ON THE GEOMETRY OF SUBSETS OF POSITIVE REACH

ALEXANDER LYTCHAK

ABSTRACT. We prove that sets of positive reach in Riemannian manifolds and more generally, almost convex subsets in spaces with an upper curvature bound have an upper curvature bound with respect to the inner metric.

1. INTRODUCTION

The aim of this paper is to prove that almost convex subsets of spaces with an upper curvature bound have an upper curvature bound with respect to the inner metric. As special case we obtain:

THEOREM 1.1. *Let M be a smooth Riemannian manifold, Z a compact subset of M that has positive reach. Then Z has an upper curvature bound with respect to the inner metric.*

Recall that a subset Z of a manifold M is said to have positive reach $\geq r$ if for all $x \in M$ with $d(x, Z) \leq r$ there is a unique point $p \in X$ with $d(x, p) = d(Z, x)$ ([Fed59]). For example compact convex subsets or $\mathcal{C}^{1,1}$ submanifolds with boundary have positive reach. Thus our theorem is a generalization of [ABB93].

In [Lyt] we prove that a subset Z of M has positive reach $r > 0$ iff it is $(C, 2, \rho)$-embedded in the sense of the next definition for positive C, ρ depending on r and the curvature bounds of M.

Definition 1.1. A subset Z of a metric space X is called (C, α, ρ)-embedded if for all $z_0, z_1 \in Z$ with $s = d(z_0, z_1) < \rho$ there is a point $m \in Z$ with $d(z_i, m) \leq \frac{s}{2}(1 + Cs^{\alpha})$.

Therefore Theorem 1.1 is implied by the following more general:

THEOREM 1.2. *Let X be a space with an upper curvature bound κ. Let Z be a $(C, 2, \rho)$-embedded subset of X. Then Z with respect to its inner metric is a $CAT(k)$ space with $k = k(C, \kappa, \rho)$.*

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A regular sublevel set $Z = f^{-1}[-\infty, t]$ of a semi-convex function f on a $CAT(\kappa)$ space X satisfies the assumptions of Theorem 1.2. Since $CAT(\kappa)$ spaces admits many semi-convex functions we get a lot of new spaces with an upper curvature bound. As an example we get:

Corollary 1.3. *Let X be a $CAT(1)$ space, $x \in X$, $r < \pi$. Then the closed ball $B_r(x)$ around x has an upper curvature bound with respect to the inner metric.*

Remark 1.1. As the proof shows the curvature bound of Z depends only on C and κ. We have not tried to get the optimal curvature constants as it was done in [ABB93]. However, it seems to be possible to go more carefully through the proofs and to obtain optimal results.

Remark 1.2. Theorem 1.1 can be proved in an easier way. One has to approximate sets of positive reach by tubular neighborhoods and apply the result of [ABB93] to these tubes observing that they are manifolds with boundary.

To prove Theorem 1.2 we proceed as follows. An upper curvature bound for Riemannian manifolds can be expressed by some precise semi-convexity of normal Jacobi fields. In general metric spaces no differential tools such as Jacobi fields are available and one has to work directly with distance functions. One observes that the existence of an upper curvature bound on a metric space X is essentially equivalent to the statement that the distance d, considered as a function on the geodesic space $X \times X$, is $B \cdot d$ semi-convex for some constant $B \in \mathbb{R}$, i.e. for the restriction of d to each geodesic $d'' \geq Bd$ holds.

The idea is now to work not with the inner metric d^Z on Z but to consider the restriction of the metric $d : X \times X \to \mathbb{R}$ to $Z \times Z$. This restriction is (up to a factor) equal to the distance to the diagonal Δ in $X \times X$. We know from [Lyt] that the restriction h of d to $Z \times Z$ is semi-convex, i.e. we have $h'' \geq A$ for some (negative) constant A. However, this is not enough to conclude the proof, and we need the deeper estimation $h'' \geq Bh$.

To achieve this we observe that if the restriction of h onto a geodesic γ in the inner metric of $Z \times Z$ is not convex, we get a curve μ in $X \times X$ with the same endpoints of a smaller length. Choosing this curve in the right way we see that the distance between this new curve μ and $Z \times Z$ is very small (much smaller then the distance between μ and γ). Thus projecting μ onto $Z \times Z$ we get a curve $\bar{\gamma}$ in $Z \times Z$ with the same endpoints, and the statement that its length is not less then the length of γ implies the right semi-convexity constant for h.

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2. Preliminaries

In a metric space X a geodesic or more precisely an X-geodesic will denote always a (globally) isometric embedding of an interval into X. For a geodesic space X and a subset Z of X we will denote by d the metric on X and its restriction to $Z \times Z$ and by d^Z the inner metric on Z. For a subset K of a metric space X we denote by d_K the distance function $d_K(x) = d(K, x)$.

For Lipschitz functions $h, \tilde{h}$ on a geodesic space X the function h will be called $\tilde{h}$-convex (and written $h'' \geq \tilde{h}$), if for each geodesic γ in X holds $(h \circ \gamma)'' \geq \tilde{h} \circ \gamma$ in the weak sense. We refer to [BBI01] and [AB03] for more on this subject.

By a $CAT(\kappa)$ space we denote a complete metric space, in which all points with distance at most $\frac{\pi}{\sqrt{\kappa}}$ are connected by a geodesic and such that triangles are not thicker than triangles in the space M^2_κ of constant curvature κ. We refer to [BH99] for the theory of $CAT(\kappa)$ spaces.

We will use the following easy observation. Let X be a $CAT(0)$ space, μ a geodesic between x_0 and x_1, $p \in X$ a point. If $d(p, \mu) = r$, then $d(x_0, p) + d(x_1, p) \geq \sqrt{L(\mu)^2 + 4r^2}$.

3. Simplifications

First of all we simplify our task a little bit and recall some results from [Lyt]. Assume that we are under the assumptions of Theorem 1.2. By rescaling the metric we may assume that X is a $CAT(1)$ space. Considering now Z as a subset of the Euclidean cone CX and using the fact that X is a $(1, 2)$-embedded subset of the Euclidean cone CX, we may assume that X is a $CAT(0)$ space. (The constants C and ρ will certainly change by this procedure). The intersection of Z with each ball $B_r(x)$ of sufficiently small radius r is $(C, 2, \rho)$-embedded in $B_r(x)$, hence we may assume that the diameter of X is arbitrary small. In particular we may assume that the diameter is smaller than ρ and forget about ρ.

By [Lyt] there is a number $r_0 = r_0(C)$, such that for each $x \in X$ with $d(x, Z) < r_0$ there is a unique point $p = P(x) \in Z$ such that $d(x, Z) = d(x, p)$. Moreover the map P is Lipschitz and the Lipschitz constant of P at x is not bigger than $1 + A \cdot d(x, Z)$ for some $A = A(C) > 0$. We will assume that the diameter of X is smaller than r_0.

We know ([Lyt]) that the inner metric d^Z on Z is bounded by $d^Z \leq d(1+\bar{C}d^2)$ for some $\bar{C} = \bar{C}(C)$. We may assume (making A bigger), that $\bar{C}$ coincides with the constant A from above and that $d^Z \leq 2d$ holds.

Replacing X by its ultraproduct X^ω and Z by $Z^\omega \subset X^\omega$ we may assume that (Z, d^Z) is a geodesic space. In this step we use the fact that a complete metric space Z is $CAT(k)$ if and only if its ultraproduct X^ω is $CAT(k)$. This step is redundant if X is a proper space.

Each convex subset (in particular each Z-geodesic) in (Z, d^Z) is again $(A, 2)$-embedded. For each 1-Lipschitz convex function $f : X \to \mathbb{R}$ its restriction to (Z, d^Z) is $(-\tilde{A})$-convex, for some $\tilde{A} = \tilde{A}(A)$.

By [Lyt] we know that each Z-geodesic γ starting in a point x defines a unique X-direction $v = \gamma' \in C_x X$. The angle between each two Z-geodesics starting at the same point x measured in the inner metric space (Z, d^Z) is well defined and coincides with the angle between the starting directions of these two curves in X. For some constant A_1 the angle between each Z-geodesic $\gamma : [0, t] \to Z$ and the X-geodesic $\bar{\gamma}$ connecting $\gamma(0)$ and $\gamma(t)$ satisfies $\angle(\gamma^+, \bar{\gamma}^+) \leq A_1 t$.

Finally the distance function to $d_Z : X \to \mathbb{R}$ to the subset Z is A_2-convex in the δ-tube around Z for some small number δ. We assume that X has diameter $\leq \delta$ and increasing A we assume $A = A_1 = A_2 = \tilde{A}$. These assumptions imply that for $x_0, x_1 \in Z$ with $d(x_0, x_1) = t$ and the X-geodesic γ between x_0 and x_1 we have $d(Z, \gamma(\epsilon)) \leq 2A\epsilon t$, if δ has been chosen sufficiently small.

4. Generalizing the theorem of Pythagoras

Let X be a $CAT(0)$ space, K a closed convex subset of X, $\gamma : [a, b] \to X$ an arclength parametrized curve between $x_0 = \gamma(a)$ and $x_1 = \gamma(b)$. Denote by f the distance function d_K and by h its restriction $h = f \circ \gamma$ to γ. Let $\tilde{\gamma}$ be the projection of γ onto K, i.e. we have $d(\tilde{\gamma}(t), \gamma(t)) = h(t)$. Let η_t be the geodesic between $\gamma(t)$ and $\tilde{\gamma}(t)$.

Lemma 4.1. *In the above notations assume that* $r = h(\frac{a+b}{2}) - \frac{h(a)+h(b)}{2} > 0$. *Then there is a curve* $\mu : [a, b] \to X$, *such that* $\mu(t) \in \eta_t$, $\mu(a) = \gamma(a) = x_0$, $\mu(b) = \gamma(b) = x_1$ *and such that* $L(\gamma) \geq \sqrt{L(\mu)^2 + 4r^2}$.

Proof. Consider the union of the geodesics η_t as a ruled surface $i : D \to X$. Denote by $\tilde{d}$ the pulled-back metric $i^* d$ on D. Due to [Ale57] (see also [Pet99] for a more general case) the space $(D, \tilde{d})$ is a compact $CAT(0)$ space. Since the projection of X onto K is 1-Lipschitz the curve $\tilde{\gamma}$ becomes a (not arclength parametrized) geodesic in D. Moreover the function $\tilde{h}(t) := \tilde{d}(\gamma(t), \tilde{\gamma})$ coincides with f. Since the map

$i : D \to X$ is 1-Lipschitz and preserves the length of γ we may replace X by D.

Let μ be the (arclength parametrized) geodesic (in D) between x_0 and x_1. Then μ certainly lies on the union of the geodesics η_t and can be parametrized in the needed way. Thus we only have to estimate $L(\mu)$.

Since the distance function (in D) to the convex subset $\tilde{\gamma}$ is convex, for the midpoint m of μ holds $d(\tilde{\gamma}, m) \leq \frac{h(a)+h(b)}{2}$. Hence for $\bar{m} = \gamma(\frac{a+b}{2})$ we obtain $d(m, \bar{m}) \geq r$.

Now $\angle \bar{m} m x_i \geq \frac{\pi}{2}$ for $i = 0$ or $i = 1$. For this i we get $\tilde{d}(x_i, \bar{m}) \geq \sqrt{\tilde{d}(x_i, m)^2 + r^2}$. Since $L(\gamma) \geq 2\tilde{d}(x_i, m)$ and $L(\mu) = 2\tilde{d}(x_i, m)$ we obtain the desired inequality. □

Let X, K, γ, μ be as above. Assume that γ is in addition $(C, 2)$-embedded in X. Then γ, considered as a curve in D, is also $(C, 2)$-embedded. Therefore if the length of γ is small enough, then for each $s \in [a, b]$ the angles between γ and the geodesics in D connecting two points on γ are at most $\frac{\pi}{12}$.

Denote by $l : [a, b] \to \mathbb{R}$ the nonnegative function $l(t) = d(K, \gamma(t)) - d(K, \mu(t))$. Set $l = \max_{t \in [a,b]} l(t)$.

Lemma 4.2. *Under the assumptions of Lemma 4.1, assume in addition that h is $\frac{1}{\sqrt{2}}$-Lipschitz and that γ is $(C, 2)$-embedded and of sufficiently small length. Then $L(\gamma) \geq \sqrt{L(\mu)^2 + l^2}$ holds.*

Proof. Again we may replace X by the ruled surface D and assume that μ is a geodesic (up to reparametrization). Let t be such that $l(t) = l$. Set $p = \gamma(t)$ and $q = \mu(t)$. Let α_0 resp. α_1 be the angle between pq and qx_0 resp. between pq and qx_1. Let β_0 resp. β_1 be the angles between qp and γ. The assumption $|h'(t)| \leq \frac{1}{\sqrt{2}}$ implies that $\beta_i \leq \frac{3\pi}{4}$. Since $\beta_0 + \beta_1 \geq \pi$ we obtain $\beta_i \geq \frac{\pi}{4}$.

In the triangle px_iq we see that $\angle x_i pq \geq \frac{\pi}{4} - \frac{\pi}{12} = \frac{\pi}{6}$. Thus we obtain $\alpha_i \leq \pi - \angle x_i pq \leq \pi - \frac{\pi}{6}$. Since $\alpha_0 + \alpha_1 \geq \pi$ we get $\alpha_i \geq \frac{\pi}{6}$.

Therefore $\tilde{d}(p, \mu) \geq l \sin(\frac{\pi}{6}) = \frac{l}{2}$. The remark in the preliminaries implies $L(\gamma) \geq \tilde{d}(x_0, p) + \tilde{d}(x_1, p) \geq \sqrt{L(\mu)^2 + l^2}$. □

5. The projections

5.1. **Setting.** We assume that we are under the assumptions made in Section 3.

Consider the direct product $X \times X$ with the product metric. Denote by Δ the diagonal of $X \times X$. It is a convex subset of the $CAT(0)$ space

$X \times X$. The distance function $f := d_\Delta$ to the diagonal is given by $f((x,y)) = d^{X\times X}((x,y),\Delta) = \frac{d(x,y)}{\sqrt{2}}$. The projection $P^\Delta : X \times X \to \Delta$ is given by $P^\Delta((x,y)) = (m,m)$ where m is the midpoint of the geodesic $\eta : [0,a] \to X$ between x and y. Moreover the geodesic $\tilde{\eta} : [0, \frac{a}{\sqrt{2}}] \to X \times X$ between (x,y) and (m,m) is given by $\tilde{\eta}(t) = (\eta(\frac{t}{\sqrt{2}}), \eta(\frac{a-t}{\sqrt{2}}))$.

The subset $Z \times Z$ is $(A,2)$-embedded in $X \times X$ and $(Z \times Z, d^Z \times d^Z) = (Z, d^Z) \times (Z, d^Z)$. By the last remark in Section 3 we see, that for a point $z = (z_1, z_2) \in Z \times Z \subset X \times X$ the geodesic $\eta : [0, \frac{d(z_1,z_2)}{\sqrt{2}}] \to X \times X$ between z and Δ satisfies $d(\eta(s), Z \times Z) \leq 2Asd(z_1, z_2)$.

Since the function f is convex the restriction of f to each geodesic γ in $Z \times Z$ is A-convex. To prove Theorem 1.2 we have to improve the convexity constant of f.

5.2. **First projection.** Let $\gamma : [t-\epsilon, t+\epsilon] \to Z \times Z$ be a $d^{Z\times Z}$-geodesic for a small ϵ. Assume that $|(f \circ \gamma)'| \leq \frac{1}{\sqrt{2}}$. Denote by r the value $r = f(t) - \frac{f(t+\epsilon)+f(t-\epsilon)}{2}$ and assume $r > 0$.

Due to Lemma 4.1 we obtain a curve $\mu : [t-\epsilon, t+\epsilon] \to X \times X$ connecting $\gamma(t+\epsilon)$ and $\gamma(t-\epsilon)$ such that $L(\gamma[t-\epsilon, t+\epsilon]) \geq \sqrt{L(\mu)^2 + 4r^2}$. Moreover for $\mu : [a,b] \to X \times X$ the point $\mu(t)$ lies on the geodesic η_t between $\gamma(t)$ and Δ.

Due to Lemma 4.2 for $l = \max(f(\gamma(t)) - f(\mu(t)))$ holds $L(\gamma) \geq \sqrt{L(\mu)^2 + l^2}$. Since $L(\gamma) \leq L(\mu) + AL(\mu)^3$, we see that r and l are very small in comparison to $L(\mu)$.

Hence we obtain $L(\gamma) \geq L(\mu)\sqrt{1 + \frac{4r^2}{L(\mu)^2}} \geq L(\mu)(1 + \frac{r^2}{L(\mu)^2})$. In the same way we see $L(\gamma) \geq L(\mu)(1 + \frac{l^2}{4L(\mu)^2})$. Using $L(\mu) \leq 2\epsilon$ we obtain

(1) $L(\mu) \leq L(\gamma)(1 - \frac{r^2}{10\epsilon^2})$;
(2) $L(\mu) \leq L(\gamma)(1 - \frac{l^2}{20\epsilon^2})$.

5.3. **Second projection.** Since the point $\mu(t)$ lies on the geodesic η_t between $\gamma(t) \in Z \times Z$ and Δ we obtain from Subsection 5.1: $d(\mu(t), Z \times Z) \leq 2Al\sqrt{2}f(\gamma(t))$. Therefore if $f(\gamma(t)) \leq D$ we see that the projection of μ onto $Z \times Z$ is a curve $\bar{\gamma}$ connecting $\gamma(t-\epsilon)$ and $\gamma(t+\epsilon)$ and satisfying $L(\bar{\gamma}) \leq L(\mu)(1 + 2\sqrt{2}A^2 lD)$. Since $L(\bar{\gamma}) \geq L(\gamma)$ we obtain

(1) $(1 - \frac{r^2}{10\epsilon^2}) \cdot (1 + 2\sqrt{2}A^2 lD) \geq 1$;
(2) $(1 - \frac{l^2}{20\epsilon^2}) \cdot (1 + 2\sqrt{2}A^2 lD) \geq 1$.

Separating the cases $r \geq l$ and $r \leq l$ we obtain in both cases that $r \leq BD\epsilon^2$, for some constant B. Thus we have proved:

Lemma 5.1. *Let $\gamma : [a,b] \to Z\times Z$ be a $d^{Z\times Z}$-geodesic. If the inequality $|(f\circ\gamma)'| \leq \frac{1}{\sqrt{2}}$ holds for all $t \in [a,b]$, then $(f\circ\gamma)'' \geq -B(f\circ\gamma)$ for some fixed positive constant B.*

6. The conclusion

Let Z, X be as in the last section. Since the function f is a multiple of the metric $d : X \times X \to \mathbb{R}$ we see, that the restriction of d to $(Z\times Z, d^{Z\times Z})$ is a $\sqrt{2}$-Lipschitz function, that satisfies $d'' \geq -A$ and for each $Z\times Z$-geodesic γ the assumption $|(d\circ\gamma)'| \leq 1$ implies the much stronger semi-convexity $(d\circ\gamma)'' \geq -B(d\circ\gamma)$ for some positive constant $B = k^2$. We may assume that the diameter of X is at most $\frac{\pi}{2\sqrt{k}}$. We are going to prove that Z is a $CAT(k)$ space.

Remark 6.1. Actually we prove that Z is a $CAT(\frac{k}{2})$ space.

Let $\gamma_1 : [0,a] \to Z, \gamma_2 : [0,b] \to Z$ be d^Z-geodesics starting at the same point x. Recall that the angle at x in the inner metric space (Z, d^Z) between γ_1 and γ_2 is well defined and coincides with the angle between these two curves in X. We have to prove that $d^Z(\gamma_1(a), \gamma_2(b))$ is not less then the corresponding distance for the comparison hinge in the simply connected surface M_k^2 of constant curvature k.

Let x be a fixed point and let γ_1 be fixed, $y = \gamma_1(a)$. It is enough to prove that for some $\epsilon > 0$ (that may depend on x and γ_1) each point $z \in X$ with $d(y,z) < \epsilon$ and each Z-geodesic γ_2 connecting x and z the comparison angle at x in the space M_k^2 is not less than the angle between γ_1 and γ_2.

Consider $\bar\gamma_1 : [0,a] \to M_k^2$ and $\bar\gamma_2 : [0,b] \to M_k^2$ geodesics in M_k^2 starting at the same point $\bar x$ and such that $d(\bar\gamma_1(a), \bar\gamma_2(b)) = d(\gamma_1(a), \gamma_2(b))$. Remark that on the right hand side of the equation we consider the induced distance d of X and not the inner distance d^Z. Since $d^Z \geq d$ it is enough to show that the angle between γ_1 and γ_2 is not bigger than the angle between $\bar\gamma_1$ and $\bar\gamma_2$.

Consider the $Z\times Z$-geodesic $\gamma : [0, \sqrt{a^2+b^2}] \to Z\times Z$ given by $\gamma(t) = (\gamma_1(\frac{at}{\sqrt{a^2+b^2}}), \gamma_2(\frac{bt}{\sqrt{a^2+b^2}}))$. Denote by $\bar\gamma$ the corresponding geodesic in $M_k^2 \times M_k^2$. Finally set $h = d^{X\times X}\circ\gamma$ and $\bar h = d\circ\bar\gamma$.

By definition we have $h(0) = \bar h(0) = 0$ and $h(\sqrt{a^2+b^2}) = \bar h(\sqrt{a^2+b^2})$. If ϵ was chosen small enough in comparison to a, we get $\bar h'' \leq -k^2 \bar h$ for all t. Moreover choosing ϵ small enough we may achieve $|\bar h'| \leq \frac{1}{2}$.

On the other hand we have $h'' \geq -A$ and if $|h'| \leq 1$ on some interval, then $h'' \geq -k^2 h$ on this interval. Assume $h'(t_0) < -1$ for some t_0. Then the semi-convexity and the small size of X show, that $h'(t) < -\frac{1}{2}$ for all $t < t_0$. In particular $h(0) > h(t_0)$. But $h(0) = 0$ and this is not possible. On the other hand if $h'(t_0) \geq 1$ (which certainly can occur), we get $h'(t) \geq \frac{1}{2} \geq \bar{h}'(t)$ for all $t > t_0$.

Assume that the angle between γ_1 and γ_2 is bigger than the angle between $\bar{\gamma}_1$ and $\bar{\gamma}_2$. This amounts to $h'(0) > \bar{h}'(0)$. Taking now all the comparison results, the standard arguments provide $h(t) > \bar{h}(t)$ for all t and thus a contradiction to $h(\sqrt{a^2+b^2}) = \bar{h}(\sqrt{a^2+b^2})$.

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Mathematisches Institut, Universität Bonn, Beringstr.1, 53115 Bonn, Germany,

E-mail address: lytchak@math.uni-bonn.de