STRUCTURE OF SUBMETRIES

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ABSTRACT. We investigate the geometric and topological structure of equidistant decompositions of Riemannian manifolds.

1. INTRODUCTION

1.1. **Subject of investigations.** An *equidistant decomposition* $\mathcal{F}$ of a metric space X is a decomposition of X into a collection of pairwise equidistant closed subsets $L_i, i \in I$, called the *leaves* of the decomposition. The *space of leaves* I of $\mathcal{F}$ can be equipped with a natural distance, such that the canonical projection $P : X \to I$ is a *submetry*, that is a map that sends metric balls in X to metric balls in I of the same radius. On the other hand, the fibers of any submetry $P : X \to Y$ provide an equidistant decomposition of the space X. Basic examples of submetries are given by Riemannian submersions and quotient maps under proper isometric group actions.

Submetries were defined by V. Berestovskii in [Ber87]; in [BG00] it was proved that a map $P : M \to N$ between complete, smooth Riemannian manifolds is a submetry if and only if P is a $\mathcal{C}^{1,1}$ Riemannian submersion. Other large classical sources of equidistant decompositions are provided by the decompositions into orbits of isometric group actions and *singular Riemannian foliations* with closed leaves. Singular Riemannian foliations, defined by P. Molino, [Mol88], include as subclasses many famous foliations in Riemannian geometry, like the isoparametric foliations, and have been actively investigated recently from geometric, topological, analytic and algebraic points of view, [Tho10], [LT10], [Rad14], [GGR15], [AR17], [MR19], [MR20]. Submetries often appear in connections with rigidity phenomena, see [GG87], [Per94b], [Lyt05b], [Wil07]. Conjecturally, collapsing of manifolds with lower curvature bounds is modelled by submetries, see

[Yam91], [CFG92], [GK95] [Kap05], [SY00], [Yam12]. Recent appearance of submetries in several completely unrelated settings, [BN13], [LPZ18], [GW18], [GGKMS18], [BN19], [MR20], further motivates a systematic study of the subject.

1.2. **Main results.** The main objective of the present paper is the description of the structure of equidistant decompositions of Riemannian manifolds. Equivalently, we describe the structure of possible spaces of leaves Y and of submetries $P : M \to Y$ where M is a Riemannian manifold. All Riemannian manifolds appearing in the paper are assumed to be sufficiently smooth, in particular, they have *local two sided curvature bounds* in the sense of Alexandrov. A sufficient (and almost necessary) condition is that the Riemannian metric is $\mathcal{C}^{1,1}$ in some coordinates, [BN93], [KL20]. Most results are local and do not require completeness of M. In fact they are valid for *local submetries*, see Subsection 2.4.

The first theorem provides a characterization of possible leaves, see Proposition 6.3 and Remark 6.4 for a local converse statement.

Theorem 1.1. *Let M be a Riemannian manifold. Any fiber L of any submetry $P : M \to Y$ is a set of positive reach in M.*

Recall that a subset L of M has *positive reach* if for some neighborhood U of L in M and any $x \in U$ there exists a unique *foot point* $\Pi^L(x) \in L$ closest to x in L, [Fed59]. The structure of sets of positive reach is well understood, [Fed59], [Kle81], [Ban82], [Lyt04b], [Lyt05c], [RZ17]; some features are summarized in Section 6.

In general, even for very nice manifolds M, some leaves of P may be non-manifolds and have rather complicated local topological structure, see Section 6 for examples. However, most leaves are manifolds, and any submetry is a Riemannian submersion on a large set:

Theorem 1.2. *Let M be a Riemannian manifold and let $P : M \to Y$ be a submetry. Then Y has an open, convex, dense subset Y_{reg}, locally isometric to a Riemannian manifold with a Lipschitz continuous Riemannian metric, and $P : P^{-1}(Y_{reg}) \to Y_{reg}$ is a $\mathcal{C}^{1,1}$ Riemannian submersion.*

Here and below a $\mathcal{C}^1$ map is called $\mathcal{C}^{1,1}$ if its differentials depend *locally* Lipschitz continuously on the point. Even for smooth manifolds M and Y, $\mathcal{C}^{1,1}$ regularity in Theorem 1.2 is optimal, see Example 6.9.

It was already observed in [BGP92], that a lower bound on curvature in the sense of Alexandrov is preserved under submetries. Localizing the argument, we see that a space of leaves Y as in Theorem 1.1 is an

Alexandrov region, a length space in which every point has a compact convex neighborhood isometric to an Alexandrov space, see Section 3 below and [LN20]. We can therefore use all the terminology established in Alexandrov spaces. In particular, we have spaces of directions $\Sigma_y Y$ and tangent cones $T_y Y$ at $y \in Y$, which are Alexandrov spaces of curvature ≥ 1, respectively ≥ 0. Also the notions of *boundary* and *extremal subsets*, [Pet07], on any Alexandrov region are well-defined.

The space of leaves has a much more special structure than a general Alexandrov space.

Theorem 1.3. *Let M be a Riemannian manifold, let $P : M \to Y$ be a submetry and let $y \in Y$ be a point. Then there exists some $r = r(y) > 0$ such that the following holds true.*

    (1) *A geodesic of length r starts in every direction $v \in \Sigma_y Y$.*
    (2) *For any $s \leq r$, the closed ball $\bar{B}_s(y)$ is strictly convex in Y.*
    (3) *For any $s \leq r$, the boundary $\partial B_s(y)$ is an Alexandrov space.*

Theorem 1.3.(2) seems to be new even for orbit spaces under isometric group actions, see, however, [PP93], [Kap02], [Nep19], for related weaker statements valid in general Alexandrov spaces.

Note that in the formulation of Theorem 1.3 and below, *geodesic* will always be globally length minimizing curve parametrized by arclength.

Also the structure of tangent cones and spaces of directions turns out to be very restrictive, as they are spaces of leaves of equidistant decompositions of Euclidean spaces and spheres, respectively:

Theorem 1.4. *Let $P : M \to Y$ be a submetry, where M is an n-dimensional Riemannian manifold. Let $y \in Y$ be arbitrary. If $\Sigma_y Y \neq \emptyset$ then there exists some $n > k \geq 0$ and a submetry $\mathbb{S}^k \to \Sigma_y Y$.*

This result, [BG00] and Browder's theorem, [Bro63], imply that the Euclidean cone $Y = C(CaP^2)$ over the Cayley plane cannot be the base of a submetry $P : M \to Y$. This should be compared to the conjectured impossibility to obtain Y as a collapsed limit of Riemannian manifolds with a lower curvature bound, [Kap05].

From Theorem 1.4 we deduce:

Corollary 1.5. *Let M be a Riemannian manifold and $P : M \to Y$ be a submetry. Let $y \in Y$ be arbitrary. Then, for some $l \geq 0$, the tangent space $T_y Y$ has a canonical decomposition*

$$T_y Y = \mathbb{R}^l \times T_y^0 \ ,$$

where $T_y^0 Y$ is the Euclidean cone over an Alexandrov space $\Sigma_y^0 Y$ of diameter at most $\frac{\pi}{2}$.

By the *l-dimensional stratum* Y^l of the space of leaves Y we denote the set of points $y \in Y$ whose tangent space $T_y Y$ splits off as a direct factor $\mathbb{R}^l$ but not $\mathbb{R}^{l+1}$, as in Corollary 1.5. The strata Y^l define a topologically and geometrically well behaved stratification of Y:

Theorem 1.6. *Let $P : M \to Y$ be a submetry from a Riemannian manifold M. Set $m = \dim(Y)$. For any $m \geq l \geq 0$, the stratum Y^l is an l-dimensional topological manifold which is locally closed and locally convex in Y. The maximal stratum Y^m is open and globally convex.*

For any $y \in Y^l$, the closure $\bar{E}^y$ of the connected component E^y of y in Y^l is the smallest extremal subset of Y which contains y.

It turns out that, the distance on the strata Y^l is locally induced by a Lipschitz continuous Riemannian metric, see Theorem 11.1 below. Moreover, the strata Y^l have positive reach in Y and the topological structure of a submetry over any stratum is rather simple:

Theorem 1.7. *Let M be a Riemannian manifold and let $P : M \to Y$ be a submetry. The preimage $P^{-1}(Y^l)$ of any stratum Y^l is a locally closed subset of positive reach in M.*

If M is complete, then for any connected component E of Y^l, the restriction $P^{-1}(E) \to E$ is a fiber bundle.

We mention, that *quasigeodesics* in the space of leaves Y are much simpler than in general Alexandrov spaces: they are concatenations of geodesics, Corollary 7.4. Moreover, the quasigeodesic flow exists almost everywhere and preserves the Liouville measure, Section 12.1.

Non-manifold fibers of P are related to the boundary ∂Y of Y:

Theorem 1.8. *Let $P : M \to Y$ be a submetry from a Riemannian manifold M. For any point $y \in Y \setminus \partial Y$, the fiber $P^{-1}(y)$ is a $\mathcal{C}^{1,1}$-submanifold of M.*

Of particular importance are submetries for which *all* fibers are $\mathcal{C}^{1,1}$-submanifolds. They are defined under the name *manifold submetry* in [CG16] and investigated further in a more specific situation in [MR20]. We call them *transnormal submetries*, borrowing the term *transnormality* from the theory of singular Riemannian foliations, [Mol88].

Theorem 1.9. *Let M be a Riemannian manifold and $P : M \to Y$ be a submetry. Then the following are equivalent:*

(1) *P is a transnormal submetry.*
(2) *All fibers of P are topological manifolds.*
(3) *Any local geodesic in M which starts normally to any fiber of P remains normal to all fibers it intersects.*

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Transnormal submetries are stable under some limit operations, Corollary 12.6. In particular, if $P\colon M \to Y$ is a transnormal submetry then, for any $x \in M$, the differential $D_x P : T_x M \to T_y Y$ is a transnormal submetry as well, Proposition 12.5. Moreover, transnormal submetries have the property known as *equifocality* in the theory of singular Riemannian foliations, see Proposition 12.7. Finally, for any leaf L of a transnormal submetry $P : M \to Y$, the foot point projection $\Pi^L : U \to L$ in a neighborhood of L restricts as a fiber bundle to any leaf $L' \subset U$, see Theorem 12.8.

For a transnormal submetry, the preimage $P^{-1}(Y^l)$ of any stratum is a locally closed $\mathcal{C}^{1,1}$ submanifold of M and the restriction of P to this submanifold is a $\mathcal{C}^{1,1}$ Riemannian submersion. Even if M is smooth one cannot expect higher regularity of the stratification of P. However, $\mathcal{C}^{1,1}$ submanifolds of a manifold with curvature locally bounded from both sides again has curvature locally bounded from both sides, [KL20], so we stay in the category of manifolds we have chosen to work with.

1.3. **Questions.** There are many open questions about finer structural properties of submetries. We would like to collect a few of them below.

We do not know if base spaces of transnormal submetries are different from base spaces of general submetries. Even if $\dim(Y) = 1$ the following question is absolutely non-trivial:

Question 1.10. *Given a submetry $P : \mathbb{S}^n \to Y$, does there exist a transnormal submetry of the round sphere (possibly of different dimension) with the same quotient space Y?*

One obtains a closely related question if $\mathbb{S}^n$ is replaced by a general manifold M.

The positive answer to the following question in the non-collapsed case is provided in Section 12. We expect that in the collapsed case the answer is affirmative as well.

Question 1.11. *Let a sequence $P_i : M_i \to Y_i$ of submetries converge in the Gromov–Hausdorff sense to a submetry $P : M \to Y$. Assume that M_i and M have the same dimension and curvature bounds and that P_i are transnormal. Is P transnormal?*

The following question is natural in view of the structural results Theorem 11.1 and Corollary 9.2:

Question 1.12. *Do the strata Y^l defined in Theorem 1.6 have curvature locally bounded from both sides?*

It should be possible to derive a positive answer to the following question as a consequence from Theorem 1.7 and Corollary 8.4:

Question 1.13. *Given a transnormal submetry* $P : M \to Y$, *does the decomposition of* M *into connected components* M_i^l *of preimages of strata* $P^{-1}(Y^l)$ *satisfy Whitney's conditions (A) and (B)?*

The following question is related to the previous one and Theorem 12.8 but is likely much more challenging:

Question 1.14. *Let* $P : M \to Y$ *be a transnormal submetry. Does a version of the slice theorem hold in* M, *compare* [MR19]*?*

A closely related question is the Lipschitz version of a well-known problem in singular Riemannian foliations, [Mol88], [Wil07]:

Question 1.15. *Given a transnormal submetry* $P : M \to Y$, *do there exist Lipschitz vector fields everywhere tangent to the fibers and generating the tangent spaces to the fibers at all points?*

The continuous dependence of differentials of a submetry along a manifold fiber, Lemma 8.2, leads to the following question for $M = \mathbb{S}^n$:

Question 1.16. *How large can be the set of transnormal submetries* $P : M \to Y$ *modulo isometries of* M, *for a fixed compact manifold* M *and a fixed Alexandrov space* Y?

For some non-compact manifolds or for non-transnormal submetries, the space of submetries can be infinite-dimensional, Examples 6.9, 6.10.

At least for quotient spaces Y of Riemannian manifolds, the answer to the next question, related to [GL20], should be affirmative.

Question 1.17. *Given an Alexandrov space* Y, *does there exist a description of all discrete submetries* $P : X \to Y$, *with* X *an Alexandrov space, similar to the Riemannian orbifold case,* [Lan20]*?*

There are many basic topological questions. For instance:

Question 1.18. *Which manifolds admit non-trivial (transnormal) submetries for some Riemannian metric?*

See [GR15], for related results for singular Riemannian foliations.

1.4. **Structure of the paper.** In Section 2 we fix notation and collect basic facts about submetries. In particular, we introduce the notion of local submetries and discuss horizontal lifts of curves. In Section 3 we discuss some basic facts about submetries between general Alexandrov regions. In particular, we recall that submetries preserve lower curvature bounds and have differentials at all points. In Section 4 we observe that a submetry between Alexandrov spaces lifts many semiconcave functions to semiconcave functions and commutes with the corresponding gradient flows and discuss the first structural consequences

of this fact. In Section 5 we discuss the structure of differentials of submetries between Alexandrov spaces. All these sections are of general and auxiliary character, most statements contained in them have appeared in [Lyt01] and might be known to specialists. The findings of Section 5 include the proofs of Theorem 1.4 and Corollary 1.5.

Only in Section 6 we turn to the main subject of this paper, submetries of Riemannian manifolds and prove Theorem 1.1.

In Section 7 we begin to investigate the structure of the base space Y and prove a part of Theorem 1.3. In the most technical Section 8 we collect some observation about (semi-) continuity of differentials of a submetry. These are used in Section 9 to prove that small balls in the base space are convex and to finish the proof of Theorem 1.3. In Section 10, the structure of the regular part is investigated and Theorem 1.2 is verified. In Section 11 we study the properties of the natural stratification of the base space and prove Theorem 1.6 and Theorem 1.7. In Section 12 we discuss manifold fibers and transnormal submetries and prove Theorem 1.8 and Theorem 1.9.

1.5. **Acknowledgements.** The study was initiated many years ago in the PhD thesis of A.L., but the results were not brought into a final form. In the meantime some results were found and used by other authors and the interest in the subject seem to have increased, justifying a systematic investigation.

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2. Preliminaries and basics

2.1. **Notations.** By d we denote the distance in metric spaces. A metric space is *proper* if its closed bounded subsets are compact.

For a subset A of a metric space X we denote by $d_A : X \to \mathbb{R}$ the distance function to the set A and by $B_r(A)$ the open r-neighborhood around A in X.

The length of a curve γ will be denote by $\ell(\gamma)$. Curves of finite length are called *rectifiable*. For a locally Lipschitz curve $\gamma : I \to X$ in a metric space X we denote by $|\gamma'(t)| \in [0, \infty)$ the *velocity* of γ in $t \in I$. The velocity is defined for almost all $t \in I$ and $\ell(\gamma) = \int_I |\gamma'(t)| \, dt$.

A metric is a *length metric* if the distance between any pair of points equals the infimum of the lengths of curves connecting the points.

A *geodesic* will denote an isometric (i.e. distance preserving) embedding of an interval. In particular, all geodesics are parametrized by arc-length. A metric space X is *geodesic* if any pair of its points is connected by a geodesic.

2.2. **Main definitions.** Two subsets L_1, L_2 of a metric space X are called *equidistant* if d_{L_i} is constant on L_j, for $i, j = 1, 2$.

Recall from the introduction that a map $P : X \to Y$ is a *submetry* if for any $x \in X$ and any $r > 0$ the equality $P(B_r(x)) = B_r(P(x))$ holds true. We call X the *total space* and Y the *base* of a the submetry P.

The following observation is a direct consequence of the definition.

Lemma 2.1. *A map $P : X \to Y$ between metric spaces is a submetry if and only if P is surjective and, for any $y \in Y$, we have $d_y \circ P = d_{P^{-1}(y)}$.*

In particular, for any pair of points $y_1, y_2 \in Y$ with fibers $L_i := P^{-1}(y_i)$, the function d_{L_i} is constantly equal $d(y_i, y_j)$ on L_j. Hence L_1 and L_2 are equidistant. On the other hand, if a metric space X is decomposed in a family of closed, pairwise equidistant subsets $L_i, i \in I$, then the *set of leaves* I becomes a metric space, when equipped with the natural distance between the corresponding subsets in X. The canonical projection $P : X \to I$ sending a point x to the leaf $L_x := P^{-1}(P(x))$ through x is a submetry with respect to this metric.

Hence, there is a one-to-one correspondence (up to isometries between base spaces) of submetries with total space X and decompositions of X into closed equidistant subsets, [GGKMS18, Lemma 8.1]. In particular, any isometric group action on a space X, with all orbits closed, determines a unique submetry, the *quotient map*, whose fibers are the orbits of the action.

Remark 2.2. *In [BG00], submetries are defined by the slightly stronger requirement that the images of all* closed *balls are closed balls of the same radius. For proper spaces both notions coincide.*

2.3. **Basic properties and operations with submetries.** Any submetry is 1-Lipschitz and surjective.

For a submetry $P : X \to Y$, a point $y \in Y$ is isolated in Y if and only if the fiber $P^{-1}(y)$ has non-empty interior. Thus, if X is connected then either Y is a singleton, or any fiber P is nowhere dense in X.

Many properties of the total space are inherited under submetries, [Ber87, Proposition 1]:

Lemma 2.3. *Let $P : X \to Y$ be a submetry. If X is compact or proper or complete or length space then Y has the corresponding property.*

For any submetry $P : X \to Y$ and $A \subset Y$ we get from Lemma 2.1:

$$(2.1) \qquad\qquad d_A \circ P = d_{P^{-1}(A)} \; .$$

A composition of submetries is a submetry. Moreover, if $P : X \to Y$ is a submetry and, for some map $Q : Y \to Z$, the composition $Q \circ P$ is a submetry then Q must be a submetry too.

Any isometry $P : X \to X$ and the projection $P : X \to \{0\}$ to a singleton are submetries. A direct product of submetries is a submetry. In particular, the projection of a direct product onto a factor is a submetry.

If $P : X \to Y$ is a submetry then the cone $C(P) : C(X) \to C(Y)$ between the Euclidean cones over X and Y is a submetry as well. Any Euclidean cone $C(X)$ admits a submetry $C(X) \to [0, \infty)$ given by $v \to |v|$, i.e. by the distance to the vertex of the cone.

For any submetry, $P : X \to Y$ and any subset $Y' \subset Y$ the restriction $P : P^{-1}(Y') \to Y'$ is a submetry.

Submetries are stable under convergence:

Lemma 2.4. *Let $P_j : (X_j, x_j) \to (Y_j, y_j)$ be a sequence of submetries between pointed proper spaces. If the sequence of spaces (X_j, x_j) converges in the pointed Gromov–Hausdorff topology to a space (X, x) then, after choosing a suitable subsequence, (Y_j, y_j) converge to a space (Y, y), the submetries P_j converge to a submetry $P : (X, x) \to (Y, y)$. Finally, under this convergence, the fibers $P^{-1}(y_j)$ converge to $P^{-1}(y)$.*

Proof. The uniform compactness of balls of fixed radius around x_j in X_j imply the uniform compactness of the corresponding balls around y_j. Since the maps P_j are 1-Lipschitz, we can choose a subsequence and assume that (Y_j, y_j) converges to a space (Y, y) and that P_j converges to a map P.

By the definition of Gromov–Hausdorff convergence, P is a submetry. Clearly, the limit of any sequence of points in $P^{-1}(y_j)$ is contained in $P^{-1}(y)$. On the other hand, any $z \in f^{-1}(y)$ is a limit of a sequence of points $z_j \in X_j$ such that $P_j(z_j)$ converges to y. Consider $\hat{z}_j \in P^{-1}(y_j)$, with $d(\hat{z}_j, z) = d(y_j, y)$ and observe that $\hat{z}_j$ converge to z. $\qquad\square$

Readers familiar with the ultralimits will easily verify the more general statement that any ultralimit of submetries is a submetry.

Basic examples of submetries were mentioned in the introduction:

Example 2.5. *For any isometric action of a group G on a metric space X the orbits are pairwise equidistant. Thus, the closures of the orbits of G define an equidistant decomposition of X in closed subsets and, therefore, a submetry onto the quotient space.*

The properties of isometric actions of a closed Lie group on a Riemannian manifold is a classical object of investigations [Bre72], see also [GGG13], [HS17] for similar results on Alexandrov spaces. The present paper aims at the generalization of the starting points of the theory to the non-homogeneous setting.

Example 2.6. *The leaves of a singular Riemannian foliation $\mathcal{F}$ of any complete Riemannian manifold M are equidistant, [Mol88]. If all leaves are closed then $\mathcal{F}$ defines a submetry with total space M.*

2.4. **Localization, horizontal lifts of curves and globalization.** Since we would like to restrict submetries to open subsets, we localize the definition of a submetry.

Definition 2.7. *Let $P : X \to Y$ be a map between metric spaces. We say that P is a local submetry if for any point $x \in X$ there exists some $r > 0$ with the following property. For any point $x' \in B_r(x)$ and any $s < r - d(x, x')$ we have $P(B_s(x')) = B_s(P(x'))$.*

While a restriction of a submetry $P : X \to Y$ to an open subset U of X is rarely a submetry, it is always a local submetry.

Any local submetry $P : X \to Y$ is an open map which is locally 1-Lipschitz. In particular, P does not increase length of curves and Hausdorff dimension of subsets.

For a local submetry $P : X \to Y$ we call a rectifiable curve $\gamma : I \to X$ *horizontal* (with respect to P) if $\ell(\gamma) = \ell(P \circ \gamma)$. In this case, we call γ a *horizontal lift* of $P \circ \gamma$.

Let $I \subset \mathbb{R}$ be an interval. A locally Lipschitz curve $\gamma : I \to X$ is horizontal if and only if for almost all $t \in I$ the velocities $|\gamma'(t)|$ and $|(P \circ \gamma)'(t)|$ coincide.

For 1-Lipschitz curves the following Lemma is the special case of [Lyt05a, Lemma 4.4]. For general rectifiable curves, the result follows after a reparametrization of the curve by arclength.

Lemma 2.8. *Let $P : X \to Y$ be a local submetry. Assume that for some $x \in X$ and $r > 0$, the closed ball $\bar{B}_r(x)$ is compact. Then, for any curve $\eta : I \to Y$ of length at most r which starts at $P(x)$, there exists a horizontal lift of η starting at x.*

In particular, for a local submetry $P : X \to Y$ between proper spaces, any rectifiable curve in Y admits a horziontal lift with the prescribed lift of a starting point. From Lemma 2.8 we deduce the following local-to-global property:

Corollary 2.9. *Let $P : X \to Y$ be a local submetry between length spaces. Assume that for some $x \in X$ and $r > 0$, the closed ball $\bar{B}_r(x)$ is compact. Then $P(B_s(x)) = B_s(P(x))$, for any $s \le r$.*

Moreover, equality (2.1) holds on $B_{\frac{r}{3}}(x)$ for any subset $A \subset Y$ with non-empty intersection $A \cap B_{\frac{r}{3}}(P(x))$.

A local submetry between proper length spaces is a submetry.

In fact, [Lyt05a, Proposition 4.3] shows that the property of being a submetry can be recognized not only locally, but infinitesimally.

Another consequence of Lemma 2.8 is the following statement that allows us to replace the induced metric on a subset by the intrinsic one:

Corollary 2.10. *Let X be locally compact and $P : X \to Y$ be a local submetry. If any pair of point in X is connected by a rectifiable curve, then equipping X and $Z = P(X)$ with their induced length metrics d^X and d^Z, we obtain a local submetry $P : (X, d^X) \to (Z, d^Z)$.*

2.5. **Gradient curves and submetries.** We refer to [Amb18, Chapter 2], [AGS14] for gradient curves of general functions in general metric spaces and to [Pet07], for the case of semiconcave functions in Alexandrov spaces, important for us.

Recall that for a locally Lipschitz function $g : Z \to \mathbb{R}$ on a metric space Z the *ascending slope* of g at a point x is defined as

$$|\nabla^+ g|(x) = \limsup_{y \to x} \frac{\max\{g(y) - g(x), 0\}}{d(x, y)} \in [0, \infty) .$$

A locally Lipschitz continuous curve $\gamma : [0, t) \to Z$ is called a gradient curve of g starting at $x = \gamma(0)$ if for almost all $s \in [0, t)$

$$|\gamma'(s)| = |\nabla^+ g|(\gamma(s)) \text{ and } (g \circ \gamma)'(s) = |\nabla^+ g|^2(\gamma(s)) .$$

Let now $P : X \to Y$ be a local submetry, let $g : Y \to \mathbb{R}$ be a locally Lipschitz function. Then $P \circ g$ is locally Lipschitz and, for all $x \in X$,

$$|\nabla^+(P \circ g)|(x) = |\nabla^+ g|(P(x)) .$$

It follows from the definition of gradient curves that a locally Lipschitz curve $\gamma : [0, t) \to X$ is a gradient curve of $P \circ g$ if and only if γ is a horizontal curve and $P \circ \gamma$ is a gradient curve of g.

3. Submetries and lower curvature bounds

3.1. **Alexandrov spaces, Alexandrov regions.** Following the notation in [AKP19], for $\kappa \in \mathbb{R}$ and points x, y, z in a metric space X, we denote by $\tilde{\angle}_\kappa(x_y^z)$ *the κ-comparison angle at x*, whenever it is defined.

The metric space X is $CBB(\kappa)$ if for any $p, x, y, z \in X$ the following inequality holds true, whenever all κ-comparison angles are defined:

$$\tilde{\angle}_\kappa(p_x^y) + \tilde{\angle}_\kappa(p_y^z) + \tilde{\angle}_\kappa(p_z^x) \le 2\pi \ .$$

An *Alexandrov space of curvature* $\ge \kappa$ is a complete length space of finite Hausdorff dimension which is $CBB(\kappa)$. Any such space is proper, [BGP92], in particular it is geodesic. For an extensive literature on such spaces see [BGP92], [Pet07], [AKP19] and the bibliography therein. We will assume some familiarity with the theory of Alexandrov spaces.

A metric space X is an *Alexandrov region* if it is a length space of finite Hausdorff dimension in which every point x has a $CBB(\kappa)$ neighborhood, with κ possibly depending on x.

For instance, any smooth Riemannian manifold is an Alexandrov region. Due to [LN20], a length space X is an Alexandrov region if and only if every point $x \in X$ admits a compact neighborhood U of x in X which is an Alexandrov space.

3.2. Basic geometric objects in Alexandrov spaces and regions.
For any point x in an Alexandrov space X (and therefore in an Alexandrov region X) we have a well-defined *tangent space* $T_x X$. This tangent space is a Euclidean cone with vertex 0_x over the *space of directions* $\Sigma_x X$.

We refer to [Pet07] for a detailed discussion of semiconcave functions on Alexandrov spaces and their gradient flows. We will only use the following facts. For any (in the sequel always locally Lipschitz continuous) semiconcave function $g : U \to \mathbb{R}$ on an Alexandrov region U, there is a unique maximal gradient curve starting at any point of U. The local gradient flow Φ_t is locally Lipschitz continuous on U.

A subset E of an Alexandrov space Z is called an *extremal subset* if it is invariant under the gradient flow of any semiconcave function, [Pet07], [PP93]. Equivalently, E is extremal if it is invariant under the gradient flows of all functions d_q^2, $q \in Z$. The same definition provides a notion of an extremal subset in an Alexandrov region.

The *boundary* ∂X of an Alexandrov region is defined inductively on dimension as the set of all points, for which $\Sigma_x X$ has non-empty boundary. The boundary ∂X is an extremal subset of X and any extremal subset is closed in X, [PP93].

A *quasigeodesic* in an Alexandrov region X is a curve $\gamma : I \to X$ parametrized by arclength such that, for any $t \in I$, we have the following inequality for $q \in X$ converging to $\gamma(t)$:

$$(3.1) \qquad (\tfrac{1}{2}(d_q \circ \gamma)^2)''(t) \le 1 + o(d(q, \gamma(t))).$$

This is equivalent to the more common definition, [PP94, 1.7], that the restriction of distance functions to all points is as concave as the restriction of a distance function to a geodesic in the comparison space. Any *local geodesic* in an Alexandrov region is a quasigeodesic, but in general quasigeodesics can be much more complicated. We refer to [Pet07] and [PP94] for the theory of quasigeodesics in Alexandrov spaces.

3.3. **Local submetries preserve lower curvature bounds.** The following result is essentially contained in [BGP92].

Proposition 3.1. *Let $P : X \to Y$ be a surjective local submetry. If X is an Alexandrov region (of curvature $\geq \kappa$) then Y is an Alexandrov region (of curvature $\geq \kappa$), if we equip Y with the induced length metric.*

If $P : X \to Y$ is a submetry and X is an Alexandrov space of curvature $\geq \kappa$ then so is Y.

Proof. Due to Corollary 2.10, we may assume that Y is a length space. For any $y \in Y$, choose an arbitrary $x \in P^{-1}(y)$. Find $r > 0$ such that $\bar{B}_{3r}(x)$ is compact and $CBB(\kappa)$. We claim that $B_r(y)$ is $CBB(\kappa)$. Indeed, for any $p, y_1, y_2, y_3 \in B_r(y)$ we consider any $\bar{p} \in P^{-1}(p)$ and then $\bar{y}_j \in P^{-1}(y_j)$ such that $d(x, \bar{p}) = d(y, p)$ and $d(\bar{p}, \bar{y}_j) = d(p, y_j)$.

Then $d(\bar{y}_i, \bar{y}_j) \geq d(y_i, y_j)$, Thus, the κ-comparison angles at $\bar{p}$ are not smaller than the corresponding κ-comparison angles at p. Therefore,

$$\sum_{i=1}^{3} \tilde{\angle}_\kappa(p_{y_i}^{y_{i+1}}) \leq \sum_{I=1}^{3} \tilde{\angle}_\kappa(p_{\bar{y}_i}^{\bar{y}_{i+1}}) \leq 2\pi .$$

Since local submetries do not increase the Hausdorff dimension, Y is a finite-dimensional, locally compact, length space which is locally $CBB(\kappa)$. By [LN20], Y is an Alexandrov region.

The global statement follows by Toponogov's globalization, in fact, it is already contained in [BGP92]. $\qquad\square$

3.4. **Lifts and images of geodesics.** Let $P : X \to Y$ be a local submetry between Alexandrov regions. We call a curve $\gamma : [a, b] \to X$ a *P-minimal geodesic* if γ is parametrized by arclength and

$$b - a = d(P(\gamma(a)), P(\gamma(b))) .$$

If γ is a *P*-minimal geodesic, then γ is a geodesic in X, $P \circ \gamma$ is a geodesic in Y and γ is a horizontal lift of $P \circ \gamma$. On the other hand, any horizontal lift γ of a geodesic $\hat{\gamma} : [a, b] \to Y$ is a *P*-minimal geodesic.

The image of a horizontal geodesic γ in X under a submetry does not need to be a geodesic. However, cf. [Proposition 4][GW11]:

Proposition 3.2. *Let $P : X \to Y$ be a local submetry between Alexandrov regions. Let $\gamma : I \to X$ be a geodesic. Then γ is horizontal if and only if the composition $P \circ \gamma : I \to Y$ is a quasigeodesic.*

Proof. If $P \circ \gamma$ is a quasigeodesic, then it is parametrized by arclength. Hence γ must be a horizontal in this case.

If γ is horizontal then $P \circ \gamma$ is parametrized by arclength and the property (3.1) follows from the equality (2.1) and Corollary 2.9. $\quad\square$

3.5. **Differentiability.** The following result in the case of submetries can be found in [Lyt04a, Proposition 11.3]. For local submetries, the differentiability is a direct consequence of Corollary 2.9 and [Lyt04a].

Proposition 3.3. *Any local submetry $P : X \to Y$ between Alexandrov regions is differentiable at each point $x \in X$. In other words, for $y = P(x)$, there exists a map $D_x P : T_x X \to T_y Y$, such that for every sequence $t_i \to 0$, the submetry P seen as a map between rescaled spaces $P : (\frac{1}{t_i} X, x) \to (\frac{1}{t_i} Y, y)$ converge to the map $D_x P$.*

The map $D_x P : T_x X \to T_y Y$ is a submetry and commutes with natural dilations of the Euclidean cones $T_x X$ and $T_y Y$.

By definition, for any curve $\gamma : [0, \epsilon) \to X$ starting in x in the direction $v \in T_x X$ the curve $P \circ \gamma$ starts in the direction $D_x P(v)$.

For submetries the following result is a direct consequence of Lemma 2.4. For local submetries the statement follows, by adapting the proof of Lemma 2.4:

Corollary 3.4. *Let $P : X \to Y$ be a local submetry between Alexandrov regions. Consider $x \in X$ and $y = P(x)$ and the fiber $L = P^{-1}(y)$. Then, under the Gromov–Hausdorff convergence $(\frac{1}{t} X, x) \to T_x X$ the sets $(\frac{1}{t} L, x) \subset (\frac{1}{t} X, x)$ converge for $t \to 0$ to the fiber $D_x P^{-1}(0_y)$ of the differential $D_x P : T_x X \to T_y Y$.*

The last statement can be interpreted as the fact that the tangent cone $T_x L \subset T_x X$ is well-defined and coincides with $(D_x P)^{-1}(0_y)$.

3.6. **Measure and coarea formula.** We will denote here and below by $\mathcal{H}^m$ the m-dimensional Hausdorff measure. For any Alexandrov region Y the Hausdorff dimension is a natural number m. The set Y_{reg} of points $y \in Y$ with $T_y Y = \mathbb{R}^m$ has full $\mathcal{H}^m$ measure and is contained in an m-dimensional Lipschitz manifold Y^δ, [BGP92].

In particular, m-dimensional Alexandrov regions are countably m-rectifiably metric spaces and the metric differentiability theorem, the area and coarea formula applies to Lipschitz maps between Alexandrov regions [AK00], [Kar08].

Let now $P : X \to Y$ be a local submetry. From Proposition 3.3 and [AK00] directly follows:

Lemma 3.5. *Let $P : X \to Y$ be a local submetry, where X and Y are n- and m-dimensional Alexandrov regions. Then, for $\mathcal{H}^n$-almost every point $x \in X$ the point $y = P(x)$ is a regular point of Y and the submetry $D_x P : T_x X = \mathbb{R}^n \to T_y Y \cong \mathbb{R}^m$ is a linear map.*

In this situation the coarea formula [AK00], [Kar08] reads as:

Corollary 3.6. *Let $P : X \to Y$ be a local submetry, where X and Y are n- and m-dimensional Alexandrov regions. Then, for $\mathcal{H}^m$-almost every point $y \in Y$, the preimage $P^{-1}(y)$ is countably $(n-m)$-rectifiable. For every Borel subset $A \subset X$ we have the equality*

$$(3.2) \qquad \mathcal{H}^n(A) = \int_Y \mathcal{H}^{n-m}(A \cap P^{-1}(y)) \, d\mathcal{H}^m(y) \ .$$

4. LIFTS OF SEMICONCAVE FUNCTIONS

4.1. **Special semiconcave functions.** Let Y be an Alexandrov region. For $A \subset Y$ and $y \in Y$ with $t = d_A(y) > 0$, let $0 < 2r < t$ be such that $\bar{B}_{2r}(y)$ is compact. Then on $B_r(y)$ we have the equality

$$d_A = d_{S_{t-r}} + r \ ,$$

where S_{t-r} is the set of points $p \in \bar{B}_{2r}(y)$ with $d_A(p) = t - r$.

The semicontinuity of the squared distance functions on *Alexandrov spaces* and the previous observation show that for any *Alexandrov region* Y and any subset $A \subset Y$ the squared distance function d_A^2 is semiconcave on Y.

Let Y be an Alexandrov region and let $A_1, ..., A_k \subset Y$ be closed. Let $\Theta : \mathbb{R}^k \to \mathbb{R}$ be semiconcave and non-decreasing in each argument. The function $q_{\Theta, A_1, ..., A_k} := \Theta(d_{A_i}^2, ..., d_{A_k}^2) : Y \to \mathbb{R}$ will be called *special semiconcave* on Y.

Since $d_{A_i}^2$ is semiconcave, it follows that any special semiconcave function is semiconcave, cf. [Pet07, Section 6].

4.2. **Lifts of special semiconcave functions.** For us, the importance of special semiconcave functions is due to the following

Lemma 4.1. *Let $P : X \to Y$ be a local submetry between Alexandrov regions. Let $f : Y \to \mathbb{R}$ be a special semiconcave function. Then $P \circ f : X \to \mathbb{R}$ is semiconcave.*

Proof. If $f = \Theta(d_{A_i}^2, ..., d_{A_k}^2) : Y \to \mathbb{R}$ then

$$f \circ P = \Theta((d_{A_1})^2 \circ P, ..., (d_{A_k})^2 \circ P)$$

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and it suffices to prove that $(d_A \circ P)^2$ is semiconcave for every subset $A \subset Y$. But this follows from equality 2.1, Corollary 2.9 and the first observation in Subsection 4.1. $\qquad\square$

As explained in Subsection 2.5, the gradient curves of $f \circ P$ are exactly the horizontal lifts of the gradient curves of f. Thus, P sends the gradient flow of $f \circ P$ to the gradient flow of f. More precisely,

Lemma 4.2. *In the above notation, let Φ_t be the gradient flow of a special semiconcave function f on Y and let $\hat{\Phi}_t$ be the gradient flow of the function $f \circ P$ on X. Then, for all (z, t) in the domain of definition of $\hat{\Phi}$, we have*

$$P(\hat{\Phi}_t(z)) = \Phi_t(P(z)).$$

4.3. **Perelman's function and its lift.** For any Alexandrov region Y and any point $y \in Y$ there exists a strictly concave function in a neighborhood of y which has its maximum at y. More precisely, [Pet07, Theorem 7.1.1], there exists $\epsilon = \epsilon(y) > 0$ and a special semiconcave function $f = f_y$ on Y with the following properties:

The restriction of f to the ball $B_\epsilon(y)$ is a strictly concave function and has a unique maximum at the point y. Moreover, the ascending slope at any point $z \in B_\epsilon(y) \setminus \{y\}$ satisfies $|\nabla^+ f|(z) \geq \frac{1}{2}$.

Using this function we can now easily derive:

Proposition 4.3. *Let $P : X \to Y$ be a local submetry between Alexandrov regions. Let $y \in P(X)$ be arbitrary. Then there exist a neighborhood U of the fiber $L := P^{-1}(y)$ in X and a special semiconcave function $f : Y \to \mathbb{R}$ with the following properties.*

The function $g = f \circ P$ is semiconcave on X. The set of maximum points of g in U is exactly L and $|\nabla^+ g|(q) \geq \frac{1}{2}$, for any $q \in U \setminus L$.

The gradient flow $\hat{\Phi}$ of g is defined in U for all times and for some $\delta > 0$ we have $\hat{\Phi}_\delta(U) = L$.

Proof. Consider a small relatively compact ball $B_\epsilon(y)$ around y and Perelman's function $f = f_y : Y \to \mathbb{R}$ as described above.

The function $g = P \circ f$ is semiconcave on X, by Lemma 4.1. Clearly, L is exactly the set of maximum points of g in $\breve{U} := P^{-1}(B_\epsilon(y))$.

For any $z \in B_\epsilon(y) \setminus \{y\}$ we have $|\nabla^+ f|(z) \geq \frac{1}{2}$. By the definition of gradient curves, the point y is the unique fixed point of the partial gradient flow Φ of f on B_ϵ. Moreover, for

$$\delta := 2 \cdot \inf_{z \in B_\epsilon(y)} (f(y) - f(z)),$$

any flow line of Φ defined at least for time δ ends in y.

Let $U = U_\delta \subset \tilde{U}$ be the set of points $p \in \tilde{U}$ at which the flow line of $\hat{\Phi}$ on $\tilde{U}$ is defined at least for the time δ. Since L is the set of fixed points of $\hat{\Phi}$ in $\tilde{U}$, the set U is an open neighborhood of L. By construction and Lemma 4.2, $\hat{\Phi}_\delta(p) \in L$, for any $p \in U$. In particular, $\hat{\Phi}$ is defined in U for all times and $\hat{\Phi}_\delta(U) = L$.

By Subsection 2.5, $|\nabla^+ g|(q) \geq \frac{1}{2}$, for any $q \in U \setminus L$. $\qquad \square$

4.4. **Extremal subsets and dimension of fibers.** While subsequent results have local versions, we prefer to state them only for "global" submetries, for the sake of simplicity. The first result follows directly from the definition of extremal subsets, via gradient flows of semiconcave functions, Lemma 4.1 and Lemma 4.2. An alternative more direct proof of the following statement can be found in [GW11]:

Proposition 4.4. *Let $P : X \to Y$ be a submetry between Alexandrov spaces. Let $E \subset X$ be an extremal subset. Then the image $P(E)$ is an extremal subset of Y.*

However, extremal subsets in the quotient are often much more numerous than in the total space. For instance:

Lemma 4.5. *Let $P : X \to Y$ be a submetry between Alexandrov spaces. Then, for any $k \geq 0$, the set $Y(k)$ of points $y \in Y$ such that $\mathcal{H}^k(P^{-1}(y)) = 0$ is an extremal subset of Y.*

Proof. Fix $y \in Y(k)$ and a special semiconcave function f on Y with gradient flow Φ. Then the gradient flow $\hat{\Phi}$ of $g = f \circ P$ is defined for all times and sends fibers of P surjectively onto fibers of P, due to Lemma 4.1 and Lemma 4.2. Since this gradient flow is locally Lipschitz, $\mathcal{H}^k(P^{-1}(\Phi_t(y))) = 0$, for all t. Thus, $Y(k)$ is invariant under Φ. $\qquad \square$

Similarly, one shows that the set of points $y \in Y$ whose P-fibers have at most k connected components are extremal subsets of Y.

We can draw the following consequence of the coarea formula:

Corollary 4.6. *Let X and Y be n- and m-dimensional Alexandrov spaces and let $P : X \to Y$ be a submetry. A Borel subset $B \subset Y$ satisfies $\mathcal{H}^m(B) = 0$ if and only if $\mathcal{H}^n(P^{-1}(B)) = 0$.*

Proof. If $\mathcal{H}^m(B) = 0$ then $\mathcal{H}^n(P^{-1}(B)) = 0$ as one directly sees from the coarea formula (3.2).

To prove the other implication we only need to show, due to (3.2), that the set $Y(n - m)$ defined in Lemma 4.5 has $\mathcal{H}^m$-measure 0. But by Lemma 4.5, the set $Y(n - m)$ is an extremal subset of Y. Thus, if it has positive measure, it must contain an open subset. Then, by the

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coarea formula, its P-preimage has $\mathcal{H}^n$-measure zero in X and contains an open subset, which is impossible. $\qquad\square$

5. Infinitesimal submetries

5.1. **Lines and rays.** From Toponogov's splitting theorem we derive:

Proposition 5.1. *Let X, Y be Alexandrov spaces with non-negative curvature, let $P : X \to Y$ a submetry and assume that Y splits as $Y = Y_0 \times \mathbb{R}$. Then there exists an isometric splitting $X = X_0 \times \mathbb{R}$, such that P is given as $P(x,t) = (P_0(x), t)$ for a submetry $P_0 : X_0 \to Y_0$.*

Proof. Consider the composition Q of P and the projection $Y \to \mathbb{R}$. This is a submetry. Lifting $\mathbb{R}$ to a horizontal line in X and using the splitting theorem, we see that X splits as $X_0 \times \mathbb{R}$ such that Q is the projection onto the second factor.

Therefore, P has the form $P(x,t) = (P^t(x), t)$ for a map $P^t : X_0 \to Y_0$, a priori, depending on t. However, since P is 1-Lipschitz, P^t does not depend on t and equals a map $P_0 : X_0 \to Y_0$. Since P is a submetry, P_0 must be a submetry as well. $\qquad\square$

The first statement of the next observation follows from (2.1), the second from the concavity of Busemann functions in non-negative curvature or by a direct comparison argument:

Lemma 5.2. *Let X be a non-negatively curved Alexandrov space and let $P : X \to [0, \infty)$ be a submetry and set $L = P^{-1}(0)$. Then $P = d_L$. The function P is convex, in particular, L is a convex subset of X.*

5.2. **Infinitesimal submetries, horizontal and vertical vectors.**
In this section we call points of a Euclidean cone $X = C(S)$ vectors and denote, for $h \in X$, by $|h|$ the distance $|h| := d(h, 0_x)$.

We will call a submetry $P : X = C(S) \to Y = C(\Sigma)$ between two Euclidean cones over Alexandrov spaces S, Σ of curvature ≥ 1 an *infinitesimal submetry* if it commutes with the natural dilations of the cones. In other words, if the equidistant decomposition of X induced by P is equivariant under dilations. Equivalently, we can require that P sends the origin of the cone X to the origin of Y and coincides with its own differential at the origin 0_X.

Given such an infinitesimal submetry $P : X = C(S) \to Y = C(\Sigma)$ we define a vector $v \in X$ to be *vertical* if $P(v) = 0_Y$. We call $h \in X$ *horizontal* if $|P(h)| = |h|$. The set of vertical and horizontal vectors in X will be usually denoted by V and H, respectively. Clearly V and H are subcones of X. By definition, $V = P^{-1}(0_y)$.

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By (2.1), for any point $h \in X$, we have

$$|P(h)| = d(0_Y, P(h)) = d(V, h) .$$

This immediately implies:

Lemma 5.3. *If $V = \{0\}$ then $H = X$. If $V \neq \{0\}$ then the set of unit horizontal vectors $H \cap S$ is the* polar set *of $V \cap S$, i.e., the set of all points $x \in S$ such that the distance in S between x and any point in $V \cap S$ is at least $\frac{\pi}{2}$.*

The following proposition is essentially contained in [Lyt01]. For convenience of the reader we include a proof here.

Proposition 5.4. *Let $P : X = C(S) \to Y = C(\Sigma)$ be an infinitesimal submetry. Then the vertical and horizontal spaces V, H are convex subcones of X. The distance functions d_V and d_H are convex functions on X. The foot-point projections Π^V and Π^H from X on V and H, respectively, are well defined and 1-Lipschitz.*
We have $P = P \circ \Pi^H$. The restriction $P : H \to Y$ is a submetry, which is a cone over the submetry $P : H \cap S \to \Sigma$.

Proof. If $V = \{0\}$ the proposition follows by Lemma 5.3.
Suppose $V \neq \{0\}$. We claim that any $x \in X \setminus (V \cup H)$ is contained in a unique subcone $C(\Gamma)$, where Γ is a geodesic of length $\frac{\pi}{2}$ in the unit sphere S of X connecting a point in $H \cap S$ with a point in $V \cap S$.
The uniqueness of Γ follows from the fact that $H \cap S$ and $V \cap S$ have distance $\frac{\pi}{2}$ in S and that geodesics in S do not branch.
In order to find such a quarter-plane $C(\Gamma)$ bounded by a vertical and a horizontal radial rays, consider the radial ray γ in Y through $P(x)$ and its unique horizontal lift $\bar{\gamma}$ through x, Subsection 3.4.
Then $v = \bar{\gamma}(0)$ is contained in V and, by horizontality of $\bar{\gamma}$, we have

$$d(x, v) = d(x, V) = |P(x)| .$$

Therefore, the radial vertical ray η through v meets $\bar{\gamma}$ at v orthogonally.
Since P commutes with dilations, the radial rays $\bar{\gamma}_t$ through $\bar{\gamma}(t)$ converge, as $t \to 0$, to a radial ray $\bar{\gamma}_\infty$ such that

$$P \circ \gamma_\infty(t) = P \circ \bar{\gamma}(t) = \gamma(t) .$$

Moreover, this ray $\bar{\gamma}_\infty$ makes an angle of $\frac{\pi}{2}$ with η and the union of the rays $\bar{\gamma}_t$ is the required quarter-plane spanned by $\bar{\gamma}_\infty$ and η.
By construction, $P(x) = P(h)$, where $h \in H$ is the closest point on $\bar{\gamma}_\infty$ to x. Moreover, h is the closest point to x in H (otherwise, we would find a horizontal vector making an angle less than $\frac{\pi}{2}$ with η).

Now, we easily see

$$d_H(x) = \sup_{w \in V \cap S} \{-b_w(x)\} \,,$$

where b_w is the Busemann function of the radial ray through w.

If we replace $-b_w$ by $\max\{-b_w, 0\}$ then the equality remains to hold on V and H as well. Thus, we have represented the distance function d_H as a supremum of convex functions, since any Busemann function is concave in non-negative curvature. This proves the convexity of d_H.

The convexity of d_V is proved similarly (or directly by Lemma 5.2).

Now, the gradient flows of $-d_H$, respectively of $-d_V$, converge to the closest point projection Π^H and Π^V. The contractivity of such gradient flows proves that Π^H and Π^V are well-defined and 1-Lipschitz.

The statement that $P = P \circ \Pi^H$ has been obtained in the proof above on $X \setminus (V \cup H)$. The equality is clear on $V \cup H$ as well.

Since P is a submetry and Π^H is 1-Lipschitz, this equality implies that the restriction $P : H \to Y$ is a submetry. $\qquad\square$

5.3. Infinitesimal submetries of Euclidean spaces. We now specialize the above structural results to the case $X = \mathbb{R}^n$.

Proposition 5.5. *Let $P : \mathbb{R}^n \to Y$ be an infinitesimal submetry to a Euclidean cone $Y = C(\Sigma)$. Let $H \subset \mathbb{R}^n$ be the subcone of horizontal vectors as above. Let $H^E \ni 0$ be the maximal Euclidean subspace of H. If $H^E \neq \{0\}$ then the restriction $P : H^E \to Y$ is a submetry and the restriction of P to the unit sphere in H^E is a submetry onto Σ.*

If H is not a Euclidean space then there exists a round hemisphere $\mathbb{S}^k_+ \subset H$ such that the restriction $P : \mathbb{S}^k_+ \to \Sigma$ is a submetry.

Proof. Let $V = P^{-1}(0_Y) \subset \mathbb{R}^n$ be the vertical cone. If $V = \{0\}$ then $H = \mathbb{R}^n$ and the claim of the Proposition is trivial.

Suppose $V \neq \{0\}$. Then H is the polar cone of V by Lemma 5.3.

Let $v \in V$ be arbitrary. Then $T_v V$ is a convex subcone of the linear span V^+ of V. The cone $T_v V$ contains V, so that its polar cone H_v is a subset of H.

The differential $Q = D_v P : T_v \mathbb{R}^n = \mathbb{R}^n \to C_{0_y} Y = Y$ is an infinitesimal submetry. Its vertical subspace is $T_v V$ by Proposition 3.3; hence H_v is the horizontal subspace of the infinitesimal submetry Q.

By Proposition 5.4, the map $Q : H_v \to Y$ is a submetry. Identifying H_v with the starting directions of P-horizontal rays in $\mathbb{R}^n$ starting at v, we see that under the canonical identification of H_v with a subset of H, the map Q is just the restriction of P to H_v.

Choosing v to be an inner point of V in its linear span V^+, we get $T_v V = V^+$ and $H_v = H^E$ proving the first statement.

On the other hand, $V \neq V^+$ if and only if $H \neq H^E$. In this case, we can always find a point v in the boundary of V in V^+ at which $T_v V$ is a Euclidean halfspace. Then H_v is a Euclidean halfspace as well.

Restricting to the unit spheres in the so obtained cones H_v, we find the desired submetries with base Σ. $\square$

Recall, that any Alexandrov space Z has a unique decomposition $Z = \mathbb{R}^k \times Z_0$, where Z_0 does not admit any $\mathbb{R}$-factor, see [FL08]. If Z is non-negatively curved, then Z_0 does not contain lines, by Toponogov's splitting theorem. Now we state:

Proposition 5.6. *Let* $P : \mathbb{R}^n \to C(\Sigma)$ *be an infinitesimal submetry, let* $H \subset \mathbb{R}^n$ *be the cone of horizontal vectors and let* $C(\Sigma) = \mathbb{R}^l \times C(\Sigma_0)$ *be the canonical decomposition, so that* $C(\Sigma_0)$ *does not contain lines.*
There is a natural splitting $\mathbb{R}^n = H^0 \times \mathbb{R}^{n-l}$ *with* $H^0 = \mathbb{R}^l$ *and*

$$P = (Id, P_0) : \mathbb{R}^l \times \mathbb{R}^{n-l} \to \mathbb{R}^l \times C(\Sigma_0) .$$

The set H^0 *consists of all points* $h \in H$ *such that* $P^{-1}(P(h) \cap H) = \{h\}$.
The space Σ_0 *has diameter at most* $\frac{\pi}{2}$.

Proof. The first statement follows from Proposition 5.1. By definition of horizontal points, $H^0 \subset H$.

Using the product structure of P, we reduce the statement to the case $l = 0$, hence $\Sigma = \Sigma_0$. Thus, we may assume $\mathrm{diam}(\Sigma) < \pi$. We then need to prove that the diameter of Σ is at most $\frac{\pi}{2}$ and that for all $h \neq 0$ in H, the fiber through h of the submetry $P : H \to C(\Sigma)$ has more than one point.

Due to Proposition 5.5, there exists a submetry $\hat{P} : \mathbb{S}^k \to \Sigma$.

Assume that the diameter of Σ is larger than $\frac{\pi}{2}$. Consider $p, q \in \Sigma$ such that $d(p,q) > \frac{\pi}{2}$ equals the diameter of Σ.

Let $L_p = \hat{P}^{-1}(p)$ and $L_q = \hat{P}^{-1}(q)$. By (2.1), L_q is contained in the set of points in $\mathbb{S}^{k-1}$ with maximal distance to L_p, and since this maximal distance is larger than $\frac{\pi}{2}$, we see that L_q is a singleton. Then the antipodal point of L_q is also a fiber of $\hat{P}$, by the same argument. Therefore, $\mathrm{diam}(\Sigma) = \pi$ in contradiction to the assumption $\Sigma = \Sigma_0$.

Consider the restriction $P : H \cap \mathbb{S}^{n-1} \to \Sigma$ and assume that some fiber of this restriction is a singleton. By Proposition 5.5, we find a sphere $\mathbb{S}^k \subset H \cap \mathbb{S}^{n-1}$ such that the restriction $P : \mathbb{S}^k \to \Sigma$ is a submetry. Clearly, this restriction has also a singleton fiber. But this implies, $\mathrm{diam}(\Sigma) = \pi$.

This contradiction finishes the proof. $\square$

The above proof shows that if $P : \mathbb{S}^n \to \Sigma$ is a submetry then either $\mathrm{diam}\, \Sigma = \pi$ or $\mathrm{diam}\, \Sigma \leq \pi/2$, see also [MR20], [CG16].

6. Fibers have positive reach

6.1. Distance functions in manifolds.
From now on let M be a Riemannian manifold with local two sided bounds on curvature. We will always equip M with the $\mathcal{C}^{1,1}$ atlas of distance coordinates, [BN93].

For any subset $A \subset X$ the distance function $f = d_A$ is semiconcave on $M \setminus A$ and f^2 is semiconcave on M. For an open subset $O \subset M \setminus A$ the function f is $\mathcal{C}^{1,1}$ in O if and only if for any $x \in O$ there exists at most one geodesic $\gamma_x : [0, \epsilon) \to O$ starting at x such that $f(\gamma(\delta)) = f(x) - \delta$ for all $0 < \delta < \epsilon$, [KL20]. In this case, the gradient curves of f are geodesics.

We denote by $\mathcal{U}(A)$ the largest open subset of M on which d_A^2 is $\mathcal{C}^{1,1}$. Due to the previous considerations $\mathcal{U}(A) \setminus A$ is foliated by geodesics, the gradient curves of d_A.

6.2. Positive reach.
Recall that a subset L of a Riemannian manifold M has *positive reach* if the foot-point projection on L is uniquely defined in some open neighborhood U of L, [Fed59]. This is a local property, which is moreover independent of the Riemannian metric and any $\mathcal{C}^{1,1}$ submanifold has this property, [Ban82, KL20].

A locally closed subset L of M has positive reach if and only if the open set $\mathcal{U}(L)$ of points at which d_L^2 is $\mathcal{C}^{1,1}$ contains L.

Structure of sets of positive reach is rather well understood. For any set L of positive reach, the topological and the Hausdorff dimensions of L coincide and L is locally contractible [Fed59, Remark 4.15]. The intrinsic and the induced metrics on L are locally equivalent and in the intrinsic metric the space L locally has curvature bounded from above in the sense of Alexandrov, [Lyt04b, Theorem 1.1].

For all $x \in L$, there exists a well-defined *tangent cone* $T_x L$ which is a convex subset of $T_x M$, [Fed59, Theorem 4.8]. The *normal cone* $T_x^\perp L$ is the convex cone of all vectors in $T_x M$ enclosing angles at least $\frac{\pi}{2}$ with all vectors in $T_x L$, i.e. it is the polar cone of $T_x L$ in $T_x M$. For any $x \in L$ and unit $h \in T_x^\perp L$ the geodesic γ_h starting in the direction of h satisfies $d_L(\gamma_h(s)) = s$, for all s with $\gamma_h(s) \in \mathcal{U}(L)$, [KL20].

Moreover, we have, [Lyt05c, Proposition 1.4]:

Proposition 6.1. *Let L be a set of positive reach in M. Then L contains a $\mathcal{C}^{1,1}$ submanifold K of M which is dense and open in L.*

Moreover, the following are equivalent:

(1) *The set L is a $\mathcal{C}^{1,1}$ submanifold;*
(2) *The set L is a topological manifold;*
(3) *All tangent spaces $T_x L$ for $x \in L$ are Euclidean spaces.*

However, a connected subset L of positive reach does not need to have locally constant dimension nor does it have to admit a triangulation.

Example 6.2. *Let $L \subset \mathbb{R}^2$ be the set of points (x, y) with $0 \leq y \leq f(x)$, where $f : \mathbb{R} \to [0, \infty)$ is smooth with $f^{-1}(0)$ being the union of the rays $(-\infty, 0]$ and $[1, \infty)$ and a Cantor set $C \subset [0, 1]$. Then L has positive reach in $\mathbb{R}^2$.*

6.3. **Connection to submetries.** It is now not difficult to see that sets of positive reach are intimately related to submetries.

Proposition 6.3. *Let $L \subset M$ be a closed subset of positive reach, nowhere dense in M. Then $P = d_L : \mathcal{U}(L) \to [0, \infty)$ is a local submetry with $L = P^{-1}(0)$.*

Proof. P is 1-Lipschitz, thus $P(B_r(x)) \subset B_r(P(x))$ for all $x \in \mathcal{U}(L)$.

Let $x \in \mathcal{U}(L)$ be such that $\bar{B}_r(x)$ is compact in $\mathcal{U}(L)$.

For $x \in \mathcal{U}(L) \setminus L$, we restrict P to the maximal gradient curve γ of d_L through x, and use that $|(d_L \circ \gamma)'| = 1$ to deduce $P(B_r(x)) = B_r(P(x))$.

If $x \in L$, consider a unit vector $h \in T_x^\perp$, and the geodesic $\gamma = \gamma_h$ starting in the direction of h. Since $d_L \circ \gamma_h(t) = t$ for all $t \leq r$, we see $P(B_r(x)) = [0, r) = B_r(P(x))$, as desired. $\qquad\square$

Remark 6.4. *Changing the metric in a neighborhood of L by some conformal factor depending on the distance function d_L^2, it is easy to see the following. Any compact subset L of positive reach in M admits a complete metric in a neighborhood U of L in M, such that this metric still has local two sided curvature bounds and such that the distance function to L in this metric is a (global) submetry $P : U \to [0, \infty)$. This should be possible for noncompact L as well.*

The following converse contains Theorem 1.1 as a special case:

Theorem 6.5. *Let M be a Riemannian manifold and let $P : M \to Y$ be a surjective local submetry. Then any fiber $L = P^{-1}(y)$ is a subset of positive reach in M.*

Proof. Applying Proposition 4.3, we find a neighborhood U of L and a semiconcave function $g : U \to \mathbb{R}$, which has L as its set of maximum points. Moreover, $|\nabla^+ g|(p) > \frac{1}{2}$ for $p \in U \setminus L$.

Thus, L is a regular sublevel set of the semiconvex function $-g$, in the sense of [Ban82], and by the main result of [Ban82] (see also [KL20]) L has positive reach in M. $\qquad\square$

6.4. **Examples.** A subset L of positive reach in M has *reach* $\geq r$ if the set $\mathcal{U}(L)$ contains the r-tube $B_r(L)$ around L.

Closed convex sets in $\mathbb{R}^n$ are exactly sets of reach ∞ in $\mathbb{R}^n$. By Proposition 6.3 this implies:

Example 6.6. *Let $C \subset \mathbb{R}^n$ be a closed subset. The distance function $d_C : \mathbb{R}^n \to [0, \infty)$ is a submetry if and only if $C = f^{-1}(0)$ is convex and nowhere dense in $\mathbb{R}^n$.*

If L is a co-oriented $\mathcal{C}^{1,1}$ hypersurface in M then we can consider in Proposition 6.3 the oriented distance function instead of the non-oriented one and see:

Example 6.7. *If L is a $\mathcal{C}^{1,1}$ hypersurface in M with oriented normal bundle then the signed distance P to L defines a local submetry $P : \mathcal{U}(L) \to \mathbb{R}$.*

The following example is a prominent theorem in the theory of nonnegative curvature:

Example 6.8. *Let M be a complete open manifold of nonnegative sectional curvature. Then any soul S of M is a subset of reach ∞ and the distance function $d_S : M \to [0, \infty)$ is a submetry, [Per94b], [Wil07].*

The next examples provide large infinite-dimensional families of submetries $P : H^2 \to \mathbb{R}$ and $P : \mathbb{S}^4 \to [0, \frac{\pi}{2}]$ with non-smooth leaves.

Example 6.9. *If H^2 is the hyperbolic plane and $L \subset H^2$ is a complete $\mathcal{C}^{1,1}$-curve with geodesic curvature bounded by 1 at every point. Then the signed distance function to L is a submetry $P : H^2 \to \mathbb{R}$.*

Example 6.10. *Let $C \subset \mathbb{S}^3$ be a convex subset without interior points. Let $L = \mathbb{S}^0 * C \subset \mathbb{S}^4$ be the suspension of C. Then the distance function $d_L : \mathbb{S}^4 \to [0, \frac{\pi}{2}]$ is a submetry.*

7. Local structure of the base

7.1. **Injectivity radius in the base.** We get a local version of Theorem 1.3, (i), (iii):

Theorem 7.1. *Let M be a Riemannian manifold and $P : M \to Y$ a surjective local submetry. For any $y \in Y$, there exists $r > 0$, such that any $v \in \Sigma_y Y$ is the starting direction of a geodesic of length r in Y.*
For any $s < r$ the distance sphere $\partial B_s(y)$ is an Alexandrov space.

Proof. We may assume that y is not isolated. Then $L = P^{-1}(y)$ is a nowhere dense set of positive reach in M and the distance function d_L has ascending slope 1 at all points in the neighborhood $\mathcal{U}(L)$ of L.

For $x \in L$ consider $r > 0$ such that $\bar{B}_{3r}(x)$ in $\mathcal{U}(L)$ is compact. Then, by Subsection 2.5 and Corollary 2.9, the distance function d_y has

ascending slope 1 at all points $z \in B_r(y)$. Thus, d_y is semiconcave, 1-Lipschitz and $|\nabla d_y| = 1$ on $B_r(y) \backslash \{y\}$. Therefore its maximal gradient curves in $B_r(y) \backslash \{y\}$ are unit speed geodesics of length r starting at y. Thus, any $v \in \Sigma_y$ is the starting direction of a geodesic γ_v of length r.

For $s < r$ we set

$$N^s = P^{-1}(\partial B_s(y)) \cap B_{3r}(x) .$$

Then $P : N^s \to \partial B_s(y)$ is a surjective local submetry.

On the other hand, N^s is the level set $d_L^{-1}(s)$ of the $\mathcal{C}^{1,1}$-submersion $d_L : B_{3r}(x) \setminus L \to \mathbb{R}$. Thus, N^s is a $\mathcal{C}^{1,1}$-submanifold of M. Hence in its intrinsic metric, N^s has curvature locally bounded from both sides, see [KL20].

By Proposition 3.1 and Corollary 2.10, ∂B_s is an Alexandrov region. Due to compactness, $\partial B_s(y)$ is an Alexandrov space. $\qquad\square$

7.2. Infinitesimal structure.

Let $P : M \to Y$ be a local submetry, let $x \in M$ be arbitrary and set $y = P(x)$. By Proposition 3.3, the differential $D_x P : T_x M \to T_y Y$ is an infinitesimal submetry. We will denote vectors in $T_x M$ which are vertical, respectively horizontal, with respect to $D_x P$, *vertical*, respectively *horizontal*, with respect to P.

By Proposition 3.3, the vertical space $D_x P^{-1}(0_y)$ is exactly the tangent space $T_x L$, where $L := P^{-1}(y)$ is the fiber of P through x.

By Lemma 5.3, a vector $h \in T_x M$ is a horizontal vector if and only if it is contained in the *normal space* $T_x^\perp L$ of the set of positive reach L at the point x. Note that the convex cone $T_x^\perp L$ is a Euclidean space if and only if $T_x L$ is a Euclidean space.

Theorem 7.2. *Let M be an n-dimensional Riemannian manifold and $P : M \to Y$ a surjective local submetry. If $y \in Y$ is non-isolated then there exists a submetry $P' : \mathbb{S}^k \to \Sigma_y$, for some $k < n$.*

If $L = P^{-1}(y)$ is not a $\mathcal{C}^{1,1}$-submanifold of M then, for some $k < n$, there exists a submetry of the closed hemisphere $\mathbb{S}_+^k$ onto Σ_y.

Proof. Consider an arbitrary $x \in L$. If L is not a $\mathcal{C}^{1,1}$ submanifold of M, we may choose x such that the vertical space $T_x L$ is not a Euclidean space, by Proposition 6.1. In this case the horizontal space $T_x^\perp L$ is not a Euclidean space as well.

The differential $D_x P : T_x M \to T_y Y$ is an infinitesimal submetry, Proposition 3.3. The claim now follows from Proposition 5.5. $\qquad\square$

The first part of the above theorem contains as a special case Theorem 1.4. Now, Proposition 5.6 implies Corollary 1.5.

7.3. **Standard local picture.** Let M be a Riemannian manifold and $P : M \to Y$ be a local submetry. For a point $x \in M$ we denote by L_x the fiber $P^{-1}(P(x))$. As before let $\mathcal{U}(L_x)$ be the open neighborhood of L_x on which $d^2_{L_x}$ is $\mathcal{C}^{1,1}$. Consider $r > 0$ such that the set $\bar{B}_{10r}(x)$ is a compact, convex subset of $\mathcal{U}(L_x)$. Moreover, we require that geodesics in $\bar{B}_{10r}(x)$ are uniquely determined by their endpoints and such that $P(\bar{B}_{10r}(x))$ is contained in a compact Alexandrov space $Y' \subset Y$.

Then, for $y = P(x)$, the restriction $P : B_{10r}(x) \to B_{10r}(y)$ will be called the *standard local picture at* x. Eventually, we will later adjust the choice of $r = r_x$. Note that $r = r_x$ in the standard local picture around x satisfies the statements of Corollary 7.1.

Proposition 7.3. *Let $P : M \to Y$ be a local submetry. For any $x \in M$, let $r = r_x$ be as in the standard local picture around x. For a unit vector $h \in T_x M$ and the geodesic γ_h starting in the direction of h the following are equivalent:*

  (1) *The vector h is horizontal.*
  (2) *The geodesic $\gamma_h : [0, r] \to M$ is a horizontal curve.*
  (3) *The geodesic $\gamma_h : [0, r] \to M$ is P-minimal.*

Proof. Set $y = P(x)$ and set $L = P^{-1}(y)$.

If h is horizontal, then $h \in T_x^\perp L$ and $d(\gamma_h(r), L) = r$. Hence, $P \circ \gamma_h$ is a geodesic starting in y. Thus,(1) implies (3) .

Clearly (3) implies (2) and (1) .

Given (2) , the image $P \circ \gamma_h$ is a quasi-geodesic of length r starting in y, by Proposition 3.2. By Corollary 7.1, there exists a geodesic in Y of length r with the same starting vector as $P \circ \gamma_h$. Due to [PP94], the quasigeodesic $P \circ \gamma_h$ coincides with this geodesic, proving (3). $\square$

The last argument in the above proof also shows that quasigeodesics in Y are of a much more special form than in general Alexandrov spaces:

Corollary 7.4. *Let $P : M \to Y$ be a local surjective submetry. If $\gamma : [a, b] \to Y$ is a quasigeodesic then there exists a finite subdivision $a = t_1 \leq t_2... \leq t_k = b$ such that the restriction of γ to any of the subintervals $[t_i, t_{i+1}]$ is a geodesic.*

8. Some technical statements

8.1. **Setting for semicontinuity questions.** In this section we fix a local submetry $P : M \to Y$. As before we denote by L_x the fiber $P^{-1}(P(x))$ through a point $x \in M$. We are going to analyze the (semi)-continuity of vertical spaces $T_x L_x$, as x varies over M.

The following example should be seen as a warning:

Example 8.1. *For $C = [0, \infty) \subset \mathbb{R} \subset \mathbb{R}^2$, consider the submetry $P = d_C : \mathbb{R}^2 \to [0, \infty)$. Then $C = L_{(0,0)}$ and $T_{(0,0)} L_{(0,0)} = C$. On the other hand, for $x = (-t, 0) \in \mathbb{R}^2$, the vertical space $T_x L_x = \{0\} \times \mathbb{R}$ is orthogonal to C.*

We fix a sequence $x_j \in M$ converging to $x \in M$ and $r = r_x > 0$ in the standard local picture around x, Section 7.3. We may assume $x_j \in B_r(x)$, for all j. Set $y_j = P(x_j)$ and $y = P(x)$.

We set $\Sigma_j = \Sigma_{y_j} Y$ and $\Sigma = \Sigma_y Y$. We denote by H_j and H the set of unit horizontal vectors in $T_{x_j} M$ and $T_x M$ respectively. By Q_j and Q we denote the differentials $Q_j := D_{x_j} P$ and $Q = D_x P$ of P and their restrictions $Q_j : H_j \to \Sigma_j$ and $Q : H \to \Sigma$.

After replacing x_j by a subsequence, we assume that the infinitesimal submetries Q_j converge to a submetry $\hat{Q} : T_x M \to C(\hat{\Sigma})$, where $\hat{\Sigma}$ is a Gromov–Hausdorff limit of Σ_j. By $\hat{H} \subset T_x M$ we denote the set of unit horizontal vectors of the infinitesimal submetry $\hat{Q}$.

We fix some $0 < \epsilon < r$ and consider the compact subsets $\Sigma_j^\epsilon \subset \Sigma_j$ of all starting directions of geodesics of length ϵ. The preimages H_j^ϵ of Σ_j^ϵ under the submetries Q_j are exactly the subsets of unit vectors at x_j, which are the starting directions of P-minimal geodesics of length ϵ. Choosing a subsequence, we may assume that the subsets H_j^ϵ converge in the Hausdorff topology to a subset $\hat{H}^\epsilon$ of $\hat{H}$ in $T_x M$.

All subsequent statements are based on the simple observation that P-minimal geodesics converge to P-minimal geodesics. In particular, such limits are horizontal. Thus, $\hat{H}^\epsilon$ is contained in H.

8.2. **Semicontinuity.** The situation is easily described if x_j vary along the same fiber. The first statement of the next Lemma just means that tangent (and normal) spaces of a set of positive reach vary semicontinuously. The second statement, related to Question 1.16, means that along any manifold fiber the differentials vary continuously.

Lemma 8.2. *In the notations above, assume in addition that x_j are contained in the leaf $L = L_x$. Then $\hat{H} \subset H$.*

If, in addition, x is contained in a $\mathcal{C}^{1,1}$-submanifold K, open in L, then $\hat{H} = H$ and the submetries $Q_j = D_{x_j} P : T_{x_j} M \to T_y Y$ converge to $Q = D_x P : T_x M \to T_y Y$.

Proof. Since $x_j \in L_x$, we have $y_j = P(x_j) = P(x) = y$. Thus, $\Sigma_j = \Sigma_{y_j} Y = \Sigma_y Y = \Sigma$. Hence, by the choice of $\epsilon < r$, $\Sigma_j^\epsilon = \Sigma_j$ and $H_j^\epsilon = H_j$ for all j. Therefore, $\hat{H}^\epsilon = \hat{H} \subset H$, proving the first statement.

If x_j, x are contained in K, a $\mathcal{C}^{1,1}$-submanifold open in L, then $T_{x_j} L = T_{x_j} K$ are vector spaces converging to $T_x K$. Therefore, $T_{x_j}^\perp L$ converge

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to $T_x^\perp L$. Due to Proposition 5.4, it suffices to prove that $Q_j : H_j \to \Sigma$ converge to $Q : H \to \Sigma$.

For $h_j \in H_j$ converging to $h \in H$, the geodesics $\gamma_{h_j}, \gamma_h : [0, r] \to M$ are P-minimal. Hence, $P \circ \gamma_{h_j}$ are geodesics starting in y and converging to $P \circ \gamma_h$. Thus, $Q_j(h_j)$ converges to $Q(h)$. $\qquad\square$

If x_j vary in different fibers we still have:

Lemma 8.3. *In the notations above, the linear span W_j of H_j^ϵ contains H_j and the linear span of $\hat{H}^\epsilon$ contains $\hat{H}$.*

Proof. Let $m = \dim Y$. By the Bishop–Gromov inequality in $B_r(y) \subset Y$, we have a uniform lower bound $\mathcal{H}^{m-1}(\Sigma_j^\epsilon) \geq \delta > 0$ for some $\delta > 0$ and all j large enough. Due to the continuity of the Hausdorff measure, [BGP92], we have the same lower bound $\mathcal{H}^{m-1}(\hat{\Sigma}^\epsilon) \geq \delta$.

Assume that the linear space W_j of H_j^ϵ does not contain the convex set H_j. Then $\mathcal{H}^{n_j}(H_j^\epsilon) = 0$, where $n_j = \dim(H_j)$. Due to Corollary 4.6, it implies $\mathcal{H}^{m-1}(\Sigma_j^\epsilon) = 0$, in contradiction to the previous observations.

The same reasoning shows that the linear span of $\hat{H}^\epsilon$ contains $\hat{H}$. $\qquad\square$

As a consequence we deduce a first general analogue of Lemma 8.2:

Corollary 8.4. *In the notations above, assume in addition that $T_x L$ is a vector space. Then $\hat{H} \subset H$.*

Proof. We have $\hat{H}^\epsilon \subset H$. By Lemma 8.3, the linear hull of $\hat{H}$ is contained in the linear hull of H. Since $T_x L$ is a vector space, H is a round sphere, thus the unit sphere in its linear hull. Hence, $\hat{H} \subset H$. $\qquad\square$

If the local submetry P is *transnormal*, Corollary 8.4 applies to all points x; see also [MR20, Lemma 45].

Using Lemma 8.2 we prove that foot-point projections from fibers to manifold fibers are open maps:

Corollary 8.5. *Let $P : M \to Y$ be a surjective local submetry. Let $y', y \in Y$ be connected in Y by a unique geodesic γ of length s. Assume that the starting direction $v \in \Sigma_{y'} Y$ of the geodesic γ has in $\Sigma_{y'} Y$ an antipodal direction. Set $L = P^{-1}(y)$ and $L' := P^{-1}(y')$. Let $x \in L$ be such that the closed ball $\bar{B}_{3s}(x)$ in M is compact.*

Then the foot-point projection

$$\Pi^L : L' \cap B_{2s}(x) \to L$$

is uniquely defined and continuous. If, in addition, $L \cap B_{3s}(x)$ is a $\mathcal{C}^{1,1}$-submanifold, then this foot-point projection is an open map.

Proof. Due to compactness of $\bar{B}_{3s}(x)$ a horizontal lift of γ starts at any $x' \in L' \cap B_{2s}(x)$. Moreover, it is unique by the assumption on v and Proposition 5.6. The map Π^L on $L' \cap B_{2s}(x)$ just assigns to x' the endpoint of this P-minimal geodesic. Thus, the foot-point projection

$$\Pi^L : L' \cap B_{2s}(x) \to L$$

is uniquely defined and continuous.

The image of this map is contained in $K = L \cap B_{3s}(x)$. Assume now that K is a $\mathcal{C}^{1,1}$ submanifold.

In order to prove the openness of Π^L, set $z = \Pi^L(x')$ and let $z_j \to z$ be a sequence in K. Consider the starting direction h of the geodesic zx'. Now we apply Lemma 8.2 and find unit horizontal directions $h_j \in T_{z_j}M$ converging to h such that $D_{z_j}P(h_j) = D_zP(h)$ is the starting direction of γ in $\Sigma_y Y$.

The points $x_j = \exp(s \cdot h_j)$ lie on L', converge to x' and satisfy $\Pi^L(x_j) = z_j$. This finishes the proof. $\qquad\square$

8.3. **Continuity of vertical spaces.** We start with the simple

Corollary 8.6. *Assume that the vertical spaces $T_{x_j}L_{x_j}$ and T_xL_x are vector spaces of the same dimension. Then $T_{x_j}L_{x_j}$ converges to T_xL_x.*

Proof. The statement is equivalent to the equality $\hat{H} = H$. Due to Corollary 8.4, $\hat{H} \subset H$. By assumption, H_j and H are round spheres of the same dimension, hence so is $\hat{H}$. The required equality follows. $\quad\square$

The next continuity and stability statement is more involved and more surprising. It is the key to Theorem 1.2.

Theorem 8.7. *Let $P : M \to Y$ be a surjective local submetry. Let y_j be a sequence of points in Y converging to y. Assume that the spaces of directions $\Sigma_{y_j}Y$ converge in the Gromov–Hausdorff topology to $\Sigma_y Y$.*

Then there exists some $\delta > 0$ and some j_0 such that, for all $j > j_0$, the following holds true. The spaces $\Sigma_{y_j}Y$ and $\Sigma_y Y$ are isometric and any direction in $\Sigma_{y_j}Y$ is a starting direction of a geodesic of length δ.

Proof. Choose $x \in P^{-1}(y)$ and $x_j \in P^{-1}(y_j)$ converging to x. In the following we use the notation introduced in Subsection 8.1.

Define $F_j : \Sigma_j^\epsilon \to \Sigma$ by sending $v \in \Sigma_j^\epsilon \subset \Sigma_{y_j}Y$ to the starting direction $w \in \Sigma_y Y = \Sigma$ of the unique geodesic connecting y and $\gamma_v(\epsilon)$.

Since $\epsilon < r$ and satisfies Corollary 7.1, $F_j(\Sigma_j^\epsilon) \subset \Sigma_y Y$ converge in the Hausdorff topology to $\Sigma_y Y$.

Due to the semicontinuity of angles in Alexandrov spaces, [AKP19, Section 7.7.4], the maps F_j converge (after choosing a subsequence), to

a 1-Lipschitz map $F : \hat{\Sigma}^\epsilon \to \Sigma$, where $\hat{\Sigma}^\epsilon \subset \hat{\Sigma}$ is the limit of the subsets Σ_j^ϵ in the Gromov–Hausdorff limit $\hat{\Sigma}$ of the sequence Σ_j.

The map F is a surjective and 1-Lipschitz and $\hat{\Sigma}$ is isometric to Σ. Hence, $\hat{\Sigma}^\epsilon = \hat{\Sigma}$ and F is an isometry, [Pet98, Section 1.2].

In particular, for any $\rho > 0$ and all sufficiently large j, the set Σ_j^ϵ is ρ-dense in Σ_j. We fix some $\rho < \frac{\pi}{2}$.

Consider $L_j = P^{-1}(y_j)$. For any $z \in L_j \cap B_r(x)$ the preimage H_z^ϵ of Σ_j^ϵ in the set of unit horizontal vectors $H_z \subset T_z M$ under the submetry $D_z P : H_z \to \Sigma_{y_j}$ is ρ-dense in H_z. Moreover, for any vector $h \in H_z^\epsilon$ the geodesic $\gamma_h : [0, \epsilon] \to M$ is P-minimal, thus, it satisfies $d(L_j, \gamma_h(\epsilon)) = \epsilon$.

Applying now [Lyt05c, Proposition 1.8, Theorem 1.6], we find some $\delta > 0$ depending only on ϵ and the curvature bounds of $B_r(x)$ such that L_j has reach $\geq \delta$ in $B_{\frac{r}{3}}(x)$.

This implies the last statement of the Theorem.

For all j, such that $d(y_j, y) < \delta$, the unique geodesic between y and y_j can be extended beyond both endpoints as a geodesic. Due to [Pet98], this implies that Σ_{y_j} and Σ_y are isometric. $\qquad\square$

8.4. **The most technical statement.** Now we turn to the final continuity statement. Its proof uses a part of Theorem 1.2, which will be proved later in Section 10 not relying on the present Subsection. Another simple ingredient in the proof will be the following variant of the classical theorem of Hurwitz in complex analysis:

Lemma 8.8. *Let U be a locally compact space, let $f_j, f : U \to B$ be continuous and open maps to a Euclidean ball B. Assume that for any $x_j \to x$ in U the points $f_j(x_j)$ converge to $f(x)$.*

Let $p \in B$ be such that $C := f^{-1}(p)$ is non-empty and compact. Then, for all j large enough, the preimage $f_j^{-1}(p)$ is not empty.

Proof. Assume the contrary. Going to a subsequence we assume that $p \notin f_j(U)$, for all j. Consider a compact neighborhood V of C in U.

Find x_j in V such that $p_j := f_j(x_j)$ is the closest point to p in $f_j(V)$, which exists, since V is compact. Since f_j is an open map on U, the point x_j must be contained in the boundary ∂V. Since f_j converges to f on C, the points p_j converge to p.

Going to a subsequence we may assume that x_j converges to a point $x \in \partial V$. Then $f(x) = p$ but $x \notin C$, which is impossible. $\qquad\square$

The statement we are going to prove is reminiscent of [Pet98] and, in view of [Pet98], could have been expected for all Alexandrov spaces. However, this expectation is wrong in general, as has been shown by a clever 3-dimensional counterexample by Nina Lebedeva.

Proposition 8.9. *Let $P : M \to Y$ be a surjective local submetry. Let $y, z \in Y$ be connected by a geodesic $\gamma : [a, b] \to Y$. Let $v \in \Sigma_y Y$ and $w \in \Sigma_z Y$ be the starting and ending directions of γ. Assume that there exist $v' \in \Sigma_y Y$ and $w' \in \Sigma_z Y$ such that $d(v, v') = d(w, w') = \pi$.*

Then $\Sigma_y Y$ and $\Sigma_z Y$ are isometric.

Proof. Due to [Pet98], all spaces of directions $\Sigma_{\gamma(t)} Y$ for $t \in (a, b)$ are pairwise isometric. Thus, by symmetry, it suffices to prove that $\Sigma_{\gamma(t)} Y$ is isometric to $\Sigma_y Y$ for some (and hence any) $t \in (a, b)$.

Choose a Lipschitz submanifold $K \subset L := P^{-1}(y)$, open in L. Let $x \in K$ be arbitrary. Choose $r > 0$ as in the standard local picture around x. In addition, we may assume that $B_{10r}(x) \cap L \subset K$.

Consider a P-minimal lift of $\bar{\gamma} : [a, a + r] \to M$ of γ starting in x. Choose $x_j = \bar{\gamma}(t_j)$ for $t_j > a$ converging to a. Set $y_j = P(x_j) = \gamma(t_j)$.

Now we are in the situation described in Subsection 8.1. Moreover, $\Sigma_j = \Sigma_{y_j} Y$ are pairwise isometric and, therefore, isometric to their limit space $\hat{\Sigma}$. We need to prove that Σ is isometric to $\hat{\Sigma}$.

Set $K_j = P^{-1}(y_j) \cap B_r(x)$, which is an open subset in the fiber $P^{-1}(y_j)$. Due to Corollary 8.5, the foot-point projection $\Pi^L : K_j \to L$ is a continuous, open map $\Pi^L : K_j \to K$ which sends x_j to x. Moreover, by our assumption on the starting direction of γ and Proposition 5.6 the vector $\gamma'(a)$ has a *unique* horizontal lift at any $z \in K$. Therefore the map $\Pi^L : K_j \to K$ is injective.

Thus $\Pi^L : K_j \to \Pi^L(K_j)$ is a homeomorphism onto an open subset of K. Hence, K_j is a $\mathcal{C}^{1,1}$ submanifold of the same dimension as K.

In particular, the assumptions of Corollary 8.6 are satisfied. We deduce, that $T_{x_j} K_j$ converge to $T_x K$ and that H_j converge to H. Thus $\hat{H} = H$. Therefore, it suffices to prove that the submetries $Q : H \to \Sigma$ and $\hat{Q} : H \to \hat{\Sigma}$ have the same fibers.

Remark 8.10. *A word of caution: The convergence of H_j to H is not (!) sufficient to conclude that Σ_j converge to Σ. The counterexample one should have in mind is given by a submetry with discrete fibers, for instance $\mathbb{R} \to \mathbb{R}/\mathbb{Z}_2 = [0, \infty)$, or a product of such a submetry with a Riemannian submersion. The remaining part of the proof excludes a behaviour such as in these examples. At a final instance, the proof relies on Lemma 8.8 and the injectivity of the projection of K_j to K.*

For any $u \in H$ we denote by F_u respectively $\hat{F}_u$ the Q-fiber, respectively the $\hat{Q}$-fiber through u. We already have seen, that there exists a subset $\hat{H}^\epsilon \subset H$ of positive measure, such that $\hat{F}_u \subset F_u$ for all $u \in H^\epsilon$.

The rest of the proof proceeds in three steps.

(1) There exists a subset $\tilde{H} \subset \hat{H}^\epsilon$ of positive measure, such that, for any $u \in \tilde{H}$, the fiber $\hat{F}_u$ is a union of connected components of F_u.

(2) For any $u \in \tilde{H}$ the fiber $\hat{F}_u$ intersects all components of F_u.

(3) For any $u \in H$ the fibers F_u and $\hat{F}_u$ coincide.

Starting with Step 1, we apply Theorem 1.2 to Q and $\hat{Q}$ and find a subset $\tilde{H}$ of full measure in H^ϵ, such that for any $u \in \tilde{H}$ the fibers F_u and $\hat{F}_u$ are closed submanifolds of the unit sphere H, both of the same dimension $\dim(H) - \dim(\Sigma) = \dim(H) - \dim(\hat{\Sigma})$. By construction, $\hat{F}_u$ is contained in F_u for all $u \in \tilde{H}$. This implies (1).

We proceed with step (3), assuming that (2) has already been verified. Combining with (1), we see that $F_u = \hat{F}_u$ for all $u \in \tilde{H}$.

For any $h, h' \in H$ we have $F_h = F_{h'}$ if and only if $d(F_u, h) = d(F_u, h')$ for all $u \in \tilde{H}$. This, is due to (2.1), to the positive measure of $Q(\tilde{H})$, Corollary 4.6, and to the fact that distance functions to points in a set of positive measure separate points in any Alexandrov space.

Similarly, for any $h, h' \in H$ we have $\hat{F}_h = \hat{F}_{h'}$ if and only if $d(\hat{F}_u, h) = d(\hat{F}_u, h')$ for all $u \in \tilde{H}$. These two statements together imply that for any $h \in H$ the fibers F_h and $\hat{F}_h$ coincide.

It remains to prove (2), for a possibly smaller subset $\tilde{H}^0$ of positive measure of $\tilde{H}$ chosen in Step (1). In order to find this subset, we call a point $p \in B_{3\epsilon}(y) \setminus B_{2\epsilon}(y)$ a *good point*, if $T_p Y = \mathbb{R}^m$ and there exists exactly one geodesic connecting y_j and p, for all j. Almost all points in $B_{2\epsilon}(y) \setminus B_\epsilon(y)$ are good points (in any Alexandrov space). Since $T_p Y = \mathbb{R}^m$, for any j the geodesic $y_j p$ satisfies the assumptions of Corollary 8.5.

For any good point p and any j, consider P-minimal lifts η_j^p starting at x_j of the geodesics $y_j p$. The set of starting directions of all such lifts η_j^p is a fiber of the submetry $Q_j : H_j \to \Sigma_j$. The limit of these fibers exists and it is a fiber of $\hat{Q}$. The union $\tilde{H}^1$ of all such fibers of $\hat{Q}$ is, by construction, contained in $\hat{H}^\epsilon$. The argument based on the Bishop–Gromov volume comparison, that we used to verify that $\hat{H}^\epsilon$ has positive measure in H, shows that $\tilde{H}^1$ has positive measure in H.

We now set $\tilde{H}^0 := \tilde{H} \cap \tilde{H}^1$ and are going to verify (2) for any $u \in \tilde{H}^0$.

By construction, we have a good point $p \in Y$ such that the following holds true. The direction u is the starting direction of a P-minimal geodesic from x to $L' := P^{-1}(p) \cap B_{5\epsilon}(x)$. The Q-fiber F_u through u is the set of all starting directions of P-minimal geodesics from x to L'.

Moreover, letting $E_j \subset L'$ be the set of endpoints of P-minimal geodesics from x_j to L', the $\hat{Q}$-fiber $\hat{F}_u$ is described as follows. The sets

E_j converge in the Hausdorff topology to $E \subset L'$ and $\hat{F}_u$ is the set of starting directions of x to points in E.

The set E_j is the preimage of x_j under the open, continuous foot-point projection $\Pi^{K_j} : L' \to K_j$, Corollary 8.5.

Set $f = \Pi : L' \to K$ and $f_j := \Pi \circ \Pi^{K_j} : L' \to K$. Note that f and f_j are open maps, by Corollary 8.5. Clearly for any $q_j \to q$ in L' the images $f_j(q_j)$ converge to $f(q)$.

The map $\Pi : K_j \to K$ is an open embedding, hence fibers of f_j and of Π^{K_j} coincide. Therefore, the claim that $\hat{F}_u$ intersects all components of F_u just means that any connected component of the fiber $f^{-1}(x)$ contains a limit point of a sequence $q_j \in f_j^{-1}(x)$.

The fiber F_u, homeomorphic to $f^{-1}(x)$, has finitely many components. Thus, assuming the contrary, we find a component C of $f^{-1}(x)$, and a small neighborhood W of C in L' such that $f_j^{-1}(x)$ does not intersect W, for all j large enough. But this contradicts Lemma 8.8.

This contradiction finishes the proof of Step 2.

Thus, we have proved that Q and $\hat{Q}$ have the same fibers. Therefore, Σ is isometric to $\hat{\Sigma}$. This finishes the proof of the Proposition. $\qquad\square$

9. STRICT CONVEXITY

In this section we prove that small balls in the base space Y are convex. This somewhat technical section is not used for other results.

Let $N \subset M$ be a closed $\mathcal{C}^{1,1}$ submanifold of a manifold M with two sided bounded curvature. Then the distance function d_N is semiconvex in a small neighborhood O of N, and the function $f = d_N^2$ is $\mathcal{C}^{1,1}$ in O, [Lyt05c], [KL20]. We are going to see that f is strictly convex in directions almost orthogonal to N.

Lemma 9.1. *Let $N \subset M$ be a $\mathcal{C}^{1,1}$-submanifold, let $x \in N$ be arbitrary. Set $f = d_N^2 : M \to \mathbb{R}$ and fix some distance coordinates around x. Then there exists $\epsilon > 0$ with the following property. For any geodesic $\gamma : [0, \epsilon] \to B_{5\epsilon}(x)$, with $d(\gamma'(0), T_x^\perp N) < \epsilon$, we have*

$$(f \circ \gamma)''(t) \geq 1$$

for almost all $t \in [0, \epsilon]$.

Proof. Fix a sufficiently small $\epsilon > 0$ and $U = B_{5\epsilon}(x)$. Then, for all $z \in B_\epsilon(x) \cap L$, the function d_z^2 is $\mathcal{C}^{1,1}$ in U and the restriction to any geodesic γ in U satisfies $||(d_z^2 \circ \gamma)''(t) - 2|| < \delta$ for almost all t, where δ goes to 0 as $\epsilon \to 0$.

Since d_N is semiconvex in U and $d_N = 0$ on N, the $\mathcal{C}^{1,1}$ function $f = d_N^2$ satisfies $(f \circ \gamma)'' \geq -\delta$ almost everywhere on any geodesic γ in

U, where again δ goes to 0 with ϵ. Together with the lower curvature bound this implies $|(f \circ \gamma)''(t)| \leq 3$, for almost all t.

Geodesics in U are uniformly $\mathcal{C}^{1,1}$, [LY06], [BN93]. Thus, for any geodesic γ in U, the assumption $d(\gamma'(0), T_x^{\perp} N) < \epsilon$ implies $d(\gamma'(t), T_x^{\perp} N) < C \cdot \epsilon$ for some fixed $C \geq 1$ and all t in the domain of definition of γ.

Denote by $H_p f : T_p M \times T_p M \to \mathbb{R}$ the Hessian of f at a point $p \in O$, whenever it exists. Since f is $\mathcal{C}^{1,1}$ in U the Hessian exists at almost all points in U. Standard measure theoretic arguments ("on almost all geodesics the Hessian $H_p f$ exists at almost all points") show that it suffices to prove the following *claim*, for all sufficiently small ϵ:

For all $p \in U$ at which the Hessian $H_p f$ exists, we have $H_p f(w, w) \geq 1$, for all unit vectors $w \in T_p M$ with $d(w, T_x^{\perp} N) \leq \epsilon$.

Consider the projection $q = \Pi^L(p)$. Consider the Lipschitz submanifold $K_p = \exp_q(O_q)$, where O_q is the open ball of radius 5ϵ in the normal space $T_q^{\perp} N$ around the origin. If ϵ is small enough, then the (unit sphere in) tangent space $J_p = T_p K_p \subset T_p M$ to K_p at p is δ-close to $T_x^{\perp} N$, where δ goes to 0 with ϵ.

We compare now the $\mathcal{C}^{1,1}$-functions d_q^2 and f at the point p. Note, that d_q^2 and f have the same value and derivative at x and that they coincide on K_p. Applying the Taylor formula at p, we get $H_p f(w, w) \geq \frac{3}{2}$, for any unit direction $w \in J_p$. Since $||H_p f|| \leq 3$ we deduce $H_p f(v, v) \geq 1$, for all unit vectors $v \in T_p M$ which are δ-close to J_p, for a sufficiently small δ. This proves the claim and the Lemma. $\square$

As a consequence we derive:

Theorem 9.2. *Let $P : M \to Y$ be a surjective local submetry. Then, for any $y \in Y$, there exists some $r > 0$ such that the function $f = d_y^2$ is a strictly convex function on $B_r(y)$.*

Proof. Consider a point $x \in L = P^{-1}(y)$ such that a small neighborhood of x in L is a $\mathcal{C}^{1,1}$ submanifold.

For sufficiently small $r > 0$ any short geodesic γ in $B_r(y)$ can be lifted to a horizontal geodesic $\hat{\gamma}$ in $B_{2r}(x)$. Due to Lemma 8.4, the starting direction of any such geodesic encloses an angle almost equal to $\frac{\pi}{2}$ with any vector in $T_x L$ (considered in some fixed distance coordinates). Then $f \circ \gamma = d_L^2 \circ \hat{\gamma}$, and this function is 1-convex by Lemma 9.1. $\square$

This result finishes the proof of Theorem 1.3.

10. Stratification, regular part

10.1. **Stratification, regular part, boundary.** As before, let $P : M \to Y$ be a surjective local submetry and let $m = \dim Y$. For

$0 \leq l \leq m$, denote by Y_+^l the set of points $y \in Y$, at which the tangent cone $T_y Y$ splits $\mathbb{R}^l$ as a direct factor.

For all Alexandrov regions, the complement $Y \setminus Y_+^l$ has Hausdorff dimension at most $l - 1$, [BGP92], see also [LN19].

In our case, the subsets Y^l defined in Corollary 1.5 are exactly

$$Y^l = Y_+^l \setminus Y_+^{l+1} .$$

Semicontinuity of spaces of directions, [BGP92], stability of tangent spaces along geodesics, [Pet98], and Corollary 1.5 directly imply:

Lemma 10.1. *For any l, the space Y_+^l is open in Y. The set Y^l is locally closed in Y. The closure $\bar{Y}^l$ of Y^l is contained in $Y \setminus Y_+^{l+1}$.*

For any geodesic $\gamma : [0, a) \to Y$ with $\gamma(0) \in Y_+^l$ we have $\gamma(t) \in Y_+^l$ for all $t > 0$.

The set of regular points is

$$Y_{reg} := Y^m = Y_+^m .$$

Thus, $y \in Y_{reg}$ if and only if $T_y Y = \mathbb{R}^m$. From Lemma 10.1 and the density of regular points in any Alexandrov space, [BGP92], we deduce:

Lemma 10.2. *Let $P : M \to Y$ be a surjective local submetry. Then the set Y_{reg} is open and dense in Y. Any geodesic $\gamma : [0, a) \to Y$ which starts at a point in Y_{reg} is completely contained in Y_{reg}.*

A point y is contained in Y^{m-1} if and only if its tangent space $T_y Y$ splits off $\mathbb{R}^{m-1}$ but is not isometric to $\mathbb{R}^m$. This happens if and only if $T_y Y$ is isometric to the Euclidean halfspace $\mathbb{R}_+^m = \mathbb{R}^{m-1} \times [0, \infty)$.

In particular, $T_y Y$ has non-empty boundary at any point $y \in Y^{m-1}$, thus $Y^{m-1} \subset \partial Y$. On the other hand, Y^{m-1} is dense in ∂Y, as it is the case in any Alexandrov region.

10.2. **Higher regularity of Y_{reg}.** From Theorem 8.7 we now deduce:

Corollary 10.3. *Let $P : M \to Y$ be a surjective local submetry. For any $y_0 \in Y_{reg}$, there exists $r_0 = r_0(y_0) > 0$, such that for any $y \in B_{r_0}(y_0)$ any $v \in \Sigma_y Y$ is the starting direction of a geodesic of length r_0.*

For any $y_0 \in Y_{reg}$, we fix $x_0 \in P^{-1}(y_0)$ and $r_0 > 0$ as in the standard local picture around x_0 and satisfying Corollary 10.3.

Now we consider *strainer maps* around y_0, [BGP92]. Namely, we choose points $y_1, ..., y_m$ at distance $r_0/2$ to y_0 such that the incoming directions of geodesics connecting y_0 to y_i enclose pairwise angles $\frac{\pi}{2}$ (or sufficiently close to $\frac{\pi}{2}$). Then, we consider the map $F : B_s(y_0) \to \mathbb{R}^m$ with a sufficiently small $s = s(r_0)$, whose coordinates are distance functions $f_i = d_{y_i}$. The map F is biLipschitz onto an open subset of

$\mathbb{R}^m$, with biLipschitz constant arbitrary close to 1, [BGP92], provided r_0 is sufficiently small. We show:

Lemma 10.4. *Strainer maps $F : B_s(y_0) \to \mathbb{R}^m$ as above define a $\mathcal{C}^{1,1}$ atlas on Y_{reg}. The distance function on Y_{reg} is defined by a Riemannian metric, which is Lipschitz continuous with respect to this atlas.*

Proof. Consider $x_0 \in P^{-1}(y_0)$, $r_0, s > 0$ and a strainer map $F : B_s(y_0) \to \mathbb{R}^m$ with coordinates $f_i = d_{y_i}$ as above.

Then $f_i \circ P = d_{L_i}$ on $B_{r_0}(x_0)$, where $L_i = P^{-1}(y_i)$. By the choice of r_0, the ball $B_{r_0}(x_0)$ is contained in the set $\mathcal{U}(L_i)$ on which $d_{L_i}^2$ is $\mathcal{C}^{1,1}$.

There exists $C > 0$ such that for any geodesic γ in $B_s(y_0)$ the restriction $f_i \circ \gamma$ is differentiable everywhere and $(f_i \circ \gamma)'$ is C-Lipschitz. This follows by restricting d_{L_i} to a horizontal lift of γ_i to $B_r(x_0)$.

This implies the existence of some $C_1 = C_1(C) > 0$ with the following property. For any geodesic $\gamma : [a, b] \to B_s(y_0)$ with midpoint q the distance between $F(q)$ and the midpoint in $\mathbb{R}^m$ between the endpoints $F(\gamma(a))$ and $F(\gamma(b))$ is at most $C_1 \cdot \ell^2(\gamma)$, compare [LY06, Lemma 2.1].

Therefore, for any Lipschitz continuous semiconcave function $g : B_s(y_0) \to \mathbb{R}$, the composition $g \circ F^{-1}$ is semiconcave. This implies that any transition map between different strainer maps as above, has coordinates which are semiconvex and semiconcave at the same time, hence they are of class $\mathcal{C}^{1,1}$.

This proves that the strainer maps define a $\mathcal{C}^{1,1}$ atlas on Y_{reg}.

The distance function on Y_{reg} is described in any distance coordinates $F : B_s(y_0) \to \mathbb{R}^m$ by a Riemannian metric g whose coordinates are expressed as algebraic functions of some distance functions to some points and their derivatives, as explained in [Per94a], see also [AB18]. Thus, the Riemannian metric g is locally Lipschitz continuous. $\qquad\square$

We can now easily prove a local generalization of Theorem 1.2:

Theorem 10.5. *Let $P : M \to Y$ be a surjective local submetry. Then Y_{reg} is open and dense in Y. Any geodesic $\gamma : [0, a) \to Y$ starting on Y_{reg} is contained in Y_{reg}.*

Y_{reg} carries a natural structure of a Riemannian manifold with a Lipschitz continuous Riemannian metric and the restriction $P : P^{-1}(Y_{reg}) \to Y_{reg}$ is a $\mathcal{C}^{1,1}$ Riemannian submersion.

Proof. Y_{reg} is open, dense and has the stated convexity property, due to Lemma 10.2. Due to Lemma 10.4, Y_{reg} has a natural $\mathcal{C}^{1,1}$ atlas of distance coordinates which makes it into a Riemannian manifold with a Lipschitz continuous metric tensor.

It remains to verify that $P : P^{-1}(Y_{reg}) \to Y_{reg}$ is a $\mathcal{C}^{1,1}$-Riemannian submersion. We fix $y_0 \in Y_{reg}$ and $x_0 \in P^{-1}(y_0)$ and consider the

standard local picture around x_0. For $y_0 = P(x_0)$ consider distance coordinates $F : B_s(y_0) \to \mathbb{R}^m$ around y_0. We have already observed, that the composition of P with any coordinate function f_i of F is given in $B_s(x)$ by the distance to a fiber of P, which is $\mathcal{C}^{1,1}$, thus $F \circ P$ is $\mathcal{C}^{1,1}$ in a small ball around x_0. Hence P is $\mathcal{C}^{1,1}$.

The differential $D_{x_0}P$ of P at x_0 is a linear map and a submetry between Euclidean spaces. Hence, it is a surjective linear map which preserves the length of any vector orthogonal to the kernel. Thus, P is a Riemannian submersion. $\qquad\square$

11. Singular strata

11.1. **Main statement.** We now turn to the singular strata and prove:

Theorem 11.1. *Let $P : M \to Y$ be a surjective local submetry. Then the set Y^l is an l-dimensional manifold.*

For every point $y \in Y^l$ there exists some $r > 0$ with the following properties. The closed ball $\bar{B}_r(y)$ is compact and convex in Y and so is the intersection $\bar{B}_r(y) \cap Y^l$. Moreover, for any $y' \in B_r(y) \cap Y^l$ any geodesic starting in y' can be extended to a geodesic of length r.

The set Y^l has a natural $\mathcal{C}^{1,1}$ atlas, such that the distance on Y^l is locally defined by a locally Lipschitz continuous Riemannian metric.

Proof. Fix $y \in Y^l$. Thus, $T_yY = \mathbb{R}^l \times C(\Sigma_0)$ with $\operatorname{diam}(\Sigma_0) \leq \frac{\pi}{2}$.

Let $r > 0$ be such that the distance function d_y^2 is convex on the compact ball $\bar{B}_{2r}(y)$, Corollary 9.2. Due to Theorem 8.7, we may choose r so that, for any $z \in B_r(y)$ such that $\Sigma_z Y$ is isometric to $\Sigma_y Y$, any geodesic starting at z can be extended to a geodesic of length $3r$.

We claim that $z \in B_r(y)$ is contained in Y^l if and only if the starting direction $v \in \Sigma_y Y$ of the geodesic γ connecting y and z lies in the direct factor $\mathbb{R}^l$ of the tangent space T_yY.

Indeed, if v is contained in $\mathbb{R}^l$ then it has an antipode in $\Sigma_y Y$. Since, z is an inner point of the geodesic γ, (by the choice of r), also the incoming direction $w \in \Sigma_z Y$ of γ has in $\Sigma_z Y$ an antipode. Due to Proposition 8.9, $\Sigma_z Y$ and $\Sigma_y Y$ are isometric. In particular, $z \in Y^l$.

On the other hand, assume v is not contained in the $\mathbb{R}^l$-factor of T_yY. Then the tangent space $T_v(T_yY)$ has $\mathbb{R}^{l+1}$ as a direct factor. Under the convergence of the rescaled spaces $(\frac{1}{j}Y, y) \to (T_yY, 0)$ the points $y_j = \gamma(\frac{1}{j})$ converge to the point $v \in T_yY$. For all j, the spaces of directions $\Sigma_{y_j}Y$ are all isometric to $\Sigma_z Y$, due to [Pet98]. By the semicontinuity of spaces of directions, there exists a distance non-decreasing map from $\Sigma_v(T_yY)$ to $\Sigma_z Y$ (the Gromov–Hausdorff limit of $\Sigma_{\gamma(\frac{1}{j})}Y$). This implies,

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that $\Sigma_z Y$ has at least l pairs of antipodal points at pairwise distance $\geq \frac{\pi}{2}$. Thus, $T_z Y$ splits off $\mathbb{R}^{l+1}$, hence $z \notin Y^l$.

The exponential map $\exp_y$ defines a homeomorphism from the r-ball around the origin in $T_y Y$ to the r-ball in Y. By above, this homeomorphism restricts to a homeomorphism from the r-ball around the origin in the Euclidean factor $\mathbb{R}^l \subset T_y Y$ with $B_r(y) \cap Y^l$. Thus, Y^l is an l-dimensional topological manifold.

We have seen that, for any $z \in B_r(y) \cap Y^l$, the space of directions $\Sigma_z Y$ is isometric to $\Sigma_y Y$. By the choice of r, for any $z \in Y^l \cap B_r(y)$, any geodesic starting in z can be extended to a geodesic of length $3r$.

We now claim that $B_r(y) \cap Y^l$ is a convex subset of Y. Consider a geodesic γ connecting two points z, z' in $Y^l \cap B_r(y)$. The geodesic γ can be extended beyond z', by the choice or r. Thus the space of direction $\Sigma_{z'} Y$ is isometric to spaces of directions at all other points on γ, by [Pet98]. Thus, all points of γ are in Y^l, proving the claim.

Now we apply the same arguments as in the proof of Lemma 10.4, to see that the distance coordinates in the Alexandrov region $B_r(y) \cap Y^l$ define a $\mathcal{C}^{1,1}$-atlas such that the distance is given in these coordinates by a Lipschitz continuous Riemannian metric. $\qquad\square$

The connected components of the sets Y^l are the so-called *primitive extremal subsets*:

Theorem 11.2. *Let $P : M \to Y$ be a surjective local submetry, with M a Riemannian manifold. Let y be a point in Y^l and let E be the connected component of y in Y^l. Then the closure $\bar{E}$ is the smallest extremal subset of Y which contains the point y.*

Proof. Clearly, the Riemannian manifold E does not contain proper extremal subsets. Thus, any extremal subset of Y which contains y must contain E. Since any extremal subset is closed, it must contain the closure $\bar{E}$.

For any point $z \in Y^k$, for any k, the tangent space $T_z Y$ splits as $T_z Y = \mathbb{R}^k \times T'$ and this direct factor $\mathbb{R}^k = T_z Y^k$ is an extremal subset of $T_z Y$ (since T' is the cone over a space with diameter $\leq \frac{\pi}{2}$). From this and [PP93, Proposition 1.4], the intersection $B_r(z) \cap Y^k$ is an extremal subset of the Alexandrov region $B_r(z)$ for all sufficiently small r.

Due to [PP93], it remains to prove, that for any natural k and any $z \in \bar{E} \cap Y^k$, the set $\bar{E}$ contains a small ball $B_r(z) \cap Y^k$. However, we can choose r (for any fixed point $z \in \bar{E} \cap Y^k$) as in Theorem 11.1. Thus, for any $z' \in B_r(z) \cap Y^k$, any geodesic from z can be extended as a geodesic to length r. Thus, for some $y' \in E$ sufficiently close to z the geodesic η from z' to y' can be extended beyond y'. By [Pet98], all

points on η but z' have the same spaces of directions as y', hence they belong to E. Therefore, $z' \in \bar{E}$. This finishes the proof. $\qquad\square$

11.2. **Topological structure.** Over any fixed stratum Y^l, the local submetry P has a local product structure.

Proposition 11.3. *Let $P : M \to Y$ be a surjective local submetry, with M a Riemannian manifold. Let $y \in Y^l$ and $x \in P^{-1}(y)$ be arbitrary. Then there exists a neighborhood O of x in $P^{-1}(Y^l)$ and a homeomorphism*

$$J : P(O) \times (O \cap P^{-1}(y)) \to O$$

such that $P \circ J$ is the projection onto the first factor.
  If M is complete then, for any connected component E of Y^l, the restriction $P : P^{-1}(E) \to E$ is a fiber bundle.

Proof. Let $r > 0$ as in the standard local picture around x and satisfying Theorem 11.1. Set $C := B_r(x) \cap P^{-1}(y)$ and $U := B_r(y) \cap Y^l$. For any $z \in U$, we have a unique geodesic γ^z from y to z. This geodesic γ^z is contained in Y^l, Theorem 11.1. Moreover, for any $x' \in C$ there exists a unique horizontal lift $\gamma^{z,x'}$ of γ^z starting in x'.
  These lifts define a map $J : U \times C \to B_{2r}(x) \cap P^{-1}(U)$ given as

$$J(z, x') := \gamma^{z,x'}(d(z, y)) .$$

The map J is continuous, injective and the composition $P \circ J$ is just the projection onto the first factor U.
  In order to see that J is an open map, consider $x_j \in B_{2r}(x) \cap P^{-1}(U)$ converging to a point x_0 in the image of J. Consider geodesics η_j in Y^l from $P(x_j)$ to y and their unique unique horizontal lifts γ_j starting in x_j. These horizontal geodesics converge to the unique shortest curve from x_0 to $P^{-1}(y)$. Therefore, the endpoints of γ_j converge to a point in C. Since C and U are open, for large j, the endpoints of γ_j are in C and the point x_j in the image of J.
  If M is complete, then the above construction, works for $C = L_x$, showing that $P : P^{-1}(U) \to U$ is a trivial fiber bundle. Since being a fibre bundle is a local condition for connected base spaces, the restriction $P : P^{-1}(E) \to E$ is a fiber bundle. $\qquad\square$

11.3. **Strata have positive reach.** We are going to prove that the preimage $P^{-1}(Y^l)$ of any stratum is a subset of positive reach and start with some preliminaries. The first two statements about differentiable manifolds are probably well-known, but we could not find a reference.

Lemma 11.4. *Let N be a k-dimensional topological submanifold of a Riemannian manifold M. Assume that, for all $x \in N$, the blow up*

$$T_x N := \lim_{t \to 0} (\frac{1}{t} N, x) \subset T_x M$$

is a well-defined k-dimensional linear subspace. Finally, let the map $x \to T_x N$ be locally Lipschitz. Then N is a $\mathcal{C}^{1,1}$ submanifold of M.

Proof. The assumptions and the claim are local. Choosing a chart around a point x, we may assume that $M = \mathbb{R}^n$ and $T_x N = \mathbb{R}^k \subset \mathbb{R}^n$.

Consider the orthogonal projection $G : \mathbb{R}^n \to \mathbb{R}^k$. We find a neighborhood O of x in N, such that $T_z N$ is close to $\mathbb{R}^k$, for all $z \in O$. In particular, $D_z G : T_z N \to \mathbb{R}^l$ has biLipschitz constant close to 1. Hence, making O smaller if needed, we see that $G : O \to \mathbb{R}^l$ is a biLipschitz map onto an open subset O' of $\mathbb{R}^l$, [Lyt05a, Proposition 1.3].

The inverse $F := G^{-1} : O' \to O$ is a biLipschitz map onto O. The map $F : O' \to \mathbb{R}^n$ is differentiable at each point $y \in O'$, and the differential is the inverse map of $D_z G$, where $z = F(y)$.

By assumption, we find Lipschitz continuous functions $b_1, ..., b_k : O \to \mathbb{R}^n$ such that for any $z \in O$ the vectors $b_j(z)$ define a basis of $T_z N$. Then $a_j(y) := D_z G(b_j(z))$, with $z = F(y)$, is a basis of Lipschitz continuous vector fields in O'. Moreover, $D_y F(a_j(y)) = b_j(F(y))$ for all $y \in O'$.

Now, we can express the canonical vector fields e_j on O' as linear combinations of the vector fields a_i with Lipschitz continuous coefficients. Therefore, $y \to D_y F(e_j)$ are linear combinations of vector fields b_i with Lipschitz continuous coefficients. Hence, the partial derivatives of F are Lipschitz continuous and the map F is $\mathcal{C}^{1,1}$. $\qquad\square$

Lemma 11.5. *Let b be a Lipschitz continuous vector field on an open subset $O \subset \mathbb{R}^k$. Let ϕ_t denote the local flow of b. Then, for any $p \in O$, there exists a neighborhood O' and $A > 0$ such that the inequality*

$$||(\phi_t(z) - \phi_t(y)) - (z - y)|| \leq A \cdot t \cdot ||z - y||$$

holds true for all $y, z \in O'$ and all t, with $|t| \leq \frac{1}{A}$.

Proof. Differentiating we see

$$\frac{d}{dt}(\phi_t(z) - \phi_t(y)) - (z - y)) = b(\phi_t(z)) - b(\phi_t(y)) .$$

The flow ϕ of the Lipschitz vector field b is locally Lipschitz. Thus, the right hand side has norm bounded by $A \cdot ||z - y||$ for some $A > 0$, all sufficiently small t and all z, y sufficiently close to x. The claim now follows by integration. $\qquad\square$

Now we can prove:

Theorem 11.6. *Let M be a Riemannian manifold and let $P : M \to Y$ be a surjective local submetry. Then, for any stratum $Y^l \subset Y$, the preimage $P^{-1}(Y^l)$ has positive reach in M.*

Proof. The claim is local. Thus, we may fix some $x \in P^{-1}(Y^l)$ and may replace M by an arbitrary small neighborhood O of x. Set $y = P(x)$ and $L = P^{-1}(y)$.

For any $x' \in L$, we consider standard local pictures around x and x'. We observe that $P^{-1}(Y^l)$ has positive reach in a neighborhood of x if and only if it has positive reach in a neighborhood of x' (and this happens if and only if Y^l has positive reach around y in Y). Thus, we may replace x by any other point x' in L. Therefore, we can assume that a neighborhood of x' in L is a $\mathcal{C}^{1,1}$ submanifold.

Restricting to a neighborhood around such $x = x'$, we can apply Proposition 11.3 and assume that $N := P^{-1}(Y^l)$ is a topological manifold of dimension $k = l + e$, where e is the dimension of L.

Since fibers converge to fibers under convergence of submetries, Lemma 2.4, we see as in Corollary 3.4, that $T_p N \subset T_p M$ exists for all $p \in N$ and equals $(D_p P)^{-1}(T_p Y^l)$. Due to Proposition 5.6, $T_p N$ is a direct product $T_p N = U_p \times V_p$ of a vector space $U_p = \mathbb{R}^l$, the set of horizontal vectors in $T_p N$ and $V_p = \mathbb{R}^e$, the tangent space to the fiber L_p.

Due to Lemma 11.4, we need to prove that U_p and V_p depend in a locally Lipschitz way on $p \in N$, since $\mathcal{C}^{1,1}$ submanifolds have positive reach, see Subsection 6.2.

We fix a small $r > 0$ as in Theorem 11.1 and fix points $y_1, ..., y_l$ in $B_{\frac{r}{3}}(y) \cap Y^l$, such that the distance functions d_{y_i} define a distance coordinate chart around y in Y^l. Set $L_i := P^{-1}(y_i)$. Then, the distance functions $f_i := d_{L_i} = d_{y_i} \circ P$ are $\mathcal{C}^{1,1}$ in a small ball O_0 around x. Moreover, the gradients $\nabla_p f_i$ build at every point $p \in O_0 \cap N$ a basis of the vector space U_p, the horizontal part of $T_p N$. This shows that the assignment $p \to U_p$ is locally Lipschitz continuous in $O_0 \cap N$.

In order to see that also the distribution of tangent spaces to fibers $p \to V_p$ is locally Lipschitz continuous on $O_0 \cap N$, we proceed as follows. The gradient flows ϕ^i of $-f_i$, thus the flows of the Lipschitz continuous vector fields $-\nabla f_i$ preserve the subset $N \cap O_o$. Moreover, this flow preserves the leaves of P, Lemma 4.2. From Lemma 11.5, we see that along the flow lines of the flows ϕ^i, the tangent spaces to leaves $V_{\phi^i_t(p)}$ depend in a Lipschitz manner on the time t.

For $p, q \in O_0 \cap N$, we find some $t_i, 1 \le i \le l$, such that $\sum |t_i| \le C \cdot d(p, q)$, for some (universal) constant C, such that the composition

of $\phi^i_{t_i}$ sends the fiber of p to the fiber of q, as we verify by looking at the projection of the flows to Y^l.

Thus, we find p' in the fiber L_p through p, such that $d(q, p') \leq 2C \cdot d(q, p)$ and such that V_q and $V_{p'}$ are at distance $C' \cdot d(q, p)$ for some constant C'.

All fibers L_p, for $p \in O_0 \cap N$ have uniformly positive reach and therefore, they are locally uniformly $\mathcal{C}^{1,1}$, [Lyt05c].

This shows that U'_p and U_p are at distance bounded by a constant times $d(p, p')$. Thus, the distance between V_p and V_q is bounded by a constant times $d(p, q)$, for all $p, q \in N \cap O_0$.

This finishes the proof of the theorem. $\qquad\square$

Note that a combination of Theorem 11.6 and Proposition 11.3 finishes the proof of Theorem 1.7.

12. MANIFOLD FIBERS

12.1. **Existence of long geodesics.** The following result can be localized and generalized, see the subsequent Remark 12.2 and compare [LT10, Theorem 1.6]. In the proof below we apply the machinery of [KLP18], to conclude that the geodesic flow on Y_{reg} preserves the Liouville measure. It is possible that this can be seen in a more direct way.

Proposition 12.1. *Let M be a Riemannian manifold and $P : M \to Y$ a submetry. Assume that Y is compact and has no boundary. Then the union of bi-infinite local geodesics $\gamma : \mathbb{R} \to Y_{reg}$ is dense in Y.*

Proof. The space Y is an Alexandrov space without boundary. The set Y_{reg} of all regular points in Y is a $\mathcal{C}^{1,1}$ manifold with a Lipschitz continuous Riemannian metric, Theorem 1.2.

Due to [KLP18, Theorem 1.6], almost every unit direction v at almost every point in Y_{reg} is the starting direction of a bi-infinite local geodesic $\gamma_v : \mathbb{R} \to Y$ completely contained in Y_{reg} provided the so-called *metric-measure boundary* of Y vanishes.

Now [KLP18, Theorem 1.7, Lemma 6.4] imply that Y has vanishing metric measure boundary. More precisely, following the notations of [KLP18], one needs to verify that any limiting measure in the space of signed Borel measures $\mathcal{M}(Y)$ on Y of the sequence

$$\nu := \lim_{r_j \to 0} \frac{\mathcal{V}_{r_j}}{r_j}$$

is the 0 measure. The signed measures $\mathcal{V}_r$ appearing in the formula is the average deviation measure from the Euclidean volume growth:

$$\mathcal{V}_r(A) = \int_A (1 - \frac{\mathcal{H}^m(B_r(y))}{\omega_m \cdot r^m}) \, d\mathcal{H}^m(y) \ ,$$

where A is a Borel subset of Y and ω_m the volume of unit ball in $\mathbb{R}^m$.

Due to [KLP18, Theorem 1.7 (3)], any such measure ν is a Radon measure concentrated on the set of regular points Y_{reg}, since Y has no boundary. Moreover, due to [KLP18, Theorem 1.7 (1)], the measure is absolutely singular with respect to the Hausdorff measure on Y_{reg}.

Since the distance on Y_{reg} is obtained from a Lipschitz continuous Riemannian metric, the *minimal metric derivative measure* $\mathcal{N}$, defined in [KLP18, Section 6.3], (essentially, just a bound on the derivatives of the coordinates of the Riemannian metric), is absolutely continuous with respect to the Hausdorff measure. Finally, due to [KLP18, Lemma 6.4], the measure ν is absolutely continuous with respect to $\mathcal{N}$. These facts together imply the vanishing of ν and therefore the result. $\qquad\square$

Remark 12.2. *Localizing the above argument and using* [KLP18, Section 3.6] *the following can be shown. For any surjective local submetry $P : M \to Y$ and almost every unit tangent vector $v \in TY_{reg}$ there exists is one maximal quasigeodesic $\gamma_v : (-a_v, b_v) \to Y$ starting in the direction of v. This quasigeodesic $\gamma_v : (-a_v, b_v) \to Y$ is completely contained in $Y^m \cup Y^{m-1}$, intersects Y^{m-1} in a discrete set of times. Outside these intersection points γ_v is a local geodesic. Finally, the local quasigeodesic flow preserves the Liouville measure.*

12.2. **Non-manifold fibers and the boundary.** The following Lemma is formulated for general Alexandrov spaces. The existence of many infinite local geodesics assumed on Y is conjecturally satisfied for all boundaryless Alexandrov spaces, [PP94], [KLP18]. For some cases it has been verified in [KLP18]; for bases of submetries of manifolds it has been shown in Proposition 12.1.

Lemma 12.3. *Let $P : X \to Y$ be a submetry between Alexandrov spaces of curvature ≥ 1. Assume that for any $y \in Y$ there exist local geodesics $\gamma_n : [0, \infty) \to Y$ such that $\gamma_n(0)$ converge to y. Then $\partial X = \emptyset$.*

Proof. Assume the contrary and choose some $x \in X \setminus \partial X$. By assumption, we find a local geodesic $\gamma : [0, \infty) \to Y$, such that

$$d(\gamma(0), P(x)) < d(x, \partial X) \ .$$

Consider a horizontal lift $\bar{\gamma} : [0, \infty) \to X$ of γ such that $d(\bar{\gamma}(0), x) = d(\gamma(0), P(x))$. Thus $\bar{\gamma}(0) \notin \partial X$.

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The lift $\bar{\gamma}$ is a local geodesic, since γ is a local geodesic. Since ∂X is an extremal subset of X, any geodesic meeting ∂X in an inner point of the geodesic must be contained in ∂X. Thus, $\bar{\gamma}$ cannot intersect ∂X.

The distance function to ∂X is strictly concave on X, [Per91]. More precisely, [AB03, Theorem 1.1] shows that

$$f := \sin \circ d_{\partial X} \circ \bar{\gamma} : [0, \infty) \to \mathbb{R}$$

satisfies in the weak sense

$$f''(t) + f(t) \leq 0 \ .$$

But such a positive function f can be defined at most on an interval of length π. This contradiction finishes the proof. $\qquad\square$

We are going to prove the following local version of Theorem 1.8.

Theorem 12.4. *Let $P : M \to Y$ be a local submetry. If a fiber $L = P^{-1}(y)$ is not a $\mathcal{C}^{1,1}$-submanifold of M then $y \in \partial Y$.*

Proof. Since L cannot be open in M, the point y cannot be isolated.

Due to Theorem 7.2 we find a submetry $Q : \mathbb{S}^k_+ \to \Sigma_y$ for a hemisphere $\mathbb{S}^k_+$ for some $k \geq 0$. Thus, Σ_y is connected. If $\dim(\Sigma_y) = 0$ then Σ_y must be a point. Hence, $T_y Y = [0, \infty)$ and $y \in \partial Y$.

Assume $\dim(\Sigma_y Y) \geq 1$. Since $\partial \mathbb{S}^k_+ \neq \emptyset$ and $\mathbb{S}^k_+$, we deduce that $\partial(\Sigma_y Y) \neq \emptyset$ from Lemma 12.3 and Proposition 12.1. Thus, $y \in \partial Y$. $\quad\square$

12.3. **Transnormal submetries.** As in the introduction, a (local) submetry $P : M \to Y$ satisfying the equivalent conditions of the next proposition will be called *transnormal*.

Proposition 12.5. *Let $P : M \to Y$ be a local submetry. Then the following are equivalent:*

(1) *All fibers of P are topological manifolds.*
(2) *All fibers of P are $\mathcal{C}^{1,1}$ submanifolds.*
(3) *For any P-horizontal unit vector h, the local geodesic γ_h in M is a P-horizontal curve, for all times of its existence.*
(4) *For any $x \in M$, the differential $D_x P : T_x M \to T_{P(x)} Y$ satisfies condition (3).*

Proof. The equivalence of (1) and (2) follows from Proposition 6.1, since all fibers of P have positive reach in M, by Theorem 1.1.

Assume (2) and let $\gamma_h : [0, a] \to M$ be a local geodesic starting in a horizontal direction. By Proposition 7.3, there exists some $0 < r \leq a$ such that $\gamma_h : [0, r) \to M$ is horizontal, and we can choose r to be maximal with this property. Applying Proposition 7.3 at $\gamma_h(r)$, we deduce that the incoming direction $-\gamma'_h(r)$ is horizontal in $T_{\gamma_h(r)} M$.

Due to (2), the horizontal space at $\gamma_h(r)$ is a vector space. Therefore, the direction $\gamma_h'(r)$ is horizontal as well. Thus, for a small $\epsilon > 0$ also the restriction of $\gamma_h : [r, r + \epsilon) \to M$ is horizontal. If $r < a$ we obtain a contradiction to the maximality of r. Hence, (2) implies (3).

Assume (3) and suppose a fiber $L = P^{-1}(y)$ not be a submanifold. Then for some $x \in L$ the tangent space $T_x L$ is not a vector space, Proposition 6.1. Thus, the horizontal space $T_x^\perp L$ is not a vector space and we find a unit vector $h \in T_x^\perp L$ such that $-h$ is not horizontal. Thus, for a small $\epsilon > 0$, the geodesic $\gamma_h : [0, \epsilon] \to M$ is horizontal, while γ_{-h} is not horizontal. Therefore, setting $w = -\gamma_h'(t)$ for some $\epsilon > t > 0$, we find a horizontal vector, such that the geodesic in the direction of this vector does not stay horizontal for all times. This contradiction shows that (3) implies (2).

Moreover, the argument above also implies that the submetry $D_x P : T_x M \to T_y Y$ does not satisfy (3) in this case, hence (4) implies (2).

It remains to prove that (3) implies (4). Thus, assume that P satisfies (3)but $D_x P$ does not satisfy (3) for some $x \in M$. Find $v, w \in T_x M$ and a point u on the segment $[v, w] \in T_x M$ such that the segment $[v, u]$ is horizontal for $D_x P$ but the segment $[u, w]$ is not horizontal.

Choosing v closer to u we may assume that $[v, u]$ is a unique $D_x P$-minimal geodesic from v to the fiber of $D_x P$ through u. We find sequences p_i in M, such that under the convergence $(\frac{1}{i} M, x) \to (T_x M, 0)$ the sequences p_i converges to v. By lifting appropriate geodesics horizontally, we find a sequence of P-minimal geodesics γ_i from p_i to some point q_i, such that under the convergence $(\frac{1}{i} M, x) \to (T_x M, 0)$ the sequence γ_i converge to a $D_x P$-minimal geodesic starting in v and going to the $D_x P$-fiber through u. By uniqueness, γ_i must converge to the segment $[v, u]$.

We extend P-minimal geodesics γ_i to geodesics $\hat{\gamma}_i$ in M by some fixed length $r > 0$ beyond p_i and q_i. Under the convergence to $T_x M$, these geodesics converge to the line $\hat{\gamma}$ extending the segment $[v, u]$.

By assumption $\hat{\gamma}_i$ is horizontal. By Proposition 3.2 the images $P \circ \hat{\gamma}_i$ are quasigeodesics in Y. By construction, they converge to the curve $P \circ \hat{\gamma}$ in $T_y Y$. But under a non-collapsed convergence, a limit of quasigeodesics is a quasi-geodesic, [Pet07]. Thus, $P \circ \hat{\gamma}$ is parametrized by arclength. Therefore, the line $\hat{\gamma}$ is horizontal in contrast to our assumption.

This contradiction finishes the proof of the final implication. $\square$

The proof of the last implication above shows the following:

Corollary 12.6. *Let $M_i \to M$ and $Y_i \to Y$ be convergent sequences in the pointed Gromov–Hausdorff topology of Riemannian manifolds*

(with locally uniform bounds on curvature) and Alexandrov spaces, re-spectively. Let $P_i : M_i \to Y_i$ be a sequence of transnormal submetries converging to a submetry $P : M \to Y$. If the the convergence $Y_i \to Y$ is non-collapsing then the submetry P is transnormal.

12.4. **Equifocality.** The following result has appeared in a slightly more special form in [MR20]. We only formulate and prove here global version of the result, which is known in the theory of singular Riemann-ian foliations under the misleading name of *equifocality*.

Proposition 12.7. *Let $P_i : M_i \to Y, i = 1, 2$ be transnormal sub-metries with the same base space. If $\gamma_i : [0, a) \to M_i$ are horizontal local geodesics such that $P \circ \gamma_i$ coincide on $[0, \epsilon)$, for some $\epsilon > 0$ then $P \circ \gamma_1 = P \circ \gamma_2$ on $[0, a)$.*

Let $Q_i : \mathbb{S}^{n_i} \to Z$ be transnormal submetries to the same base space Z. If $Q_1(v_1) = Q_2(v_2)$ the $Q_1(-v_1) = Q_2(-v_2)$.

Proof. Let $n_i = \dim M_i$ in the first statement and let $n = \max\{n_1, n_2\}$ in both statements. We will prove both statements simultaneously by induction on n. The case $n = 1$ is left to the reader.

Assume that both claims are know in dimension $n-1$. Then to prove the first claim, we find a maximal $b \leq a$ such that $P_i \circ \gamma_i$ coincide on $[0, b)$. Assume $b < a$ and consider $x_i = \gamma_i(b)$. Then $P_1(x_1) = P_2(x_2)$. Denote this point by y. Let H_i be the set of unit P_i-horizontal vectors at x_i, which are unit spheres.

The differentials $Q_i := D_{x_i} P_i : H_i \to Z := \Sigma_y Y$ are transnormal submetries which send the incoming directions v_i of γ_i at x_i to the same vector in $\Sigma_y Y$. By the inductive assumption, also $Q_i(-v_i)$ coincide. Thus, for small $r > 0$ the P_i-minimal geodesics $\gamma_i : [b, b + r) \to M_i$ are sent to geodesics in Y starting in y in the same direction. Thus $P \circ \gamma_i$ coincide on $[b, b+r)$ in contradiction to the maximality of b. This proves the first statement.

To prove the second, we take an arbitrary Q_1-horizontal direction w_1 at v_1 and the geodesic $\gamma_1 : [0, \pi] \to \mathbb{S}^{n_1}$ starting in this direction. Con-sider then a Q_2-horizontal lift $w_2 \in T_{v_2} \mathbb{S}^{n_2}$ of the direction $D_{v_1} Q_1(w_1)$. Let $\gamma_2 : [0, \pi] \to \mathbb{S}^{n_2}$ be the geodesic starting in the direction w_2.

Since Q_i are transnormal the geodesic γ_i is Q_i-horizontal. By con-struction their Q_i-images coincide initially. By the already proved first statement, $Q_1 \circ \gamma_1 = Q_2 \circ \gamma_2$ on $[0, \pi]$. At the time π we obtain $Q_1(-v_1) = Q_2(-v_2)$. $\square$

12.5. **Topological structure of projections between close fibers.** We are going to describe the topology of foot-point projections onto manifold fibers. For the sake of simplicity we only state a global result

in the case of transnormal submetries and for connected fiber. For disconnected fibers, the claim remains true for all connected components.

The proof of the next theorem heavily relies on deep results in geometric topology, characterizing fiber bundles. We refer to [LN18, Theorem 4.5, Theorem 4.8] for a more detailed discussion of these results.

Theorem 12.8. *Let M be a complete Riemannian manifold, let $P\colon M \to Y$ be a transnormal submetry. Let $L \subset M$ be a connected leaf of P.*

Then there exists $r > 0$ such that the foot point projection $\Pi^L\colon U = B_r(L) \to L$ is a fiber bundle. Moreover, for any fiber $L' \subset U$ the restriction $\Pi^L|_{L'}\colon L' \to L$ is a fiber bundle as well.

Proof. Let r be smaller than the constant provided by Theorem 1.3. Then the normal exponential map Exp^N gives a homeomorphism between the r-neighborhood of the zero section of the normal bundle to L and U which commutes with the foot-point projections. This proves the first part of the theorem.

Let L' be another fiber contained in U. Due to Corollary 8.5 the map $f := \Pi^L|_{L'}\colon L' \to L$ is open. Set $y := P(L), z := P(L')$ and $v_0 \in \Sigma_y Y$ be the starting direction of the unique geodesic from y to z. Consider the point $v \in T_y Y$ lying in direction v_0 at distance $d(y, z)$ from the origin $0_y \in T_y Y$ (thus, $\exp_y(v) = z$).

Under the homeomorphism given by the normal exponential map Exp^N the fibers $f^{-1}(x)$ of $f : L' \to L$ are sent to fibers $F_x := D_x P^{-1}(v)$, where $D_x P : H_x \to T_y Y$ is the restriction of the differential of the submetry to the horizontal part.

By Proposition 12.5, the fibers of $D_x P$ and, therefore, the fibers of f are topological manifolds. Moreover, the point v has positive injectivity radius in $T_y Y$, by Theorem 1.3. Thus, the fibers $F_x = D_x P^{-1}(v)$ all have positive reach $s > 0$, independent of x. Therefore, there exists some $\epsilon > 0$ independent of x, such that any ball of radius $\delta < \epsilon$ in F_x is contractible, [Fed59], [Lyt05c].

In other words, the fibers of the map f are compact topological manifolds which are *locally uniformly contractible*. Now a combination of results [Ung69, Theorem 1], [DH58], [Fer91, Theorems 1.1-1.4], [Ray65, Theorem 2] implies that f is a fiber bundle, see [LN18, Section 4] for a detailed discussion. $\square$

12.6. **A comment on the factorization theorem.** The following result has been proved in [Lyt01].

Theorem 12.9. *Let $P : X \to Y$ be a submetry between Alexandrov spaces. Then the connected components of fibers of P define an equidistant decomposition of X. Thus, P admits a canonical factorization*

$P = P_1 \circ P_0$, where the submetry $P_0 : X \to Y_0$ has connected fibers and the submetry $P_1 : Y_0 \to Y$ has discrete fibers.

The proof of this result is technical and remains technical if the Alexandrov space X is replaced by a complete Riemannian manifold M. However, if the submetry is transnormal, the proof is much easier.

Indeed, in this case the connected components L_p^0 of fibers L_p of P define a *transnormal decomposition* of M in the sense of [Mol88]: thus L_p^0 are $\mathcal{C}^{1,1}$ submanifolds and any local geodesic in M which starts orthogonal to any leaf of the decomposition remains orthogonal to all leaves it intersects. But such a transnormal decomposition is equidistant, as one readily verifies by the first variation formula.

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