NON-POSITIVE CURVATURE AND THE PTOLEMY INEQUALITY

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ABSTRACT. We provide examples of non-locally compact geodesic Ptolemy metric spaces which are not uniquely geodesic. On the other hand, we show that locally compact, geodesic Ptolemy metric spaces are uniquely geodesic. Moreover, we prove that a metric space is CAT(0) if and only if it is Busemann convex and Ptolemy.

1. INTRODUCTION

A metric space X is called a Ptolemy metric space, if the inequality

$$(1) \qquad |xy|\,|uv| \;\leq\; |xu|\,|yv| \;+\; |xv|\,|yu|$$

is satisfied for all $x, y, u, v \in X$.

Our interest in Ptolemy metric spaces originates from an analysis of boundaries of CAT(-1)-spaces when endowed with a Bourdon or Hamenstädt metric. Such boundaries are indeed Ptolemy metric spaces (see [7]).
Various aspects of such spaces have occasionally been studied, for instance in [4], [10], [12] and [17]. A smooth Riemannian manifold is of non-positive sectional curvature, if and only if it is locally Ptolemy, and a locally Ptolemy Finsler manifold is necessarily Riemannian (see [11] and [4]). Furthermore, CAT(0)-spaces are geodesic Ptolemy metric spaces (compare Section 2).
On first consideration, these observations might suggest that for geodesic metric spaces the Ptolemy condition is some kind of non-positive curvature condition. We show that without any further conditions this is completely wrong.

Theorem 1.1. *Let X be an arbitrary Ptolemy space, then X can be isometrically embedded into a complete geodesic Ptolemy space $\hat{X}$.*

As an application, take the four point Ptolemy space $X = \{x, y, m_1, m_2\}$ with $|xy| = 2$ and all other nontrivial distances equal to 1. By Theorem 1.1 X can be isometrically embedded into a geodesic Ptolemy space $\hat{X}$. Since m_1 and m_2 are midpoints of x and y, there are in $\hat{X}$ two different geodesics joining the points x and y. In particular, the space is not uniquely geodesic and hence far away from 'non-positively curved'.
The space $\hat{X}$ constructed in Theorem 1.1 fails to be proper. Indeed, for proper geodesic Ptolemy spaces the situation is completely different:

Theorem 1.2. *A proper, geodesic, Ptolemy metric space is uniquely geodesic.*

The proof of Theorem 1.2 also works if one replaces 'properness' through the assumption that there exists a geodesic bicombing which varies continuously with its endpoints. This will help us to prove that the property of being Ptolemy precisely distinguishes between the two most common non-positive curvature conditions for geodesic metric spaces, namely the one due to Alexandrov and the one due to Busemann.

Theorem 1.3. *A metric space is* CAT(0) *if and only if it is Ptolemy and Busemann convex.*

We finish the Introduction with a few comments on the theorems above.
The space $\hat{X}$ constructed in Theorem 1.1 has several remarkable properties. There is a huge collection of convex functions on $\hat{X}$, in particular, all distance functions to points are convex. However, from the geometrical or topological point of view the space appears rather odd. Starting with finite spaces, for instance with the four point space X as above, one obtains a space $\hat{X}$ on which the distance functions d_x to the points $x \in X \subset \hat{X}$ are affine. This is to our knowledge a first example of very strange affine functions on metric spaces. The space $\hat{X}$ is far from being a product, in contrast to the structural results obtained in [15] and [9]. Theorem 1.1 shows that the Ptolemy condition is sufficient to ensure that a metric space can be isometrically embedded into a geodesic Ptolemy metric space. This seems to be particularly interesting, as the problem of synthetic descriptions of (non-convex) subsets of CAT(0)-spaces is very difficult (cf. [8]). The properness, assumed in addition in Theorem 1.2, forces the space to be contractible; in fact an absolut neighborhood retract. Moreover, it can be shown that the distance functions to points are neither affine in this case, in fact they are strictly convex (cf. Remark 4.1). Finally, the proof of Theorem 1.3 confirms the following idea. If a Busemann convex space is not CAT(0), then it contains an infinitesimal portion of a non-Euclidean Banach space. This observation may be of some interest in its own right.

After a preliminary section, we prove Theorem 1.1 in Section 3, Theorem 1.2 in Section 4 and Theorem 1.3 in Section 5.

2. PRELIMINARIES

We start by recalling some easy examples. The real line $\mathbb{R}$ is a Ptolemy space. To show this, consider points x, y, z, w in this order on the line. A completely trivial computation shows that $|xz||yw| = |xy||zw| + |yz||wx|$, which implies that $\mathbb{R}$ is Ptolemy. Since the Ptolemy condition on four points is Möbius invariant (see below), the equality above holds for points x, y, z, w, which lie in this order on a circle in the plane $\mathbb{R}^2$. This is the classical Theorem of Ptolemy for cyclic quadrilaterals.
To show that the Euclidean space $\mathbb{R}^n$ is Ptolemy, consider again four points x, y, z, w. Applying a suitable Möbius transformation we can assume that z is a midpoint of y and w, i.e. $|yz| = |zw| = \frac{1}{2}|yw|$. For this configuration the Ptolemy inequality is equivalent to $|xz| \leq \frac{1}{2}(|xy| + |xw|)$, which is just the convexity of the distance to the point x.

Every CAT(0)-space is Ptolemy, since every four point configuration in a CAT(0)-space admits a subembedding into the Euclidean plane ([3], p. 164).

Möbius invariance: Let d, d' be two metrics on the same set X. The metrics are called Möbius equivalent, if for all quadruples x, y, z, w of points

$$\frac{d(x,y)d(z,w)}{d(x,z)d(y,w)} = \frac{d'(x,y)d'(z,w)}{d'(x,z)d'(y,w)}.$$

If d and d' are Möbius equivalent, then (X, d) is Ptolemy, if and only if (X, d') is Ptolemy.

Indeed, the Ptolemy inequality says that for all quadruples of points the three numbers of the triple

$$A = (d(x,y)d(z,w), d(x,z)d(y,w), d(x,w)d(y,z))$$

satisfy the triangle inequality. By dividing all three numbers by $d(x,w)d(y,z)$, we see that these numbers satisfy the triangle inequality if and only if the three numbers of the triple

$$B = (\frac{d(x,y)d(z,w)}{d(x,w)d(y,z)}, \frac{d(x,z)d(y,w)}{d(x,w)d(y,z)}, 1)$$

satisfy the triangle inequality. In this expression we can replace d by d', and hence we obtain the claim.

Basic properties of Ptolemy spaces: Here we collect a couple of basic properties of Ptolemy spaces which will be frequently used in the remainder of this paper.

   (P1): Every subset $Y \subset X$ of a Ptolemy metric space X, endowed with the metric inherited from X, is Ptolemy.
   (P2): A metric space X is Ptolemy if and only if for every $\lambda > 0$ the scaled space λX is Ptolemy.

Some of our arguments below will use the notions of ultrafilters and ultra-limits; a generalization of pointed Gromov-Hausdorff convergence. We refer the reader, not familiar with these methods, to [3] and [14]. The symbol $\lim_\omega(X_n, x_n)$ will denote such an ultralimit (w.r.t. a non-principal ultrafilter ω).

As every metric property, the Ptolemy condition is invariant w.r.t. ultra-convergence.

   (P3): For every sequence $\{(X_i, x_i)\}_i$ of pointed Ptolemy spaces and every non-principle ultrafilter ω, the ultralimit $\lim_\omega(X_i, x_i)$ is a Ptolemy space.

Finally, we recall another important observation, which is due to Schoenberg (see [18]). This property lies in the heart of Theorem 1.3.

   (P4): A normed vector space is an inner product space if and only if it is Ptolemy.

A subset of a normed vector space is called linearly convex, if with any two points it contains the straight line segment connecting these points. A metric space is called linearly convex, if it is isometric to a linearly convex subset of a normed vector space and called flat, if it is isometric to a convex

subset of an inner product space.

With this notation the properties above immediately yield the

Corollary 2.1. *Let X be a Ptolemy space, then every linearly convex subset $C \subset X$ of X is flat.*

Let X be a metric space and $x, y \in X$. Then a point $m \in X$ is called a midpoint of x and y if $|xm| = \frac{1}{2}|xy| = |my|$. We say that X has the midpoint property, if for all $x, y \in X$ there exists a midpoint of x and y in X. The space X is called geodesic, if any two points $x, y \in X$ can be joined by a geodesic path, i.e. to any two points $x, y \in X$ there exists an isometric embedding γ of the interval $[0, |xy|]$ of the Euclidean line into X such hat $\gamma(0) = x$ and $\gamma(|xy|) = y$. In this paper we will always assume that geodesics are parameterized affinely, i.e. proportionally to arclength. A complete space with the midpoint property is geodesic. Note that a Ptolemy metric space X which has the midpoint property satisfies

$$(2) \qquad |mz| \ \leq \ \frac{1}{2}\left[|xz| \ + \ |yz|\right]$$

for all $x, y, z, m \in X$ such that m is a midpoint of x and y (cf. [6] for a discussion of such spaces).

Non-positive curvature conditions and often convex spaces: The most common non-positive curvature conditions are due to Alexandrov and Busemann (cf. [1] and [5]).

We suppose that the reader interested in this paper's subject is familiar with the notion of CAT(0)-spaces. Roughly speaking, a geodesic space X is called a CAT(0)-space if all geodesic triangles in X are not thicker than their comparison triangles in the Euclidean plane $\mathbb{E}^2$ (for a precise definition we refer the reader, for instance, to Section II.1 in [3]).

The geodesic space X is said to be Busemann convex, if for any two (affinely parameterized) geodesics $\alpha, \beta : I \longrightarrow X$, the map $t \mapsto |\alpha(t)\beta(t)|$ is convex.

Remark 2.2. Every CAT(0)-space is Busemann convex and every Busemann convex space is uniquely geodesic. However, there are Busemann convex spaces which are not CAT(0)-spaces, as for instance all non-Euclidean normed vector spaces with strictly convex unit norm balls.

The notion of Busemann convexity is not stable under limit operations. For instance, a sequence of strictly convex norms on $\mathbb{R}^n$ may converge to a non-strictly convex norm. To overcome such a phenomenon, Bruce Kleiner introduced a weaker notion of Busemann convexity in [13], which is stable with respect to limit operations. A metric space X is called *often convex*, if there exists a convex geodesic bicombing, i.e. a map $\gamma : X \times X \times [0, 1] \to X$, $(x, y, t) \mapsto \gamma_{x,y}(t)$, such that $t \mapsto \gamma_{x,y}(t)$ is a geodesic and for all x, y, x', y' and all (not necessarily surjective) affine maps $\varphi : I \to [0, 1]$, $\psi : I \to [0, 1]$ defined on the same interval I the map $s \mapsto |\gamma_{x,y}(\varphi(s))\gamma_{x'y'}(\psi(s))|$ is convex. This convexity implies in particular, that γ is continuous and that for points $x', y' \in \gamma_{x,y}([0, 1])$ the geodesic $\gamma_{x',y'}$ is contained in $\gamma_{x,y}$. In particular, γ defines a continuous midpoint map $m(x, y) := \gamma_{x,y}(1/2)$.

We state the following properties of often convex spaces, that are direct consequences of the definitions (cf. [13]).

(OC1): For every sequence $\{(X_i, x_i)\}_i$ of often convex spaces and every non-principle ultrafilter ω, the ultralimit $\lim_\omega(X_i, x_i)$ is an often convex space.

(OC2): A metric space X is often convex if and only if for every $\lambda > 0$ the scaled space λX is often convex.

(OC3): Let X be often convex. Then X is Busemann convex if and only if X is uniquely geodesic.

3. Proof of Theorem 1.1

We explicitly construct the complete, geodesic, Ptolemy metric space $\hat{X}$. First, we subsequently add midpoints to X in order to obtain a Ptolemy metric space $\mathfrak{M}(X)$ which has the midpoint property. Then we pass to an ultraproduct of $\mathfrak{M}(X)$.

Let Σ denote the set of unordered tuples in X. Formally, $\Sigma = \{\{x_1, x_2\} \subset X | x_1, x_2 \in X\}$, i.e. Σ consists of all subsets of X with one or two elements. On Σ we define a metric via

$$|\{x_1, x_2\}\{y_1, y_2\}| \; := \; \left\{ \begin{array}{ll} \frac{1}{4}[|x_1 y_1| + |x_1 y_2| + |x_2 y_1| + |x_2 y_2|] & \text{if } \{x_1, y_1\} \neq \{y_1, y_2\} \\ 0 & \text{if } \{x_1, x_2\} = \{y_1, y_2\} \end{array} \right\}$$

for all $\{x_1, x_2\}, \{y_1, y_2\} \in \Sigma$. This indeed defines a metric on Σ. In order to verify this, one has to prove the triangle inequality

$$|\{x_1, x_2\}\{y_1, y_2\}| \; \leq \; |\{x_1, x_2\}\{z_1, z_2\}| \; + \; |\{z_1, z_2\}\{y_1, y_2\}|$$

for all $\{x_1, x_2\}, \{y_1, y_2\}, \{z_1, z_2\} \in \Sigma$. If two of the tuples coincide, the validity of the inequality above is evident, and otherwise it just follows by applying the triangle inequality in X three times.

Moreover, the space $M(X) := (\Sigma, |\cdot|)$ is Ptolemy, i.e., it satisfies

$$|\{x_1, x_2\}\{y_1, y_2\}| \cdot |\{z_1, z_2\}\{u_1, u_2\}| \leq |\{x_1, x_2\}\{z_1, z_2\}| \cdot |\{y_1, y_2\}\{u_1, u_2\}|$$
$$+ |\{x_1, x_2\}\{u_1, u_2\}| \cdot |\{y_1, y_2\}\{z_1, z_2\}|$$

for all $\{x_1, x_2\}, \{y_1, y_2\}, \{z_1, z_2\}, \{u_1, u_2\} \in \Sigma$. Once again, the validity of this inequality is evident, if two of the tuples coincide, and otherwise it follows by applying the Ptolemy inequality in X sixteen times.

Note further that X isometrically embeds into $M(X)$ via $x \mapsto \{x, x\}$. Thus we may identify X with a subset of $M(X)$.

Now we define $M^0(X) := X$ as well as $M^{k+1}(X) := M(M^k(X))$ and set $\mathfrak{M}(X) := \bigcup_{k=0}^{\infty} M^k(X)$. From the considerations above it follows that this space is a Ptolemy metric space. Moreover, it has the midpoint property. Namely, each pair $x, y \in \mathfrak{M}(X)$ is contained in some $M^k(X)$ and $\{x, y\} \in M^{k+1}(X)$ is a midpoint of x and y. Passing to an ultraproduct $\hat{X}$ of $\mathfrak{M}(X)$, i.e. $\hat{X} := \lim_\omega\{(\mathfrak{M}(X), \mathfrak{x})\}_n$, the ultralimit of the constant sequence $\{(\mathfrak{M}(X), \mathfrak{x})\}_n$ w.r.t. some ultrafilter ω, where $\mathfrak{x} \in \mathfrak{M}(X)$, we obtain a complete Ptolemy metric space which has the midpoint property and, therefore, is geodesic. $\square$

4. Proof of Theorem 1.2

In this section we prove Theorem 1.2. For two points $p^-, p^+ \in X$ we consider the set
$$C(p^-, p^+) := \left\{ x \in X \,\Big|\, |p^-p^+| = |p^-x| + |xp^+| \right\}.$$
Since the distance functions $d_{p^\pm} := d(p^\pm, \cdot) : X \to \mathbb{R}_0^+$ are convex, the set $C(p^-, p^+)$ is convex. The convexity of $d_{p^\pm}$ implies that these functions are affine on the set $C(p^-, p^+)$.

Proof of Theorem 1.2: Let $\gamma_1, \gamma_2 : [0, L] := |p^-p^+| \to C$ be geodesics connecting p^- to p^+. Set $x_s := \gamma_1(s)$ as well as $y_s := \gamma_2(s)$ and let m_s denote a midpoint of x_s and y_s. Since the functions $d_{p^\pm} : C(p^-, p^+) \to \mathbb{R}_0^+$ are affine, we obtain $|p^-m_s| = s$ and $|p^+m_s| = L - s$.
Let now $0 < s < t \le L$ be arbitrary, then the triangle inequality yields

(3) $$|m_s x_t| + |m_s y_t| \ge |x_t y_t|,$$

the Ptolemy inequality yields
$$|p^-m_t| \cdot |x_s y_s| \le |m_t x_s| \cdot |p^- y_s| + |m_t y_s| \cdot |p^- x_s|$$
and therefore

(4) $$|m_t x_s| + |m_t y_s| \ge \frac{t}{s} \cdot |x_s y_s| = |x_s y_s| + (t - s)\frac{|x_s y_s|}{s}.$$

On the other hand the Ptolemy property for the points m_s, x_t, m_t, x_s and m_s, y_t, m_t, y_s yields

(5) $$|m_s x_t| \cdot |m_t x_s| + |m_s y_t| \cdot |m_t y_s| \le \frac{1}{2}|x_s y_s| \cdot |x_t y_t| + 2(t - s) \cdot |m_s m_t|.$$

Now, fix any $0 < s < L$ with $l := |x_s y_s| \ge 0$, and choose a sequence $s_n \to s$ with $s_n > s$. Let m_s denote midpoints of x_{s_n} and y_{s_n}. By compactness, we can pass to a subsequence $m_{s_n} \to m$, where $m = m_s$ is a midpoint of x_s and y_s. Now, we set

$$\varphi_n := |m_{s_n}m|, \quad \epsilon_n := s_n - s, \quad a_n^+ := |m_{s_n}x_s|, \quad b_n^+ := |m_{s_n}y_s|$$
$$a_n^- := |m, x_{s_n}|, \quad b_n^- := |m y_{s_n}|, \quad l_n := |x_{s_n}y_{s_n}| \quad \text{and} \qquad Q := \frac{l}{s} \ge 0.$$

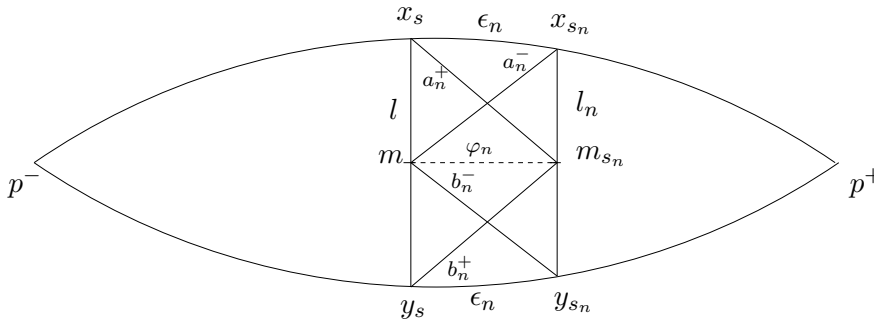
FIGURE 1. This figure visualizes the notation used in the proof.

We have by (3), (4) and (5):

(6) $$a_n^- + b_n^- \ge l_n, \quad a_n^+ + b_n^+ \ge l + \epsilon_n \cdot Q, \quad b_n^+ \cdot b_n^- + a_n^+ \cdot a_n^- \le \frac{1}{2}l \cdot l_n + 2\epsilon_n \varphi_n.$$

By triangle inequalities, we see $|l_n - l| \leq 2\epsilon_n$, $|a_n^\pm - \frac{1}{2}l| \leq 2\epsilon_n$, $|b_n^\pm - \frac{1}{2}l| \leq 2\epsilon_n$, hence passing to a subsequence, we find $A^+, A^-, B^+, B^-, C \in [-2, 2]$ such that $a_n^\pm = \frac{1}{2}l + A^\pm \epsilon_n + o(\epsilon_n)$, $b_n^\pm = \frac{1}{2}l + B^\pm \epsilon_n + o(\epsilon_n)$ and $l_n = l + C\epsilon_n + o(\epsilon_n)$. Since $2\epsilon_n \varphi_n = o(\epsilon_n)$, we derive from (6) at

$$B^- + A^- \geq C, \quad B^+ + A^+ \geq Q \geq 0, \quad (B^+ + B^-) + (A^+ + A^-) \leq C.$$

Thus $Q = 0$, which implies $\gamma_1(s) = \gamma_2(s)$. Since $s \in (0, L)$ was arbitrary, we have $\gamma_1 = \gamma_2$. $\qquad \square$

Remark 4.1. It is not difficult to prove that in Theorem 1.2 one can replace "properness" by the slightly weaker assumption of "local compactness". A more involved argument shows that a locally compact geodesic Ptolemy metric space is even strictly distance convex, i.e. that the inequality (2) is strict, whenever $|xy| > ||xz| - |zy||$. The proof of this claim will be given elsewhere.

We finish this section with an important observation. In the proof of Theorem 1.2 we use the properness of X only to show that $m_{s_n} \to m$. Clearly, the existence of a continuous midpoint map $m : X \times X \to X$ implies also this convergence. As a consequence we have

Theorem 4.2. *Let X be a geodesic, Ptolemy space which admits a continuous midpoint map. Then X is uniquely geodesic.*

Since often convex spaces admit continuous midpoint maps the first statement of the following corollary is an immediate consequence.

Corollary 4.3. *Let X be an often convex and Ptolemy space. Then X is uniquely geodesic and hence, by (OC3), Busemann convex. Moreover, for every sequence of pointed Busemann convex and Ptolemy metric spaces $\{(X_i, x_i)\}$, every non-principle ultrafilter ω and all $\lambda_i \in \mathbb{R}^+$, the ultralimit $\lim_\omega (\lambda_i X_i, x_i)$ is Busemann convex.*

The second statement goes as follows: A Busemann convex space is in particular often convex. By (OC1) this property and by (P3) the Ptolemy property is stable under limits. Thus the limit is often convex and Ptolemy and hence Busemann convex.

5. Proof of Theorem 1.3

In this section we prove Theorem 1.3. First, we recall the notion of generalized and weak angles (cf. [14]). Then we prove Proposition 5.3 (a version of the Toponogov rigidity theorem), that was shown independently by Rinow and Bowditch under slightly more restrictive conditions (see [16] and [2]), and which will finally allow us to obtain a proof of our Theorem 1.3.

5.1. **Ultrarays associated to geodesics and their enclosed angles. Ultrarays:** Let X be a geodesic metric space and let γ be a geodesic in X emanating from $p \in X$. Now take a non-principle ultrafilter ω, consider the ω-blow up $(\bar{X}, \bar{d})$ of X in p, i.e. $(\bar{X}, \bar{d}) := \lim_\omega \{(nX, p)\}_n$, and define $\bar{\gamma} : [0, \infty) \longrightarrow \bar{X}$ through $\bar{\gamma}(s) := \lim_\omega \{(\gamma(\frac{s}{n}), p)\}_n$ for all $s \in [0, \infty)$. This map indeed is a geodesic ray in $(\bar{X}, \bar{d})$ emanating in $\{p\}_n \in \bar{X}$. We call $\bar{\gamma}$

the ultraray associated to γ (and ω).

Weak angles: In order to get a grip on the interplay between geodesics and their associated ultrarays, we recall certain notions of angles.

Given three points p, x and y in a metric space X, consider corresponding comparison points p', x' and y' in the Euclidean plane $\mathbb{E}^2$. Let $[p', x']$ and $[p', y']$ denote the geodesic segments in $\mathbb{E}^2$ connecting p' to x' and p' to y'. These segments enclose an angle in p' and this angle is referred to as the (Euclidean) comparison angle of x and y at p. We write $\angle_p(x, y)$ for this angle.

Let now X be a metric space and consider two geodesic segments γ_1 and γ_2 parameterized by arclength, both initiating in some $p \in X$. Then γ_1 and γ_2 are said to enclose the angle $\angle_p(\gamma_1, \gamma_2)$ (in the strict sense) at p if the limit $\angle_p(\gamma_1, \gamma_2) := \lim\limits_{s,t \to 0} \angle_p(\gamma_1(s), \gamma_2(t))$ exists.

Recall that for instance a normed vector space is an inner product space if and only if all straight line segments emanating from the origin enclose an angle. However, even in normed vector spaces that are not inner product spaces certain so called generalized angles do exist between any straight line segments initiating in a common point. Such generalized angles were introduced in [14].

Let $a, b > 0$ and γ_1 and γ_2 be as above. then we say that γ_1 and γ_2 enclose a generalized angle $\angle_p^g(\gamma_1, a, \gamma_2, b)$ at scale (a, b), if the limit

$$\angle_p^g(\gamma_1, a, \gamma_2, b) := \lim_{s \to 0} \angle_p(\gamma_1(as), \gamma_2(bs))$$

exists. If γ_1 and γ_2 enclose generalized angles at all scales (a, b) and, moreover, these generalized angles do not depend on the particular scale, then we say that γ_1 and γ_2 enclose the weak angle

$$\angle_p^w(\gamma_1, \gamma_2) := \angle_p^g(\gamma_1, 1, \gamma_2, 1).$$

Ultrarays in Busemann convex spaces: Now let X be Busemann convex and γ_1 and γ_2 be geodesics on X with $\gamma_1(0) = p = \gamma_2(0)$. Then, for all scales (a, b), the generalized angle $\angle_p^g(\gamma_1, a, \gamma_2, b)$ exists. This is immediate, since $s \mapsto \frac{|\gamma_1(as)\gamma_2(bs)|}{s}$ is monotonously increasing.

Next consider the ultrarays $\overline{\gamma_1}$ and $\overline{\gamma_2}$ associated to γ_1 and γ_2. These ultrarays satisfy

$$(7) \qquad \bar{d}\Big(\overline{\gamma_1}(as), \overline{\gamma_2}(bs)\Big) = s \cdot \bar{d}\Big(\overline{\gamma_1}(a), \overline{\gamma_2}(b)\Big) \qquad \forall a, b, s > 0.$$

Moreover, the existence of weak angles of geodesics γ_1 and γ_2 in a Busemann convex space is equivalent to the existence of angles (in the strict sense) between their associated ultrarays $\overline{\gamma_1}$ and $\overline{\gamma_2}$ in $\bar{X}$.

Lemma 5.1. *Let X be Busemann convex, let γ_1 and γ_2 denote geodesics in X initiating in a common point $p \in X$ and let $\overline{\gamma_1}$ and $\overline{\gamma_2}$ denote their associated ultrarays. Then the following properties are mutually equivalent.*

(1) *The rays γ_1 and γ_2 enclose a weak angle.*
(2) *The ultrarays $\overline{\gamma_1}$ and $\overline{\gamma_2}$ enclose an angle (in the strict sense).*
(3) *The union $\overline{\gamma_1}(\mathbb{R}^+) \cup \overline{\gamma_2}(\mathbb{R}^+)$ admits an isometric embedding into the Euclidean plane $\mathbb{E}^2$.*

Proof. The equivalence of (2) and (3) follows immediately from Equation 7. Moreover, this equation also implies that the ultrarays $\overline{\gamma_1}$ and $\overline{\gamma_2}$ enclose an angle if and only if they enclose a weak angle. Hence the equivalence of (1) and (2) is a consequence of $\frac{1}{s}\bar{d}(\overline{\gamma_1}(as), \overline{\gamma_2}(bs)) = \lim_{\omega}\{\frac{1}{n}|\gamma_1(a/n)\gamma_2(b/n)|\}_n$, and the fact that the generalized angles between γ_1 and γ_2 exist for all scales in any case. $\square$

Proposition 5.2. *Let X be Busemann convex. Assume that for all geodesic segments γ_1 and γ_2 with $\gamma_1(0) = p = \gamma_2(0)$, the weak angle $\angle_p^w(\gamma_1, \gamma_2)$ exists. Then X is a CAT(0)-space.*

Proof. We first show that the weak angle satisfies the four axioms of an angle as formulated in [3] II.1.8 (p. 162). Thus we have to show
(A1) $\angle_p^w(\gamma_1, \gamma_2) = \angle_p^w(\gamma_2, \gamma_1)$
(A2) $\angle_p^w(\gamma_1, \gamma_3) \leq \angle_p^w(\gamma_1, \gamma_2) + \angle_p^w(\gamma_2, \gamma_3)$
(A3) if γ_2 is the restriction of γ_1 to an initial segment, then $\angle_p^w(\gamma_1, \gamma_2) = 0$
(A4) if the concatenation of $\gamma_1 = [p, x]$ and $\gamma_2 = [p, y]$ is a geodesic $[x, y]$, then $\angle_p^w(\gamma_1, \gamma_2) = \pi$
Now (A1), (A3), (A4) are trivially true by the definition. Since weak angles between rays coincide (by Lemma 5.1) with the angles of their associated ultrarays and since such angles satisfy the triangle inequality also (A2) holds. Furthermore we see that $\angle_p^w([p, x], [p, y]) \leq \angle_p(x, y)$, which follows from the Busemann convexity. Now implication (4) $\implies$ (2) on p.161 of II.1.7 in [3] remains valid, if one replaces the Alexandrov angle in condition (4) through the weak angle (cf. the paragraph after II.1.8 in [3]). This implies that X is a CAT(0)-space. $\square$

5.2. **A convex hull proposition.** The purpose of this subsection is to prove

Proposition 5.3. *(compare [16], p. 432, par. 7 and p. 463, par.20 as well as [2], Lemma 1.1. and the remark after its proof) Let X be Busemann convex and let $\gamma_1, \gamma_2 : I \longrightarrow X$ be two linearly reparameterized (finite or infinite) geodesics in X such that $t \mapsto |\gamma_1(t)\gamma_2(t)|$ is affine. Then the convex hull of γ_1 and γ_2 is a convex subset of a two-dimensional normed vector space.*

Given a geodesic metric space X, a function $f : X \longrightarrow \mathbb{R}$ is called *affine* if its restriction to each affinely parameterized geodesic γ in X satisfies $f(\gamma(t)) = at + b$ for some numbers $a, b \in \mathbb{R}$ that may depend on γ. We say that affine functions on X *separate points*, if for each pair of distinct points $x, x' \in X$ there is an affine function $f : X \longrightarrow \mathbb{R}$ with $f(x) \neq f(x')$. With this terminology the following theorem has been proven in [9].

Theorem 5.4. *(Theorem 1.1 in [9]) Let X be a geodesic metric space. If affine functions on X separate points then X is isometric to a convex subset of a normed vector space with a strictly convex norm.*

Using this result, we are able to provide the

Proof of Proposition 5.3: Let $y_t : I_t \longrightarrow X$ be the geodesic from $\gamma_1(t)$ to $\gamma_2(t)$ where $I_t = [0, |\gamma_1(t)\gamma_2(t)|]$. Let $C_0 := \bigcup_{t \in I} y_t(I_t)$.

Claim: C_0 is convex, i.e., $C_0 = C$.

In order to prove this claim, we may assume that γ_1 and γ_2 are closed. Then C_0 is closed and it is sufficient to prove that for $t_1, t_2 \in I$, $s_1 \in I_{t_1}$ and $s_2 \in I_{t_2}$ the midpoint of $y_{t_1}(s_1)$ and $y_{t_2}(s_2)$ is contained in C_0.

Let m denote the unique midpoint map on X and set

PSfrag replacements $m_0 := m(y_{t_1}(s_1), y_{t_2}(s_2))$, $\quad m_1 := \gamma_1(\frac{t_1+t_2}{2})$, $\quad$ and $\quad m_2 := \gamma_2(\frac{t_1+t_2}{2})$.
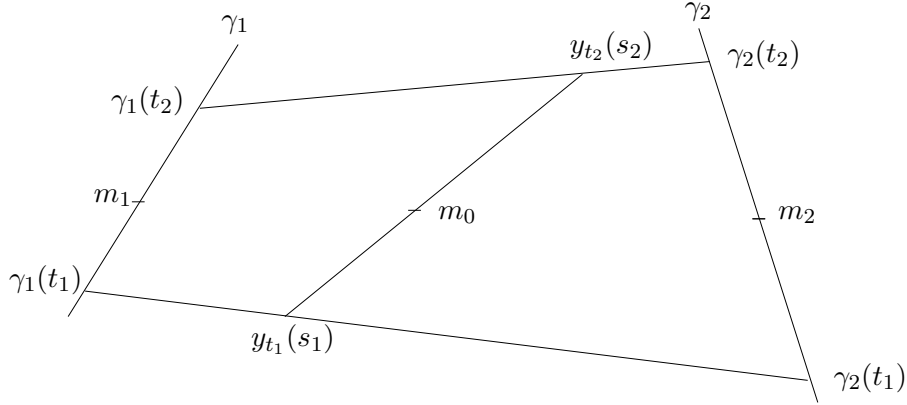

FIGURE 2. This figure visualizes our notation used in the proof of Proposition 5.3.

From the convexity of the distance function and the fact that

$$|m_1 m_2| = \frac{1}{2}\Big[|\gamma_1(t_1)\gamma_2(t_1)| + |\gamma_1(t_2)\gamma_2(t_2)|\Big],$$

we deduce that $|m_1 m_2| = |m_1 m_0| + |m_0 m_2|$. Thus m_0 is contained in the geodesic $y_{\frac{t_1+t_2}{2}}$. This proves the claim.

In fact, with the same reasoning as above we deduce more, namely that $|m_0 m_1| = \frac{s_1+s_2}{2}$. This shows that the function $F_1 : C \longrightarrow \mathbb{R}$, given through $F_1(y_t(s)) = s$ is affine on C.

Moreover, the function $F_2 : C \longrightarrow \mathbb{R}$, given through $F_2(y_t(s)) = t$ is affine as well, since the midpoint between $y_{t_1}(s_1)$ and $y_{t_2}(s_2)$ lies on the geodesic $y_{\frac{t_1+t_2}{2}}$.

Now the affine functions F_1 and F_2 separate the points of C. Thus we can apply Theorem 5.4, which finishes the proof. $\qquad\square$

5.3. **Proof of Theorem 1.3.** Every CAT(0)-space is both, Ptolemy and Busemann convex. It remains to show that a Busemann convex and Ptolemy metric space is already CAT(0).

In order to reach a contradiction, suppose that X is Ptolemy and Busemann convex but not CAT(0). Then, due to Proposition 5.2 there do exist two geodesics γ_1 and γ_2 that do not enclose a weak angle at their common starting point $p = \gamma_1(0) = \gamma_2(0)$. Let $\overline{\gamma_1}$ and $\overline{\gamma_2}$ denote the geodesic rays defined in $Y = \lim_\omega \{nX, p\}_n$ as above. Then, due to Lemma 5.1, $\overline{\gamma_1}$ and $\overline{\gamma_2}$ do not enclose an angle in $\{p\}_n \in Y$ either.

Now Y is Busemann convex by Corollary 4.3. Moreover, the function $t \mapsto$

$|\overline{\gamma_1}(t)\overline{\gamma_2}(t)|$ is linear. Thus, by Proposition 5.3, the convex hull C of $\overline{\gamma_1}$ and $\overline{\gamma_2}$ is isometric to a convex set of a two-dimensional normed vector space. Since Y is Ptolemy, C is flat by Corollary 2.1. It follows that $\overline{\gamma_1}$ and $\overline{\gamma_2}$ enclose an angle, which yields the desired contradiction. $\qquad\square$

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