2) - 9 \ge 6$ if $c(\rho_2) \ge 5$. On the other hand, for $2 \le c(\rho_2) \le 4$ the few cases given in the tables in Section 12.8 are excluded by the dimension bound.

17. FINAL ARGUMENTS

We are going to finish the proofs of Theorem 1.4 and Corollary 1.5.

Collecting the lists of Subsection 12.8 as well as the results of Sections 14, 15 and 16 yields the first two columns of the Tables 1 and 2. It remains to verify the statements about the copolarities and the boundaries.

All but the first two representations in cohomogeneity 4 have been shown to admit reductions to minimal generalized sections with the torus T^2 as the identity component of the group acting on it [GOT04]. Since these actions admit reductions, their quotients have non-empty boundary. On the other hand, for the first two representations, the quotients do not have boundary (owing to Section 10, which is independent of Theorem 1.4). Due to Proposition 5.2, these representations do not admit reductions.

In the same way all but two representations of cohomogeneity 5 admit reductions to minimal generalized sections with 3-dimensional tori as the identity component of the group acting on it (see Subsection 14.2 and [GOT04]), and the orbit space of $\mathbf{SU}(2)$ on $\mathbf{C}^4$ has no boundary.

For the remaining two presentations, one is a reduction of the second via a generalized section. Namely, for the representation of $\mathbf{U}(3) \times \mathbf{Sp}(2)$ we enlarge the group by adjoining complex conjugation of matrices to obtain an orbit equivalent representation. The new representation has non-trivial principal isotropy group whose fixed point, given by the subspace of real matrices, is a generalized section and the representation space of $\mathbf{SO}(3) \times \mathbf{U}(2)$. Therefore both quotients are isometric and have non-empty boundaries. It remains to prove

that the representation ρ of $\mathbf{SO}(3) \times \mathbf{U}(2)$ on $\mathbf{R}^3 \otimes \mathbf{R}^4$ has trivial copolarity. Since this information is not needed in the proof of Theorem 1.2, we can use this theorem to show that the *abstract* copolarity of ρ is trivial as follows. The dimension of $\mathbf{SO}(3) \times \mathbf{U}(2)$ is 7. If it had a minimal reduction to a representation of a group G with $\dim(G) \leq 6$, then the restricted representation of G^0 would be reducible, due to Proposition 9.1. Hence one could apply Theorem 1.2 to deduce that the representation of $\mathbf{SO}(3) \times \mathbf{U}(2)$ must have copolarity equal to the abstract copolarity equal to 3. One could either argue directly that ρ does not admit a reduction to a finite extension of its maximal torus, or, more elegantly, use Corollary 7.5 to deduce that ρ must have non-trivial principal isotropy groups. This provides a contradiction and finishes the proof of Theorem 1.4.

In regard to Corollary 1.5 and its assertion about representations of cohomogeneity at most 5, it remains only to check that the copolarity and abstract copolarity must be equal for the representations of copolarity 2 in Table 1 and those of copolarity 3 in Table 2. Since those representations admit reductions to finite extensions of tori, this follows from an application of Theorem 1.2. Finally, the assertion about representations of abstract copolarity at most 6 follows from Proposition 9.1 and Theorem 1.2.

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