Spectral Theory of Automorphic Forms on the Hyperbolic Plane

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Abstract

This thesis presents the spectral theory of automorphic forms (Maass forms) of arbitrary real weight $k$. We investigate automorphic Eisenstein series and state their meromorphic continuation. In addition, we develop the related functional analytic background, for instance the self-adjoint extension of the hyperbolic Laplacian and the construction of its resolvent operator. The reader is assumed to be familiar with the basic theory of holomorphic modular forms and its general notions, such as fundamental domains or cusps. Moreover, a survey of hyperbolic geometry and general linear functional analysis might be helpful. Lastly we discuss some examples and offer an outline of a proof of Duke’s famous theorem as an application of the theory.
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Glossary
Introduction

Spectral theory of automorphic forms for arbitrary real weight $k$ has its origin in the two articles [Roe66] and [Roe67], which constitute the main reference of this thesis. However, those articles aim to provide a rough survey of the topic and are not self-contained. To achieve self-containedness we rephrase the work of Roelcke and to this end we follow the approaches of Iwaniec and Kubota. Both authors illustrate the case of weight 0 and trivial multiplier. We will adapt and generalize their effort to Roelcke’s setting of weight $k \in \mathbb{R}$ and some unitary multiplier $v$. Alternatively we could have treated the topic from the modern perspective of representation theory, which is done for instance in the textbook [Bum97]. The advantage of the former strategy is to “rescue” the intermediate results of Roelcke without just filling in the missing details into his exposition. Summarizing, the goal of this thesis is to encounter Roelcke’s results from a more modern perspective.

The first chapter briefly recaps the overall setting, introduces the hyperbolic Laplacian $\Delta_k$ of weight $k$ as well as automorphic forms and computes their Fourier expansion.

Chapter two invokes the additional structure of square-integrability and proves the existence of the unique self-adjoint extension $\tilde{\Delta}_k$ of $\Delta_k$. The intermediate results proven here can be utilized already to locate the spectrum of $\tilde{\Delta}_k$.

The third chapter is devoted to the construction of the resolvent operator of $\tilde{\Delta}_k$, which we perform more or less from scratch. Its existence plays a crucial role in the upcoming two chapters.

In chapter four we move to Eisenstein series and encounter those in three variants: Automorphic, truncated and incomplete ones. The key property of automorphic Eisenstein series is the meromorphic continuation via their Fourier expansion. Unfortunately a full proof of this fact would exceed the limits of this thesis, yet is based on holomorphicity of the resolvent in $\lambda$. 
Chapter five deals with the discrete contribution to the spectrum of $\tilde{\Delta}_k$. We study invariant integral operators and point-pair invariants in greater detail than in chapter three. This enables us to apply the Hilbert-Schmidt theorem from general linear functional analysis to a suitable modified version of the resolvent of $\tilde{\Delta}_k$.

The purpose of chapter six is the inspection of the continuous spectrum of $\tilde{\Delta}_k$. As a byproduct we evolve a second discrete part of the spectrum caused by the residues of the (automorphic) Eisenstein series. Technical ingredients are the Eisenstein transform of a bump function and the Selberg / Harish-Chandra transform of a free-space point-pair invariant kernel. Examples and applications of the spectral theory are the subject of chapter seven. For instance we cover Poincaré series and the spectral expansion of the automorphic kernel function defining the resolvent of $\tilde{\Delta}_k$.

Finally we portray one more application in chapter eight, where we present a theorem of Duke and illustrate the role of the spectral theory in its proof.

I would like to thank my advisor Professor Dr. Özlem Imamoglu for our regular meetings and her continuous feedback to my work. The door to her office was always open (in the worldly sense) inviting me to come around with questions regarding this thesis.
Chapter 1

Preliminaries

We establish our overall setting and briefly recap the main facts. We follow [Roe66, §1 and §2], and [Iwa02, sections 1.6 and 1.7].

1.1 Hyperbolic geometry

The Poincaré half-plane model: Throughout this thesis we will work in the space

\[ \mathbb{H} := \{ z = x + iy \in \mathbb{C} \mid y > 0 \} \]

equipped with

\[ |dz|^2 = \frac{(dx)^2 + (dy)^2}{y^2} \quad \text{and} \quad d\mu = \frac{(dx)(dy)}{y^2} \quad (1.1) \]

as line element and surface element respectively.

Fractional linear transformations: The group \( \text{GL}_2^+ (\mathbb{R}) \) acts on \( \mathbb{H} \cup \partial \mathbb{H}^1 \) via

\[ \forall \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2^+ (\mathbb{R}) \forall z \in \mathbb{H} : \]

\[ \gamma z := \frac{az + b}{cz + d} \quad \gamma \infty := \frac{a}{c} \quad \gamma \left( -\frac{d}{c} \right) := \infty \quad (1.2) \]

due to the well known fact

\[ \text{Im}(\gamma z) = \frac{\text{det}(\gamma) \text{Im}(z)}{|cz + d|^2} > 0 \quad (1.3) \]

We restrict ourselves to the subgroup \( \text{SL}_2 (\mathbb{R}) \), because it contains the orientation-preserving elements of \( \text{GL}_2^+ (\mathbb{R}) \) and acts transitively on \( \mathbb{H} \).

\(^1\)The boundary is \( \mathbb{R} \cup \{ \infty \} \)
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**Fundamental domains:** Let $\Gamma < \text{SL}_2(\mathbb{R})$. Recall that the set of orbits $r \backslash \mathbb{H}$ of 1.2 forms a partition of $\mathbb{H}$ and the orbits are precisely the equivalence classes of $w \sim z : \iff \exists \gamma \in \Gamma : \gamma w = z$. A subset $r \backslash \mathbb{H} =: \mathcal{F} \subseteq \mathbb{H}$ is a fundamental domain for $\Gamma$ if it is closed in $\mathbb{H}$ and $\mathcal{F}$ contains each inequivalent orbit exactly once. In other words each point in $\mathbb{H}$ is equivalent to some point in $\mathcal{F}$ and no two different points in $\mathcal{F}$ are equivalent. (The boundary of $\mathcal{F}$ may be partitioned in equivalent parts.) A fundamental domain is not unique in general. Two examples are pictured below.

**Cusps:** From (1.2) we deduce that the $\text{SL}_2(\mathbb{R})$-orbit of $\infty$ is $\mathbb{Q} \cup \{\infty\}$. Take a subgroup $\Gamma < \text{SL}_2(\mathbb{R})$. Then a cusp $a$ for $\Gamma$ is an $\Gamma$-orbit of some $r_a \in \mathbb{Q} \cup \{\infty\}$, i.e. $a = \Gamma r_a$. A scaling matrix $\gamma_a \in \Gamma$ of $a$ is defined by $\gamma_a \infty = r_a$. It is convenient to work with representatives instead of the full orbit, so $\gamma_a$ has the property $2 \gamma_a \infty = a \in \mathbb{Q} \cup \{\infty\}$. Denote the stabilizer of $a$ by $\Gamma_a = \{\xi \in \Gamma \mid \xi a = a\}$. The Poincaré half-plane model has the distinguished cusp $\infty$ ([IK04, p. 355]):

$$\gamma_a^{-1} \Gamma a \gamma_a = \left\{ \begin{pmatrix} 1 & nh \\ 0 & 1 \end{pmatrix} \mid n \in \mathbb{Z} \right\} \subseteq \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \mid n \in \mathbb{Z} \right\} = \Gamma_\infty$$

(1.4)

The number $0 < h \in \mathbb{Z}$ above is unique and called width of the cusp $a$. Observe that $\Gamma_\infty$ contains precisely the integer translations.

**Discrete Subgroups:** Recall that $\text{SL}_2(\mathbb{R})$ is a topological group. A subgroup $\Gamma < \text{SL}_2(\mathbb{R})$ is called discrete if the subspace topology of $\Gamma$ is discrete, i.e. every subset of $\Gamma$ is open. Equivalently there exists an open cover of $\text{SL}_2(\mathbb{R})$ in which every open subset contains exactly one element of $\Gamma$. One can also characterize discrete (sub-)groups as zero-dimensional Lie groups. By definition a discrete subgroup is totally disconnected, hence acts discontinuously on $\mathbb{H}$. Formally

$$\forall z \in \mathbb{H} \exists U \subseteq \mathbb{H} \text{ open, such that } z \in U \text{ and } \{ \gamma \in \Gamma \mid \gamma U \cap U \neq \emptyset \} \text{ is finite.}$$

A discrete subgroup $\Gamma$ of $\text{SL}_2(\mathbb{R})$ is called co-compact if there exists a compact subset $K \subseteq \text{SL}_2(\mathbb{R})$ such that $\Gamma K = \text{SL}_2(\mathbb{R})$. Note that a co-compact group has no cusps, since its fundamental domain $\mathcal{F} = r \backslash \mathbb{H}$ is compact. We will need the existence of cusps to work with Fourier expansions at cusps, e.g. in section 1.4. Furthermore we require that a fundamental domain $\mathcal{F}$ for $\Gamma$ has finite hyperbolic volume.

Collecting our assumptions we impose $\Gamma$ to be a **Fuchsian group of the first kind**.

---

2Be aware of the fact that Roelcke uses $\gamma_a a = \infty$ instead.
3Iwa02, proposition 2.1.
4Iwa02, Proposition 2.5.
Example: The full modular group $SL_2(\mathbb{Z})$ is a discrete non co-compact subgroup of $SL_2(\mathbb{R})$, hence all subgroups of $SL_2(\mathbb{Z})$ share those two properties.

(a) $SL_2(\mathbb{Z})$ itself has exactly one cusp $a = \infty$ with width 1. The hyperbolic volume of its fundamental domain $\mathcal{F}$ is given by

$$\int_{\mathcal{F}} d\mu = \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{\sqrt{1-x^2}}^{\infty} \frac{dy}{y^2} = \frac{\pi}{3} < \infty$$

(b) The principal subgroup

$$\Gamma(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid a, d \equiv 1 \mod 2 \text{ and } b, c \equiv 0 \mod 2 \right\}$$

of level 2 has index 6 in $SL_2(\mathbb{Z})$ and three inequivalent cusps $\{0,1,\infty\}$ with equal widths $h = \frac{6}{3} = 2$. (The widths have to be equal, because $\Gamma(2)$ is a normal subgroup of $SL_2(\mathbb{Z})$ and hence all stabilizers are conjugate to each other. All widths have to sum up to the index of $\Gamma$ in $SL_2(\mathbb{Z})$ in virtue of the orbit-stabilizer theorem.) The hyperbolic volume of its fundamental domain $\mathcal{F}$ is given by

$$\int_{\mathcal{F}} d\mu = \sum_{\gamma_j \in \Gamma \backslash SL_2(\mathbb{Z})} \int_{\gamma_j \mathcal{F}} d\mu = 6 \int_{\mathcal{F}} d\mu = 2\pi$$

(c) Similarly any congruence subgroup of $SL_2(\mathbb{Z})$ enjoys all requested properties.

We will revisit those groups in chapter 7, where we discuss their spectral theories. Examples of their fundamental domains are:

![Figure 1.1: A fundamental domain for $SL_2(\mathbb{Z})$ with one cusp at $\infty$.](image1)

![Figure 1.2: A fundamental domain for $\Gamma(2)$ with cusps 0, 1 and $\infty$.](image2)

In both cases the vertical boundary lines are equivalent to each other and the
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boundary circle is equivalent to itself. A detailed discussion can be found in [DS05, sections 2.4 and 3.8], or in [Kob93, sections III.2 and III.3].

1.2 Harmonic analysis on $\mathbb{H}$

The action (1.2) behaves as follows with respect to differentiation:

$$\frac{d}{dz} \gamma z = \frac{1}{(cz + d)^2} \quad \frac{|d\gamma z|}{\text{Im}(\gamma z)} = \frac{|dz|}{\text{Im}(z)}$$  \hfill (1.6)

where we used $\det(\gamma) = 1$ for the first equation and the first equation together with (1.3) for the second one. In other words the Poincaré metric $\frac{|dz|}{y}$ from (1.1) is $SL_2(\mathbb{R})$-invariant.

From (1.1) we derive

**Definition 1.2.1** The Laplace-Beltrami operator or “hyperbolic Laplacian” or simply “Laplacian” on $\mathbb{H}$ is:

$$\Delta_0 := y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$$  \hfill (1.7)

Furthermore for any $k \in \mathbb{R}$ we define the Laplacian with weight $k$ as:

$$\Delta_k := y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) - ik \frac{\partial}{\partial x}$$  \hfill (1.8)

Thanks to (1.6) we infer the following

**Lemma 1.2.2** The linear operators $\Delta_k$ are $SL_2(\mathbb{R})$-invariant, i.e.

$$\forall f \in C^2(\mathbb{H}, \mathbb{C}) \forall \gamma \in SL_2(\mathbb{R}) : (\Delta_k f)(\gamma z) = \Delta_k(f(\gamma z))$$  \hfill (1.9)

**Proof** Recall that $SL_2(\mathbb{R})$ is generated by matrices of the form

$$\begin{pmatrix} a & 0 \\ 0 & \frac{1}{a} \end{pmatrix} \quad \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$  \hfill (1.10)

where $a \in \mathbb{R}_{>0}$ and $b \in \mathbb{R}$. One verifies the claim for those matrices directly and uses (1.6) for the general case. \hfill $\square$

Lets quickly fix some terminology from linear algebra:

**Definition 1.2.3** A function $f \in C^2(\mathbb{H}, \mathbb{C})$ is an eigenfunction of $\Delta_k$ with eigenvalue $\lambda \in \mathbb{C}$ if

$$\Delta_k f = \lambda f$$  \hfill (1.11)

The eigenfunctions of $\Delta_0$ with eigenvalue 0 are called harmonic functions.
Observe that any holomorphic function is a (hyperbolic) harmonic function as well, because holomorphic functions are characterized in virtue of the Cauchy-Riemann equations, which can be phrased shortly as
\[
\frac{\partial f}{\partial y} = i \frac{\partial f}{\partial x}
\] (1.12)
This gives \( \frac{\partial^2 f}{\partial y^2} = -\frac{\partial^2 f}{\partial x^2} \) immediately. But \( \Delta_k \) imposes regularity on \( f \) too:

**Proposition 1.2.4** Any eigenfunction of \( \Delta_k \) is real-analytic, i.e. has a locally convergent Taylor series representing it.

**Proof** This follows from the fact that \( \Delta_k \) is an elliptic operator with real-analytic coefficients, a proof is given in [Rud73, Theorem 8.12]. \( \square \)

We conclude this section with a quick example

**Example 1.2.5** The functions
\[
f(x + iy) = \frac{1}{2}(y^s + y^{1-s}) \quad \text{and} \quad g(x + iy) = \frac{1}{2s-1}(y^s - y^{1-s})
\] (1.13)
are two linearly independent eigenfunctions of \( -\Delta_k \) with eigenvalue \( \lambda = s(1-s) \). Note that \( s \mapsto \lambda \) is a double cover of \( \mathbb{C} \). In particular for \( s = \frac{1}{2} \) those eigenfunctions become \( y^{\frac{1}{2}} \) and \( y^{\frac{1}{2}} \log(y) \).

**Remark 1.2.6** A slightly more general approach to solve (1.11) in the case \( k = 0 \) uses separation of variables. Taking \( f(x + iy) = u(x)v(y) \neq 0 \) yields the equation \( \frac{d^2 u}{u(x)} = -\frac{d^2 v}{v(y)} - \frac{\lambda}{\eta^2} \). The left hand side is independent of \( y \) and the right hand side is independent of \( x \), hence both sides are equal to a constant, say \( -\eta^2 \). The resulting separate ODE’s can be solved by an exponential function for \( u \) and Bessel functions for \( v \).

### 1.3 Automorphic forms

We combine our observations from the previous two sections to define functions, which “respect” our setting in an elegant way: First we introduce the **automorphy factor**
\[
j_{\gamma}(z;k) := \left( \frac{cz + d}{|cz + d|} \right)^k \quad \text{for} \quad k \in \mathbb{R}, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{R})
\] (1.14)
and the **slash-operator**
\[(f|k\gamma)(z) := j_{\gamma}(z;k)^{-1}f(\gamma z)\] (1.15)
Second we need the auxiliary function \( w : SL_2(\mathbb{R}) \times SL_2(\mathbb{R}) \to \{-1, 0, 1\} \) characterized by
\[
w(\gamma_1, \gamma_2) + w(\gamma_1 \gamma_2, \gamma_3) = w(\gamma_1, \gamma_2 \gamma_3) + w(\gamma_2, \gamma_3)
\]
and the function
\[
\sigma_k(\gamma_1, \gamma_2) := e^{2\pi i k w(\gamma_1, \gamma_2)} \in S^1
\]
to formulate the following rules:
\[
\begin{align*}
\sigma_k(\gamma_1, \gamma_2) j_{\gamma_1 \gamma_2}(z; k) &= j_{\gamma_1}(\gamma_2 z; k) j_{\gamma_2}(z; k) \quad \text{(chain rule)} \quad (1.18) \\
(f|_k \gamma_1 \gamma_2) &= \sigma_k(\gamma_1, \gamma_2)(f|_k \gamma_1)|_{k \gamma_2} \quad \text{(associativity)} \quad (1.19)
\end{align*}
\]
More equations on \( w \) are stated in [Iwa97, section 2.6] and an explicit formula for \( w \) can be found in [Pet38, §2, Satz 4]. Additionally we infer:

**Lemma 1.3.1** The linear operators \( \Delta_k \) and \( \cdot|_k \gamma \) commute on the vectorspace of functions \( f : \mathbb{H} \to \mathbb{C} \).

**Proof** According to Lemma 1.2.2:
\[
((\Delta_k f)|_k \gamma)(z) = j_{\gamma}(z; k)^{-1}(\Delta_k f)(\gamma z) = j_{\gamma}(z; k)^{-1} \Delta_k(f(\gamma z))
\]
Hence we seek to show that \( j_{\gamma}(z; k)^{-1} \Delta_k(f(\gamma z)) = \Delta_k(j_{\gamma}(z; k)^{-1} f(\gamma z)) \). Again verify the claim directly for the generating matrices (1.10) and use the chain rule above. \( \square \)

Finally we take a discrete subgroup \( \Gamma < SL_2(\mathbb{R}) \) and a unitary multiplier\(^5\)
\[
v : \Gamma \to Aut(\mathbb{C}) \quad \text{such that:}
v(\gamma_1 \gamma_2) = \sigma_k(\gamma_1, \gamma_2)v(\gamma_1)v(\gamma_2) \quad \text{and} \quad v(-I) = e^{-i\pi k}1
\]
Now we are able to define our object of interest:

**Definition 1.3.2** An automorphic form\(^6\) associated to a 
- discrete subgroup \( \Gamma < SL_2(\mathbb{R}) \)
- unitary multiplier \( v : \Gamma \to Aut(\mathbb{C}) \)
- weight \( k \in \mathbb{R} \)
- eigenvalue \( \lambda \in \mathbb{C} \)
is a function \( f : \mathbb{H} \to \mathbb{C} \) satisfying
\[
\begin{align*}
& (i) \quad f|_k \gamma = v(\gamma)f \quad \text{(transformation law)} \\
& (ii) \quad -\Delta_k f = \lambda f \quad \text{(eigenfunction of} \ -\Delta_k) \\
& (iii) \quad \text{If } a \text{ is a cusp for } \Gamma \text{ with scaling matrix } \gamma_a \text{ then}
\quad (f|_k \gamma_a)(z) \in O(y^\kappa) \text{ as } y \to \infty \text{ uniformly in } x \text{ with a constant } \kappa \in \mathbb{R}.
\end{align*}
\]
\(^5\)We will need unitarity from chapter 2 onwards. The homomorphism-property holds up to sign.
\(^6\)Sometimes also called a Maass (wave) form, e.g. in [Bum97].
Remark 1.3.3 Observe that if $f$ is an automorphic form then the function $F(z) := -\Delta_k f(z)$ still satisfies the transformation law in virtue of lemma 1.3.1 and the eigenfunction condition according to lemma 1.2.2, but may not satisfy the growth condition.

We give a class of examples of automorphic forms. Recall the definition of a modular form:

Definition 1.3.4 Let $k \in \mathbb{Z}$. A modular form of weight $k$ for $SL_2(\mathbb{Z})$ is a function $f : \mathbb{H} \rightarrow C$ satisfying

(i) $\forall \gamma \in SL_2(\mathbb{Z}) : f(\gamma z) = (cz + d)^k f(z)$

(ii) $f$ is holomorphic on $\mathbb{H}$

(iii) $f$ is holomorphic at $i\infty$

Remark 1.3.5 In fact the third condition is a much stronger growth condition than the one we require in definition 1.3.2: We will see in remark 1.4.2 that a modular form $f$ has the following Fourier expansion (often phrased as "q-expansion" of $f$):

$$f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i nz} =: \sum_{n=0}^{\infty} a_n q^n$$

where the last sum is defined on the unit disk as a function of $q$. Now we say\textsuperscript{7} that $f(z)$ is holomorphic at $i\infty$ if and only if its q-expansion is holomorphic at $q = 0$. This implies the growth condition in definition 1.3.2, because in the case of $a = \infty$ we may simply take $\gamma_a = 1$ and obtain at most polynomial growth trivially.

Example 1.3.6 If $f$ is a modular form of necessarily\textsuperscript{8} even weight $k$ for $SL_2(\mathbb{Z})$ then $F(z) := y^k f(z)$ is an automorphic form with associated data $\Gamma = SL_2(\mathbb{Z}), k \in 2\mathbb{Z}, \nu(\gamma) = 1$ and $\lambda = \frac{k}{2}(1 - \frac{k}{2})$.

We will prove the claim that $k \geq 0$ and $k \in 2\mathbb{Z}$ in the previous example in proposition 2.3.3. But we can quickly verify the three conditions of definition 1.3.2. The transformation law is straightforward:

$$(F|\gamma)(z) = j_\gamma(z;k)^{-1} F(\gamma z) \overset{(1.3)}{=} j_\gamma(z;k)^{-1} \left( \frac{y}{|cz + d|^2} \right)^{\frac{k}{2}} f(\gamma z)$$

$$= \left( \frac{cz + d}{|cz + d|^2} \right)^{-k} \frac{y^{\frac{k}{2}}}{|cz + d|^k} (cz + d)^k f(z) = F(z)$$

\textsuperscript{7}IK04, p. 356.

\textsuperscript{8}Classical results in the theory of modular forms show that there are neither modular forms of negative weight nor modular forms of odd weight for $SL_2(\mathbb{Z})$, see e.g. [Kob93, section III.2].
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The eigenfunction-condition reads:

\[-\Delta_k F(z) = -\Delta_0 \left( y^k f(x + iy) \right) + iky \frac{\partial}{\partial x} (y^k f(x + iy)) \]

\[= -y^2 \left( ky_{k-1} \frac{\partial f}{\partial y} (x + iy) + \frac{k}{2} \frac{k}{2} - 1 \right) y^{k-2} f(x + iy) \]

\[= -y^2 \Delta_0 f(x + iy) + iky^{k+1} \frac{\partial f}{\partial x} (x + iy) \]

\[= (\ast) k \left( 1 - \frac{k}{2} \right) y^k f(x + iy) = \lambda F(z) \]

where we used (1.12) in (*) twice. The growth condition is immediate from remark 1.3.5 giving

\[F \asymp a_0 y^2 \text{ as } y \to \infty.\]

Remark 1.3.7 The converse statement to example 1.3.6 is wrong:
Let \( \Gamma = \{ \pm 1 \}, v(\gamma) = 1, k \neq 1 \). Then \( F(z) = y^{1-k} \) is an automorphic form of weight \( k \) and eigenvalue \( \lambda = \frac{k}{2} (1 - \frac{k}{2}) \) (compare (1.3) and example 1.2.5), but \( f(z) = y^{-\frac{1}{2}} F(z) \) fails to be a modular form, because a non-constant real-valued function is never holomorphic\(^{10}\). We will “repair” this counterexample in a more restrictive setting in proposition 2.3.1.

1.4 Fourier expansion

We derive the Fourier expansion of an automorphic form \( f \) at a cusp \( a \) for \( \Gamma \). Thus we have to assume that \( \Gamma \) is a non co-compact discrete group, else there are no cusps. Let \( \gamma_a \) be a scaling matrix of \( a \). Then by automorphy of \( f \) we have

\[\forall n \in \mathbb{Z} : f(\gamma_a \left( \begin{array}{cc} 1 & n \\ 0 & 1 \end{array} \right) z) = f(\gamma_a z) \quad (1.22)\]

hence \( f \) has the Fourier expansion

\[f(\gamma_a z) = \sum_{n \in \mathbb{Z}} f_n(y) e^{2\pi i n x} \quad (1.23)\]

with Fourier coefficients

\[f_n(y) = \int_0^1 f(\gamma_a z) e^{-2\pi i n x} \, dx \quad (1.24)\]

We conjugate the group \( \Gamma \) as in equation 1.4 and hence may assume \( a = \infty \) giving \( \gamma_a = 1 \). The general case follows in virtue of the transformation law

\(^9 a_0 \) denotes the constant term in the q-expansion of \( f \).

\(^{10}\)this holds again due to (1.12)
1.4. Fourier expansion

(see the formulation of the theorem below).

Recalling example 1.2.5 we write \( \lambda = s(1-s) = \frac{1}{4} + r^2 \in \mathbb{C} \). Further justification will be given after proposition 2.2.2. Then we calculate from the eigenfunction condition in definition 1.3.2:

\[
\left( \frac{1}{4} + r^2 \right) f_n(y) = -\Delta k f_n(y) = -y^2 \int_0^1 \left( \frac{\partial^2 f}{\partial x^2}(x + iy) \right) e^{-2\pi i n x} \, dx - y^2 \frac{\partial^2 f_n}{\partial y^2}(y) + iky \int_0^1 \left( \frac{\partial f}{\partial x}(x + iy) \right) e^{-2\pi i n x} \, dx
\]

\[
= -\left( 2\pi i y \right)^2 f_n(y) - y^2 \frac{\partial^2 f_n}{\partial y^2}(y) + iky(2\pi i n f_n(y))
\]

where the boundary terms from integration by parts in the last step cancel due to periodicity of \( e^{2\pi i n x} \).

The case \( n = 0 \) is solved according to example 1.2.5 as follows:

\[
f_0(y) = \begin{cases} A_0 y^{\frac{1}{2} - ir} + B_0 y^{\frac{1}{2} + ir} & , \text{if } r \neq 0 \\ A_0 y^{\frac{1}{2}} + B_0 y^{\frac{1}{2}} \log y & , \text{if } r = 0 \end{cases} \quad (1.25)
\]

The case \( n \neq 0 \) yields Whittakers equation\(^{11}\)

\[
\frac{d^2 f_n}{dy^2}(y) + \left( \frac{1}{4} - \frac{\alpha^2}{y^2} + \frac{\beta}{y} - \frac{1}{4} \right) f_n(y) = 0 \quad (1.26)
\]

where \( \alpha = ir \) is given by \( \lambda \) and \( \beta = \operatorname{sign}(n) \frac{k}{2} \) is determined by \( f \). The solution is expressed in terms of the Whittaker functions\(^{12}\):

\[
f_n(y) = A_n W_{ir,\operatorname{sign}(n)}\left( \frac{1}{2} \right) \left( 4\pi |n| y \right) + B_n M_{ir,\operatorname{sign}(n)}\left( \frac{1}{2} \right) \left( 4\pi |n| y \right) \quad (1.27)
\]

The case \( r = 0 \) is included here. We are only interested in the asymptotic behaviour of those functions as \( y \to \infty \) and give a short heuristic first\(^{13}\): If \( y \) is sufficiently large then (1.26) is a perturbation of \( \frac{d^2}{dy^2} f_n(y) - \frac{1}{4} f_n(y) = 0 \) with solutions \( e^{\pm \frac{y}{2}} \). Hence we expect one solution of (1.26) with exponential growth and one solution with exponential decay as \( y \to \infty \). Working with the explicit formulas given in [AS72, p. 508] justifies this heuristic and yields

\[
M_{ir,\operatorname{sign}(n)}\left( \frac{1}{2} \right) (y) \propto y^{-\frac{1}{2} - ir} e^{\frac{y}{2}} \quad W_{ir,\operatorname{sign}(n)}\left( \frac{1}{2} \right) (y) \propto y^{\frac{1}{2} + ir} e^{-\frac{y}{2}}
\]

as proposed. Note that the asymptotic behaviour is independent of the weight \( k \in \mathbb{R} \). This proves one direction of the following:

\(^{11}\) A modified form of the confluent hypergeometric differential equation, see [AS72, p. 505].
\(^{12}\) AS72, p. 505.
\(^{13}\) Bum97, p. 105.
Theorem 1.4.1 Let $f \in C^2(\mathbb{H}, C)$ satisfy the transformation law and the eigenfunction condition from definition 1.3.2. Then $f$ is an automorphic form, i.e. additionally satisfies the growth condition from definition 1.3.2, if and only if for every cusp $a$ for $\Gamma$ the Fourier expansion (1.23) with respect to that cusp is of the form

$$j_{\gamma_a^{-1}}(z; k)f(\gamma_az) = f_0(y) + \sum_{n\neq 0} A_n W_{ir, \text{sign}(n)}(4\pi |n| y) e^{2\pi inx}$$

(1.29)

where $f_0(y)$ is given by (1.25). In particular in this case we infer exponential decay of the sum $f(\gamma_az) - f_0(y)$ as $y \to \infty$.

Proof The growth condition (item $(iii)$ from definition 1.3.2) forces $B_n = 0$ for every $n \neq 0$ directly by (1.28). Conversely (1.29) implies the growth condition by absolute convergence of (1.23) together with (1.28). The last assertion is just a reformulation.

Again we compare our results with the special case of modular forms:

Remark 1.4.2 The same technique works in the holomorphic case too, take the Cauchy-Riemann equations (1.12) instead of $-\Delta_k f = \lambda f$. One obtains the ODE $\frac{\partial f_n}{\partial y}(y) = -2\pi nf_n(y)$ with solution $f_n(y) = B_n e^{2\pi iny}$ and (1.23) becomes $f(z) = \sum_n A_n e^{2\pi inz}$. Holomorphicity forces $\forall n < 0 : A_n = 0$. Compare again with remark 1.3.5.

1.5 Cusp forms

There is an important special case of our aforementioned discussion:

Definition 1.5.1 Let $f$ be an automorphic form. We say that $f$ is a (automorphic) cusp form if $f$ satisfies the sharpened growth condition

$$(f|k\gamma_a)(z) \in O(e^{-\varepsilon y})$$

(1.30)

for some $\varepsilon > 0$ as $y \to \infty$ and uniformly in $x$ at every cusp $a$ for $\Gamma$ with scaling matrix $\gamma_a$.

We give two characterization of cusp forms:

Proposition 1.5.2 Let $f$ be an automorphic form. Then the following are equivalent:

(a) $f$ is a cusp form

(b) $f_0(y) = 0$ in (1.29) at every cusp $a$

(c) $y^3 f$ is bounded on $\mathbb{H}$
Proof Theorem 1.4.1 gives $f(z) = f_0(y) + O(e^{-\varepsilon y})$ for some $\varepsilon > 0$ as $y \to \infty$, hence the equivalence of item (a) and (b) is immediate. If item (b) holds then $f(z) \to 0$ as $y \to \infty$, thus $y^2 f$ is bounded on a fundamental domain $\mathcal{F}$ for $\Gamma$, hence on $\mathbb{H}$ as well. Conversely item (c) gives $f(z) \to 0$ as $y \to \infty$ directly and this already forces $f_0(y) = 0$. □

The previous proposition and its proof translate verbatim to the holomorphic case, see [IK04, pp. 356, 357]. Explicit a priori bounds of Fourier coefficients can be found for example in [Iwa02, Theorem 3.1].
Chapter 2

The Hilbert space $\mathcal{H}_k$

We invoke some structure from functional analysis, prepare chapter 3 in section 2.2 and present a few observations afterwards. We follow loosely [Roe66, §3] unless stated otherwise.

2.1 Foundation

We begin with a straightforward observation:

**Proposition 2.1.1** $\Delta_k$ is symmetric\(^1\) with respect to $d\mu$, that is

$$\int_{\mathcal{H}} (\Delta_k f) g d\mu = \int_{\mathcal{H}} f(\Delta_k g) d\mu$$  \hspace{1cm} (2.1)

for every $f, g \in C^2_c(\mathcal{H}, \mathbb{C})$.

**Proof** \(^2\) Let $C$ be a closed, smooth, positively oriented and simple curve enclosing both the support of $f$ and $g$, let $\Delta^e = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ denote the euclidean Laplacian. Recall Greens theorem for $C^1$-functions $\tilde{f}, \tilde{g}$ on the domain bounded by $C$:

$$\int_{\mathcal{H}} \frac{\partial \tilde{g}}{\partial x} - \frac{\partial \tilde{f}}{\partial y} \, dx \, dy = \int_{C} \tilde{f} \, dx + \tilde{g} \, dy$$

We insert $\tilde{g} := \frac{\partial f}{\partial x} - f \frac{\partial \tilde{g}}{\partial x}$ and $\tilde{f} := \frac{\partial f}{\partial y} - f \frac{\partial \tilde{g}}{\partial y}$. Thus $f$ and $g$ enjoy the following identity:

$$\int_{\mathcal{H}} (\Delta^e f)(\tilde{g}) - f(\Delta^e \tilde{g}) \, dx \, dy = \int_{C} \left( \left( \frac{\partial f}{\partial x} \, dy - \frac{\partial f}{\partial y} \, dx \right) \tilde{g} - f \left( \frac{\partial \tilde{g}}{\partial x} \, dy - \frac{\partial \tilde{g}}{\partial y} \, dx \right) \right) = 0$$

\(^1\)Sometimes also called formally self-adjointness

\(^2\)Taken from [Bum97, Proposition 2.1.1].
where the last step follows by our choice of C. Similarly inserting \( \tilde{g} := \frac{i}{y} f \overline{g} \) and \( \tilde{f} := 0 \) in Greens theorem yields
\[
0 = \int_C \frac{i}{y} f \overline{g} \, dy = \int_H \frac{i}{y} \left( \frac{\partial f}{\partial x} \overline{g} + f \frac{\partial \overline{g}}{\partial x} \right) \, dx \, dy = \int_H \left( \left( \frac{i}{y} \frac{\partial f}{\partial x} \right) \overline{g} - f \left( \frac{i}{y} \frac{\partial \overline{g}}{\partial x} \right) \right) \, dx \, dy
\]
Now add the last two equations and use \( k \in \mathbb{R} \).

**Convention:** We will shortly phrase such an argument as “integration by parts” and omit the integration variable \( z \) if it does not change during the proof, e.g. as above. Else we write \( d\mu(z) \) to indicate integration with respect to the real and imaginary part of \( z \) as defined in (1.1). This will be needed for instance in chapter 3.

In fact proposition 2.1.1 characterizes the Laplace-Beltrami operator\(^3\) on \( \mathbb{H} \). This motivates the following definition of the *Petersson inner product*:

**Definition 2.1.2** Let \( k \in \mathbb{R} \), \( \Gamma \) be a discrete subgroup of \( \text{SL}_2(\mathbb{R}) \), \( \mathcal{F} \) a fundamental domain for \( \Gamma \). Let \( f, g: \mathbb{H} \to \mathbb{C} \) be measurable with respect to the induced measure by (1.1)\(^4\) and satisfy the transformation law \( f|_{k\gamma} = v(\gamma)f \) from definition 1.3.2 with respect to an unitary multiplier \( v: \Gamma \to \text{Aut}(\mathbb{C}) \). Then we have an inner product
\[
\langle f, g \rangle := \int_{\mathcal{F}} f \overline{g} \, d\mu
\]
inducing a norm and a Hilbert space
\[
\|f\|^2 := \int_{\mathcal{F}} |f|^2 \, d\mu \quad \mathcal{H}_k(\Gamma, v) := \{ f \text{ as above with } \|f\| < \infty \}
\]
in the usual fashion.

We implicitly fix a discrete subgroup \( \Gamma < \text{SL}_2(\mathbb{R}) \), such that a fundamental domain \( \mathcal{F} \) for \( \Gamma \) has finite hyperbolic volume, and an unitary multiplier \( v: \Gamma \to \text{Aut}(\mathbb{C}) \). Thus we omit the dependency on those inputs and abbreviate \( \mathcal{H}_k \) for \( \mathcal{H}_k(\Gamma, v) \). Nevertheless some authors put \( \mathcal{H} =: L^2(\Gamma \backslash \mathbb{H}) \).

**Remark 2.1.3** Note that

1. The inner product (2.2) is well-defined. This follows from the assumptions \( k \in \mathbb{R} \) and \( v \) unitary, because then \( f \overline{g} \) is a \( \Gamma \)-invariant function:
\[
f(\gamma z) \overline{g(\gamma z)} = (j_{\gamma}(z; k)v(\gamma)f(z))(j_{\gamma}(z; k)v(\gamma)g(z)) = f(z) \overline{g(z)}
\]
Hence (2.2) is independent of the choice of the fundamental domain \( \mathcal{F} \) for \( \Gamma \).

\(^3\)Iwa02, section 1.6.
\(^4\)Those are precisely the Lebesgue measurable functions on \( \mathbb{R}^2 \).
2. We follow the convention that \((2.2)\) is linear in its first argument and sesquilinear in its second argument. Roelckes convention is vice versa.

3. We implicitly work up to nullsets and identify \(f\) with its equivalence class modulo nullsets as usual.

4. In the theory of modular forms, e.g. in Hecke theory, one works with the slightly different transformation law from definition 1.3.4. Following example 1.3.6 this is compensated by two additional factors of \(y^2\) in the integrand of the Petersson inner product. Compare with the proof of proposition 7.3.8. This establishes item 1. again in this case.

Clearly any bounded automorphic form is in \(\mathcal{H}_k\). More precise we say that \(f\) is compactly supported mod \(\Gamma\) if the projection of \(\text{supp}(f) \subseteq \mathbb{H}\) into a fundamental domain \(\mathbb{H}/\Gamma\) is compact.

Obviously, a function \(f \in \mathcal{H}_k\) may not be differentiable, not even continuous. But since we wish to study the hyperbolic Laplacian \(-\Delta_k\), we introduce the following subspaces (utilized from the next section onwards):

**Definition 2.1.4**

\[
\mathcal{D}_k^{(2)} := \{ f \in (\mathcal{H}_k \cap C^2(\mathbb{H}, \mathbb{C})) | -\Delta_k f \in \mathcal{H}_k \} \quad (2.4)
\]

\[
\mathcal{D}_k^{(\infty)} := \{ f \in (\mathcal{H}_k \cap C^\infty(\mathbb{H}, \mathbb{C})) | f \text{ is compactly supported mod } \Gamma \} \quad (2.5)
\]

Denote the restrictions of \(\Delta_k\) to those spaces by \(\Delta_k^{(2)}\) and \(\Delta_k^{(\infty)}\) respectively.

Again we omit the dependencies on \(\Gamma\) and \(v\) of those spaces.

Their key properties are (justifying the definitions):

**Lemma 2.1.5**

(a) \(\mathcal{D}_k^{(\infty)} \subseteq \mathcal{D}_k^{(2)}\)

(b) \(\mathcal{D}_k^{(2)}\) and \(\mathcal{D}_k^{(\infty)}\) are dense in \(\mathcal{H}_k\) with respect to the norm \(\| \cdot \|\).

**Proof** Item (i) follows from Lemma 1.2.2 (\(\text{SL}_2(\mathbb{R})\)-invariance of \(\Delta_k\)) and compact support. Thus for item (ii) it suffices to prove density of \(\mathcal{D}_k^{(\infty)}\). But this is a standard mollification argument, where we work mod \(\Gamma\) and use the transformation law for \(f\). Explicitly let \(f \in \mathcal{H}_k\) and scale its support inside \(B_1(0)\) by \(\frac{f}{\|f\|}\). Take any \(0 \leq \eta \in C_0^\infty(B_1(0))\) with \(\int_{B_1(0)} \eta(w) dw = 1\) and define \(\eta_n(w) = n^2 \eta(n|w|) \in C_0^\infty(B_{\frac{1}{n}}(0))\). Hence \(\int_{B_{\frac{1}{n}}(0)} \eta_n(w) dw = 1\) and \(f_n := f * \eta_n \in \mathcal{D}_k^{(\infty)}\) approximates\(^5\) \(f\) in \(\|\cdot\|\). \(\square\)

Furthermore we can combine definition 2.1.4 with theorem 1.4.1:

\(^5\text{Str, Lemma 7.3.3.}\)
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Proposition 2.1.6 Let $f \in D_k^{(2)}$ satisfy $-\Delta_k^{(2)} f = \lambda f$. Then $f$ is an automorphic form in the sense of definition 1.3.2. If additionally $\lambda \geq \frac{1}{4}$ then $f$ is a cusp form in the sense of definition 1.5.1.

We will see in proposition 2.2.2 a) that $\lambda \in \mathbb{R}$ and in corollary 2.4.2 that $\lambda \geq \frac{|k|}{2} (1 - \frac{|k|}{2})$.

The proof of proposition 2.1.6 needs the following terminology, recall that the infinite vertical strip \{ $x + iy \mid 0 < x < 1, y > 0$ \} is a fundamental domain for $\Gamma_\infty$.

Definition 2.1.7 Let $a$ be a cusp of $\Gamma$, $\gamma_a$ be a scaling matrix, $0 < Y \in \mathbb{R}$. Then $\mathcal{C}(Y) := \gamma_a \{ x + iy \mid 0 < x < 1, y \geq Y \}$ is called a cusp sector of $a$.

Proof It remains to verify the growth condition, since the transformation law is included in the definition of $\mathcal{H}_k$ and the eigenfunction condition is assumed. Let $a$ be a cusp of $\Gamma$, $\gamma_a$ be a scaling matrix, $\mathcal{F}$ be a fundamental domain containing $\mathcal{C}(Y)$ for some $Y \geq 0^6$. Then we have in virtue of $\text{SL}_2(\mathbb{R})$-invariance of $d\mu$:

$$\int_{\mathcal{C}} |f(\gamma_a z)| \, d\mu \leq \|f\|^2 < \infty$$

Inserting the Fourier expansion (1.23) with coefficients (1.27) we see that we must have $B_n = 0$ for every $n \neq 0$, otherwise the integral would not converge. Hence $f$ is an automorphic form according to theorem 1.4.1.

The second assertion follows for instance by bounding the Fourier-coefficients of $f$ as performed in [Iwa02, Theorem 3.2]. This is possible due to $\|f\| < \infty$ and writing $\lambda = s(1-s) \geq \frac{1}{4}$ yields $\text{Re}(s) \geq \frac{1}{2}$ as required by Iwaniec. We postpone an alternative proof to proposition 7.2.2, where this claim becomes a direct consequence of the spectral decomposition of $\mathcal{H}_k$. In the meantime we shall not refer to this result either. □

2.2 Self-adjoint extension of $-\Delta_k$

We make a first step towards spectral theory and need some terminology from functional analysis:

Definition 2.2.1 $^7$ Let $H$ be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and $L : H \to H$ be a linear operator. Then $L$ is densely defined if its domain $D(L)$ is dense in $H$. In this case $L$ has a unique adjoint operator $L^*$ defined on

$$D(L^*) := \{ b \in H \mid a \mapsto \langle La, b \rangle \text{ is continuous} \} \quad (2.6)$$

---

$^6$compare section 1.1

$^7$We follow chapter 13 in [Rud73]
2.2. Self-adjoint extension of $-\Delta_k$

and characterized by the property

$$\langle La, b \rangle = \langle a, L^*b \rangle$$

(2.7)

Furthermore L is called

(i) symmetric if $\forall a, b \in D(L) : \langle La, b \rangle = \langle a, Lb \rangle$, i.e. $D(L) \subseteq D(L^*)$

(ii) self-adjoint if L is symmetric and $D(L) = D(L^*)$

(iii) essentially self adjoint if its closure $\tilde{L}$ is self-adjoint.

We quickly summarize their main properties:

**Proposition 2.2.2** (a) Any symmetric linear operator has real eigenvalues.

(b) $L$ is self-adjoint if and only if the spectrum of $L$ is real.

(c) $L$ is essentially self adjoint if and only if $L$ is symmetric and $L^*$ is self-adjoint.

**Remark 2.2.3** Recall again $\lambda = s(1 - s)$ from example 1.2.5. Now by item a) we have two possibilities in case of $-\Delta_k$: $s \in \mathbb{R}$ or $s = \frac{1}{2} + it$ for $t \in \mathbb{R}$. Thus we lost no generality in section 1.4. We will further locate the eigenvalues in section 2.4.

**Proof**  
(a) Let $Lv = \lambda v$. The claim follows from $v \neq 0$ and

$$\lambda \|v\| = \langle \lambda v, v \rangle = \langle Lv, v \rangle = \langle v, L^*v \rangle = \langle v, \lambda v \rangle = \lambda \|v\|$$

(b) See proposition A.3.7.

(c) This follows from $H^{**} = H$ and in virtue of the Riesz representation theorem then $L^{**} = L$. Finally every densely defined operator has a closed adjoint operator, see [Rud73, theorems 13.9, 13.12].

As the title suggests the goal of this section will be a proof of the following:

**Theorem 2.2.4** $\Delta_k^{(2)}$ and $\Delta_k^{(\infty)}$ are essentially self-adjoint.

Following our initial definition we have to prove two claims:

1. $\Delta_k^{(2)}$ and $\Delta_k^{(\infty)}$ are symmetric operators on their respective domains.

2. Their adjoint operators are self-adjoint.

Step 1 will be the main part. We cite a technical result first:

---

8See definition A.1.2.

9Such spaces are called reflexive, see proposition A.1.6.
Lemma 2.2.5 There exists a countable smooth partition of unity subject to \( \Gamma \):

\[
1 = \sum_{\nu \in \mathbb{N}, \gamma \in \Gamma} \varphi_\nu(\gamma z)
\] (2.8)

where \( \varphi_\nu \) is smooth on \( \mathcal{H} \) and satisfies:

(a) \( 0 \leq \varphi_\nu \leq 1 \)

(b) \( \varphi_\nu \) has compact support \( K_\nu \subseteq \mathcal{H} \)

(c) There exist open neighbourhoods \( U_\nu \) of \( K_\nu \) such that \( \{ \gamma U_\nu \mid \gamma \in \Gamma, \nu \in \mathbb{N} \} \) is an open cover of \( \mathcal{H} \) of finite type, i.e. any fixed \( \gamma_0 U_{\nu_0} \) intersects only finitely many \( \gamma U_\nu \) non-trivially.

Proof This result is proven in [Roe61, pp. 41, 42] using a standard construction via Urysohn’s lemma and performing the necessary adjustments. \( \Box \)

We obtain characterization of \( D^{(\infty)}_k \), which will be utilized in the proof of theorem 2.2.4 and proposition 2.2.9 below:

Corollary 2.2.6

\[
D^{(\infty)}_k = \left\{ \sum_{\gamma \in \Gamma} (v(\gamma)^{-1}(\psi|\kappa \gamma)) \mid \psi \in C^\infty_c(\mathcal{H}, \mathbb{C}) \right\}
\]

Proof We show both inclusions. First assume \( f \in (\mathcal{H}_k \cap C^\infty)(\mathcal{H}, \mathbb{C}) \) is compactly supported mod \( \Gamma \). Then according to lemma 2.2.5 there exists a countable smooth partition of unity subject to \( \Gamma \) as above. Denote by \( K \subseteq \mathcal{H} \) the compact support of \( f \). This gives \( \forall \gamma \in \Gamma : K \cap \gamma K_\nu \neq \emptyset \) for only finitely many \( \nu \) by compactness, hence \( f(z) \varphi_\nu(\gamma z) = 0 \) for almost every \( \nu \). The transformation law of \( f \) and the definition of \( \varphi_\nu(\gamma z) \) yield

\[
f(z) = \sum_{\nu \in \mathbb{N}, \gamma \in \Gamma} f(z) \varphi_\nu(\gamma z) = \sum_{\nu \in \mathbb{N}, \gamma \in \Gamma} v(\gamma)^{-1}((f \varphi_\nu)|\kappa \gamma)(z)
\]

and thus \( \psi := \sum_{\nu} f \varphi_\nu \) provides a representation of \( f \) given on the right hand side of the claim.

Conversely let \( \psi \in C^\infty_c(\mathcal{H}, \mathbb{C}) \) and set \( g := \sum_{\gamma \in \Gamma} (v(\gamma)^{-1}\psi|\kappa \gamma) \). Then clearly \( g \) is smooth and compactly supported mod \( \Gamma \) by construction. Using a partition of unity subject to \( \Gamma \) again gives a finite sum

\[
g(z) = \sum_{\nu \in \mathbb{N}, \gamma \in \Gamma} (v(\gamma)^{-1}(\psi|\kappa \gamma) \varphi_\nu(\gamma z))
\]

by compactness exactly as in the first part. Thus we can apply linearity of the slash-operator to the latter sum and shift the summation properly to obtain the transformation law. Finally this sum is bounded by compact support again giving \( \|g\| < \infty \), since \( \mathbb{F} \) has finite volume. \( \Box \)
Next we put

**Definition 2.2.7**

\[ R_k := iy \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + \frac{k}{2} \quad L_k := iy \frac{\partial}{\partial x} - y \frac{\partial}{\partial y} + \frac{k}{2} \] (2.9)

We investigate those operators analogously to \( -\Delta_k \):

**Lemma 2.2.8** It holds that

(a) \( R_k \) and \( L_{k+2} \) are adjoint to each other, that is

\[ \langle R_k f, g \rangle = \langle f, L_{k+2} g \rangle \quad \langle L_k f, g \rangle = \langle f, R_{k-2} g \rangle \] (2.10)

for every \( f, g \in C^1_c(\mathbb{H}, \mathbb{C}) \).

(b) We have the following identities:

\[ -\Delta_k = R_{k-2} L_k + \frac{k}{2} \left( 1 - \frac{k}{2} \right) = L_{k+2} R_k - \frac{k}{2} \left( 1 + \frac{k}{2} \right) \] (2.11)

(c) \( R_k \) and \( L_k \) behave as follows with respect to complex conjugation:

\[ R_k = -\overline{L_{-k}} \quad L_k = -\overline{R_{-k}} \] (2.12)

And for the sake of completeness: \( \Delta_k = \Delta_{-k} \).

(d) \( \forall f \in C^1(\mathbb{H}, \mathbb{C}) \, \forall \gamma \in \Gamma : \)

\[ (R_k f)|_{k+2\gamma} = R_k(f|_{k\gamma}) \quad \text{and} \quad (L_k f)|_{k-2\gamma} = L_k(f|_{k\gamma}) \] (2.13)

In other words those operators raise or lower the weight \( k \in \mathbb{R} \) by two respectively.

**Proof**

(a) This follows exactly as in Prop. 2.1.1 from integration by parts and \( k \in \mathbb{R} \), the calculations are done in [Bum97, proposition 2.1.3] \(^{11}\).

(b) These are straightforward calculations with the definitions of \( R_k \) and \( L_k \).

(c) This is obvious and stated for further reference.

(d) follows from definition 1.15 and 2.2.7, the calculations are done in [Bum97, lemma 2.1.1]. \( \square \)

\(^10\)Those operators are sometimes referred as *Maass operators*, e.g. in [Bum97, p. 129]. Do not confuse \( L_k \) (italic font) with the linear operator \( L \) (upright font) from the beginning and be aware of the different sign of \( L_k \) in [Bum97].

\(^11\)But be aware of the different sign of \( L_k \) there!
2. The Hilbert space $\mathcal{H}_k$

Now we are able to prove:

**Proposition 2.2.9** Let $f, g \in D^{(2)}_k$. Then

(a) $R_k f \in \mathcal{H}_{k+2}$ and $L_k f \in \mathcal{H}_{k-2}$

(b) $\langle -\Delta_k f, g \rangle = \langle R_k f, R_k g \rangle - \frac{k}{2} (1 + \frac{k}{2}) \langle f, g \rangle$

(c) $\langle -\Delta_k f, g \rangle = \langle L_k f, L_k g \rangle + \frac{k}{2} (1 - \frac{k}{2}) \langle f, g \rangle$

**Proof** We prove assertion (b) first. The assumption $f, g \in D^{(2)}_k$ is used implicitly via the Cauchy-Schwarz equation:

$$|\langle \Delta_k f, g \rangle| \leq \|\Delta_k f\| \|g\| < \infty \quad (2.14)$$

and similar with reversed roles of $f$ and $g$. The proof is based on lemma 2.2.8 a) and b):

$$\langle -\Delta_k f, g \rangle = \langle L_{k+2} R_k f, g \rangle - \frac{k}{2} (1 + \frac{k}{2}) \langle f, g \rangle$$

$$= \langle R_k f, R_k g \rangle - \frac{k}{2} (1 + \frac{k}{2}) \langle f, g \rangle$$

It is useful to look at the machinery behind those equations to demonstrate the “folding-” and “unfolding”-technique. It changes the integration domain between $\mathcal{F}$ and $\mathcal{H}$, which allows us to integrate by parts as in the proof of proposition 2.1.1. Hence we need $C^1_c$-functions. Differentiability is guaranteed by definition of $D^{(2)}_k$ and we utilize lemma 2.2.5 to ensure compact support. Alternatively an approximation argument with $D^{(\infty)}_k$-functions would also work$^{12}$. We emphasize the integration variable and calculate carefully:

$$-\int_{\mathcal{F}} (\Delta_k f)(z)\overline{g(z)}d\mu(z) \overset{\text{lemma 2.2.5}}{=} -\int_{\mathcal{F}} (\Delta_k f)(z) \sum_{\nu \in \mathbb{N}, \gamma \in \Gamma} (\overline{\varphi_\nu(\gamma z)g(z)})d\mu(z)$$

abs. convergence

$$= -\sum_{\nu \in \mathbb{N}, \gamma \in \Gamma} \int_{\mathcal{F}} (\Delta_k f)(z)(\overline{\varphi_\nu(\gamma z)g(z)})d\mu(z)$$

rmks 2.1.3 and 1.3.3

unfolding and $\gamma^F = (-\gamma)^F$

int. by parts

$$= -2\sum_{\nu \in \mathbb{N}} \int_{\mathcal{H}} (\Delta_k f)(z)(\overline{\varphi_\nu(z)g(z)})d\mu(z)$$

$$\overset{12}{=} 2\sum_{\nu \in \mathbb{N}} \int_{\mathcal{H}} (R_k f)(z)(\overline{\Delta_k (\varphi_\nu(z)g(z))})$$

$^{12}$See the proof of lemma 2.1.5.
2.2. Self-adjoint extension of $-\Delta_k$

\[
-\frac{k}{2} \left( 1 + \frac{k}{2} \right) f(z) \overline{(\varphi_s(z) g(z))} \right) d\mu(z)
\]

folding \[= \int_{F} \sum_{\gamma \in \Gamma} \left( (R_k f)(\gamma z) \overline{(R_k (\varphi_s g)(\gamma z))} \right) \]

lemma 2.2.8 d) \[= \int_{F} \sum_{\gamma \in \Gamma} \left( (R_k f)(\gamma z) \overline{(R_k (\varphi_s g)(\gamma z))} \right) \]

transf. law \[= \int_{F} \sum_{\gamma \in \Gamma} \left( (R_k f)(\gamma z) \overline{(R_k (\varphi_s g)(\gamma z))} \right) \]

lemma 2.2.5 \[= \int_{F} (R_k f)(z) \overline{(R_k g)(z)} d\mu(z) \]

Assertion c) follows similarly or by taking complex conjugates, see lemma 2.2.8 item c).

To verify assertion a) we take $g = f$ to get $\|R_k f\| < \infty$. The transformation law for $R_k f$ follows from the application of $R_k$ to the transformation law for $f$. Argue analogously for $L_k$ (use item c) instead of item b)). \[\Box\]

And we reached our preliminary goal to prove step 1:

**Corollary 2.2.10** $\Delta_k^{(2)}$ and $\Delta_k^{(\infty)}$ are symmetric operators on their respective domains $D_k^{(2)}$ and $D_k^{(\infty)}$.

**Proof** Recall $D_k^{(\infty)} \subseteq D_k^{(2)}$ from lemma 2.1.5, i.e. we may assume $f, g \in \Delta_k^{(2)}$.

Observe that the right hand sides in proposition 2.2.9 b) and c) are symmetric in $f, g$ up to sesquilinearity. Thus we simply apply either b) or c), reverse roles of $f$ and $g$ and apply b) or c) again (depending on our first item choice). \[\Box\]

The proof of step 2 relies on the following criterion:

**Proposition 2.2.11** Let $H$ be a Hilbert space, $L: D(L) \to H$ be a symmetric linear operator. Assume that the linear subspaces $(L \pm i1)D(L)$ are dense in $H$. Then $L$ is essentially self-adjoint.

**Proof** According to proposition 2.2.2 $\pm i$ can not be eigenvalues of $L$, since $L$ is symmetric. Thus $(L \pm i1)^{-1}$ exist, are densely defined and bounded.
2. The Hilbert space $\mathcal{H}_k$

In fact $\| (L \pm iI) f \|^2 = \|Lf\|^2 + \|f\|^2 \geq \|f\|^2$. Hence $\pm i$ are contained in the resolvent set of $L$ and we are able to follow the proof given in [Sto32, theorem 4.17]: Let $\hat{L}$ be the closure of $L$. Then we have $D(\hat{L}) \subseteq D(L^{**}) \subseteq D(L^*)$. Recall that the adjoint operator of a densely defined operator is always closed$^{14}$, i.e. it suffices to show that $D(\hat{L}) \subseteq D(L^*)$ is impossible.

First, the resolvent set of $L$ includes both half-planes of $\mathbb{C}$. To see this, note that the resolvent set of $L$ is open$^{15}$ and construct inductively open disks $D_j$, which do not intersect the real axis, $D_0$ contains $i$ or $-i$, $D_{j+1}$ contains the center of $D_j$ and $D_N$ contains a preassigned point in the upper or lower half plane. Then establish the existence of the resolvent in each disk, e.g. as in [Sto32, theorems 4.11, 4.12].

Second, $(\hat{L} - (\pm i)I)^{-1}$ is defined on all of $H$, because it is closed and bounded, i.e. continuous, and $L$ is densely defined in $H^{16}$. Consequently, the range of $\hat{L} - (\pm i)I$ is all of $H$.

Third, suppose by contradiction $\exists f \in D(L^*) \setminus D(\hat{L})$. Then

$$\exists g \in D(\hat{L}) : (\hat{L} - (\pm i)I)g = (L^* - (\pm i)I)f$$

by the previous assertion and hence $(L^* - (\pm i)I)(f - g) = 0$. But $f - g \neq 0$ by assumption and this implies that both half planes are contained in the residual spectrum of $L$ contradicting the first assertion. $\square$

**Remark 2.2.12 (Aside)** The converse statement to the previous proposition is also true: If $L$ is essentially self-adjoint then the upper and lower half plane belong to the resolvent set of $L$, see [Sto32, theorem 4.16] for details.

This gives density of $(L \pm iI)D(L)$ in $H$.

Finally we are now able to prove theorem 2.2.4:

**Proof (of theorem 2.2.4)** Recall again $D_k^{(\infty)} \subseteq D_k^{(2)}$ from lemma 2.1.5. Hence it suffices to prove that the linear subspaces $(\Delta_k \pm iI)D_k^{(\infty)}$ are dense in $\mathcal{H}_k$ in virtue of the previous proposition. We may further reduce to the case of $(\Delta_k + iI)D_k^{(\infty)}$, since the other case follows analogously or by taking complex conjugates.

Let $u \in \mathcal{H}_k$ be such that $\forall f \in D_k^{(\infty)} : \langle u, \Delta_k f + if \rangle = 0$. Our goal is to show $u \equiv 0$, then use continuity of $\langle \cdot, f \rangle$. Writing $f = \sum_{\gamma \in \Gamma} v(\gamma)^{-1} \psi|_k \gamma$ for some $\psi \in C_c^\infty(\mathbb{H}, \mathbb{C})$ by corollary 2.2.6 we infer

$$\Delta_k f = \sum_{\gamma \in \Gamma} v(\gamma)^{-1} (\Delta_k \psi)|_k \gamma$$

$^{13}$Follows by proposition A.1.3.  
$^{14}$Rud73, theorems 13.9, 13.12.  
$^{15}$See proposition A.3.4.  
$^{16}$In fact this is a corollary of the closed graph theorem.
by compact support of $\psi$ and lemma 1.3.1. Hence using linearity of the slash-operator and the transformation law for $u$

$$0 = \langle u, \Delta_k f + if \rangle = \int_{\mathcal{F}} u(z) \sum_{\gamma \in \Gamma} v(\gamma)^{-1}((\Delta_k \psi + i\psi)|_k\gamma)(z) d\mu(z)$$

$$= \sum_{\gamma \in \Gamma} \int_{\mathcal{F}} u(\gamma z)(\Delta_k \psi + i\psi)(\gamma z) d\mu(z) \overset{\text{rmk 2.1.3}}{=} \sum_{\gamma \in \Gamma} \int_{\mathcal{F}} u(z)(\Delta_k \psi + i\psi)(z) d\mu(z)$$

But now independence of the summands from $\gamma \in \Gamma$ forces

$$0 = \int_{\mathcal{F}} u(z)(\Delta_k \psi + i\psi)(z) d\mu(z)$$

Note that $u \in \mathcal{H}_k$ is a priori not differentiable, not even continuous. But the last equation also arises from the Dirichlet problem

$$\begin{cases}
\Delta_k \psi + i\psi = u & \text{in } \mathcal{F} \\
\psi = 0 & \text{on } \partial \mathcal{F}
\end{cases}$$

by integration\footnote{We will come back to this viewpoint at the beginning of chapter 3.}. Thus proposition 1.2.4 guarantees $u \in C^\infty(\mathcal{H}, \mathbb{C})$. This allows us to apply the symmetry of $\Delta_k$ (proposition 2.1.1, $\psi$ has compact support):

$$0 = \int_{\mathcal{F}} (\Delta_k u - iu)(z) \overline{\psi(z)} d\mu(z)$$

Since this holds for an arbitrary testfunction $\psi \in C^\infty_c(\mathcal{H}, \mathbb{C})$ we deduce by proposition A.1.4 that $(\Delta_k u - iu) = 0$, i.e. $\Delta_k u = iu \in \mathcal{H}_k$. Thus $u \in D_k^{(2)}$ and finally by symmetry of $\Delta_k$ on $D_k^{(2)}$ we obtain $u \equiv 0$ as desired. \hfill $\Box$

The final ingredient is taken from [Rud73, theorem 13.15]:

**Proposition 2.2.13** Any self-adjoint operator has no proper symmetric extension.

**Proof** Let $L$ be a self-adjoint operator and $H$ be a symmetric extension of $L$. By definition we have

$$D(H) \subseteq D(H^*) \subseteq D(L^*) = D(L) \subseteq D(H)$$

which proves $D(L) = D(H)$ as claimed. \hfill $\Box$

Since $\Delta_k^{(2)}$ is an extension of $\Delta_k^{(\infty)}$ and by uniqueness of a self-adjoint extension of an essentially self-adjoint operator both operators possess the same self-adjoint extension:
Definition 2.2.14 We denote by $\tilde{\Delta}_k$ the unique self-adjoint extension of $\Delta_k^{(2)}$ and $\Delta_k^{(\infty)}$ with domain of definition $\tilde{D}_k \subseteq \mathcal{H}_k$.

Remark 2.2.15 The preceding proof also shows that $\tilde{D}_k$ is obtained by “closing” $D_k^{(\infty)}$, that is:

$$\tilde{D}_k = \left\{ f \in \mathcal{H}_k \left| \exists g \in \mathcal{H}_k \forall \varepsilon > 0 \exists f_\varepsilon \in \Delta_k^{(\infty)} : \| f - f_\varepsilon \| < \varepsilon, \| g - (-\Delta_k f_\varepsilon) \| < \varepsilon \right\}$$

Then set $-\tilde{\Delta}_k f := g$ whenever $f \in \tilde{D}_k$. This is immediately well-defined by $\| g_1 - g_2 \| \leq 2\varepsilon + \| -\Delta_k (f_1^{(1)} - f_2^{(2)}) \| \xrightarrow{\varepsilon \downarrow 0} 0$.

2.3 Corollaries of section 2.2

We collect some loose ends and follow the first part in [Roe66, §5]. First, we “repair” remark 1.3.7:

Proposition 2.3.1 Let $f$ satisfy the transformation law $f |_{k\gamma} = f$ (with $v = 1$) and assume $f$ is an eigenfunction of $-\Delta_k^{(2)}$ with eigenvalue $\lambda = \frac{k}{2}(1 - \frac{k}{2})$. Then $g := y^{-\frac{k}{2}}f$ is a modular form for $\Gamma$ in the sense of definition 1.3.4.

The subtle difference to the counterexample $\tilde{f}(x + iy) = y^{1 - \frac{k}{2}}$ from remark 1.3.7 is that $\tilde{f} \notin D_k^{(2)}$. Thus we are not able to apply neither $\Delta_k^{(2)}$ nor proposition 2.2.9 to $\tilde{f}$.

Proof According to proposition 2.1.6 any eigenfunction $f$ of $-\Delta_k^{(2)}$ is an automorphic form. Thus clearly $g$ enjoys the modular transformation law $g(\gamma z) = (cz + d)^k g(z)$ for $\gamma \in \Gamma$ by the transformation law for $f$ and the same calculation as in example 1.3.6.

Next we check that $g$ is holomorphic on $\mathbb{H}$: Invoking proposition 2.2.9 item c) we have $(-\Delta_k f, f) = (L_k f, L_k f) + \frac{k}{2}(1 - \frac{k}{2}) (f, f)$, from which it follows $0 = L_k f = iy \frac{\partial f}{\partial x} - y \frac{\partial f}{\partial y} + \frac{k}{2} f$ using the assumption. We deduce

$$\frac{\partial g}{\partial y} = -\frac{k}{2} y^{-\frac{k}{2}} f + y^{-\frac{k}{2}} \frac{\partial f}{\partial y} \big|_{f=0} = i y^{-\frac{k}{2}} \frac{\partial f}{\partial x} = i \frac{\partial g}{\partial x}$$

i.e. $g$ satisfies the Cauchy-Riemann equations (1.12) on $\mathbb{H}$.

It remains to verify that $g$ is holomorphic at $i\infty$:

This follows again from the fact that $f$ is an automorphic form according to proposition 2.1.6, hence $f$ is of at most polynomial growth\(^{19}\). Thus $g \ll (y + \frac{1}{y})^\kappa$ for some $\kappa > 0$ and the q-expansion of $g$ is holomorphic on the disk $|q| < 1$, in particular at $q = 0$. \(\square\)

\(^{18}\)See remark 1.3.5.

\(^{19}\)We take $\gamma_{\infty} = 1$ again.
We have two converse statements to the preceding proof too:

**Proposition 2.3.2** Let $0 \not= f \in \mathcal{D}^{(2)}_k$.

(a) If $R_k f = 0$ then $k \leq 0$ and $-\Delta_k^{(2)} f = -\frac{k}{2} (1 + \frac{k}{2}) f$.

(b) If $L_k f = 0$ then $k \geq 0$ and $-\Delta_k^{(2)} f = -\frac{k}{2} (1 - \frac{k}{2}) f$.

First we observe that the assumption $f \in \mathcal{D}^{(2)}_k$ is again necessary: Take $\Gamma = \{ \pm 1 \}, k < 0, v = 1$ and $f := y^2 (x + iy)$. Then $f \in C^2 (\mathbb{H}, \mathbb{C})$, $f$ trivially satisfies the transformation law $f|_k \gamma = f$ and

$$L_k f = iy^{\frac{k}{2}+1} - y \left( -\frac{k}{2} y^{\frac{k}{2}-1} (x + iy) + iy^{\frac{k}{2}} \right) + \frac{k}{2} y^2 (x + iy) = 0$$

In the same fashion one calculates $-\Delta_k f = -\frac{k}{2} (1 - \frac{k}{2}) f$. But by our choice of $\Gamma$ we have $\mathbb{F} = \Gamma\backslash \mathbb{H} = \mathbb{H}$, thus $f \not\in \mathcal{H}_k$ and this gives indeed $-\Delta_k f \not\in \mathcal{H}_k$, else we would have a contradiction to item b).

**Proof** Again it suffices to prove item a), item b) then follows by complex conjugation of a). We take $g = f$ in proposition 2.2.9 giving:

\[
\langle -\Delta_k f, f \rangle = \langle R_k f, R_k f \rangle - \frac{k}{2} (1 + \frac{k}{2}) \langle f, f \rangle
\]

\[
\langle -\Delta_k f, f \rangle = \langle L_k f, L_k f \rangle + \frac{k}{2} (1 - \frac{k}{2}) \langle f, f \rangle
\]

Subtracting yields $0 = \|R_k f\|^2 - \|L_k f\|^2 - k \|f\|^2$, hence $k \leq 0$ due to $R_k f = 0$. The second assertion holds by the identity $-\Delta_k = L_{k+2} R_k - \frac{k}{2} (1 + \frac{k}{2})$ from lemma 2.2.8, where we use $R_k f = 0$ again. \(\square\)

**Corollary 2.3.3** Every modular form for $\text{SL}_2(\mathbb{Z})$ with negative or odd weight $k \in \mathbb{Z}$ is zero.

**Proof** Let $g$ be a modular form for $\text{SL}_2(\mathbb{Z})$ with negative or odd weight $k \in \mathbb{Z}$. Then $f = y^{\frac{k}{2}} g$ is an automorphic form by example 1.3.6. Since $g$ satisfies the Cauchy-Riemann equations (1.12) we further have $L_k f = 0$ and we claim $f \in \mathcal{H}_k$. Then lemma 2.2.8 b) implies $f \in \mathcal{D}^{(2)}_k$ and thus $f \equiv 0$, else we would have a contradiction to the previous proposition.

Proof of $f \in \mathcal{H}_k$: The transformation law is clear. Recall $f \asymp A_0 y^{\frac{k}{2}}$ as $y \to \infty$ by remark 1.3.5, where $A_0$ denotes the constant term in the $q$-expansion of $g$. But now $k < 0$ forces $\int_{\delta}^{\infty} \left| y^{\frac{k}{2}} \right|^2 \, dy < \infty \; \forall \delta \geq 1$. This shows that $f$ is square integrable on the cusp sector\footnote{Recall definition 2.1.7} for the cusp $a = \infty$ of $\text{SL}_2(\mathbb{Z})$, thus $f$ is square integrable on a fundamental domain for $\text{SL}_2(\mathbb{Z})$ as well. \(\square\)
Corollary 2.3.4 If \( f \) satisfies the transformation law \( f|_{0\mathcal{O}} = f \) and is an eigenfunction of \(-\Delta_0^{(2)}\) with eigenvalue \( \lambda = 0 \), then \( f \) is constant. In particular \( f \in \mathcal{H}_0 \).

**Proof** Use proposition 2.3.1 with \( k = 0 \) and recall that any modular form of weight 0 is constant.\(^{21}\)
Alternatively use proposition 2.2.9 with \( g = f \) to obtain \( R_0 f = 0 = L_0 f \) producing \( \frac{df}{dx} = 0 = \frac{df}{dy} \).
The last assertion is then just a reformulation of the requirement that \( \mathcal{F} \) has finite volume. \( \square \)

2.4 The spectrum of \(-\tilde{\Delta}_k\)

Although we are not in position to turn our interest towards a spectral resolution of \(-\Delta_k\) yet, we may locate its spectrum. We begin with a technical observation relying on corollary 2.2.6 and utilize it to sharpen proposition 2.2.2 a):

**Lemma 2.4.1** Let \( f \in \mathcal{D}_k \cap C^2(\mathbb{H}, \mathbb{C}) \). Then \( f \in \mathcal{D}_k^{(2)} \).

**Proof** Let \( g \in \mathcal{D}_k^{(\infty)} \). According to corollary 2.2.6 we may write
\[
g = \sum_{\gamma \in \Gamma} (v(\gamma)^{-1} \psi|_{\gamma})
\]
for some \( \psi \in C^\infty(\mathbb{H}, \mathbb{C}) \). Calculating analogously to the proof of Proposition 2.2.9 item b) yields:
\[
\langle \tilde{\Delta}_k f, g \rangle = \langle f, \tilde{\Delta}_k g \rangle = \left\langle f, \Delta_k^{(2)} g \right\rangle = 2 \int_{\mathbb{H}} f(\Delta_k \psi) d\mu = 2 \int_{\mathbb{H}} (\Delta_k f) \psi d\mu
\]
where we integrated by parts in \((*)\). Since \( g \in \mathcal{D}_k^{(\infty)} \) was arbitrary it follows \( \Delta_k f = \tilde{\Delta}_k f \in \mathcal{H}_k \) and hence \( f \in \mathcal{D}_k^{(2)} \) as desired. \( \square \)

**Corollary 2.4.2** The spectrum of \(-\tilde{\Delta}_k\) is contained in the interval \([\frac{|k|}{2} (1 - \frac{|k|}{2}), \infty)\).
In particular \( \forall f \in \mathcal{H}_k : \langle -\Delta_k f, f \rangle \geq \frac{|k|}{2} (1 - \frac{|k|}{2}) \).

**Proof** Let \( f \in \mathcal{D}_k \cap C^2(\mathbb{H}, \mathbb{C}) \). Then the previous lemma allows us to apply proposition 2.2.9 giving:
\[
\langle -\Delta_k f, f \rangle \geq -\frac{k}{2} \left( 1 + \frac{k}{2} \right) ||f||^2
\]
\[
\langle -\Delta_k f, f \rangle \geq \frac{k}{2} \left( 1 - \frac{k}{2} \right) ||f||^2
\]
The claim follows. \( \square \)

\(^{21}\)This follows from the “\( k^2 \)-formula.”
Motivated by remark 2.2.3 we are able to prove even more:

**Proposition 2.4.3** \(^{22}\) Let \(f \in \mathcal{D}_k^{(2)}\) be an eigenfunction of \(-\tilde{\Delta}_k\) with eigenvalue \(\lambda \in \mathbb{R}\) (by proposition 2.2.2 a)). We have

(a) If \(k = 1\) then \(\lambda \geq \frac{1}{4}\)

(b) If \(k \geq 0\) then either \(\lambda = \frac{l}{2} \left( 1 - \frac{l}{2} \right)\), where \(1 \leq l \leq k\) is an integer congruent to \(k\) modulo 2, or else \(\lambda \geq 0\) and in fact if \(k\) is odd then \(\lambda \geq \frac{1}{4}\).

(c) If \(k \leq 0\) then either \(\lambda = -\frac{l}{2} \left( 1 + \frac{l}{2} \right)\), where \(k \leq l \leq -1\) is an integer congruent to \(k\) modulo 2, or else \(\lambda \geq 0\) and in fact if \(k\) is odd then \(\lambda \geq \frac{1}{4}\).

**Proof** We reuse the results of lemma 2.2.8:

(a) On the one hand \(\langle -\tilde{\Delta}_1 f, f \rangle = \lambda \langle f, f \rangle\) and on the other hand

\[
\langle -\tilde{\Delta}_1 f, f \rangle = \langle R_{-1} L_1 f, f \rangle + \frac{1}{4} \langle f, f \rangle = \langle L_1 f, L_1 f \rangle + \frac{1}{4} \langle f, f \rangle \\
\geq \frac{1}{4} \langle f, f \rangle
\]

proving the claim.

(b) If \(L_k f = 0\) then just use proposition 2.3.2 (an eigenfunction is never zero.). Hence we may assume \(L_k f \neq 0\) and calculate

\[-\Delta_{k-2} L_k f = -L_k \Delta_k f = \lambda L_k f\]

Thus \(\lambda\) is an eigenvalue of \(L_k f\) as well and one can conclude by induction, since the case \(k = 0\) is precisely proven in the previous corollary.

(c) Follows by complex conjugation. \(\square\)

We will revisit proposition 2.4.3 in the proof of proposition 7.2.6. Furthermore we “reserve”:

**Definition 2.4.4**

\[\rho(k) := \max \left\{ \frac{l}{2} \left( 1 - \frac{l}{2} \right) \mid l \equiv k \mod 2 \right\}\]

Observe that the maximum may be computed as follows: If \(k \geq 0\) then take \(l \in (0, 2]\) congruent to \(k\) mod 2. If \(k \leq 0\) then take \(l \in [-2, 0)\) congruent to \(k\) mod 2. This shows \(\rho(k) \leq \frac{1}{4}\) with equality if \(k\) is an odd integer.

Finally we obtain:

**Corollary 2.4.5** The continuous spectrum of \(-\tilde{\Delta}_k\) begins at \(\rho(k)\) and is half-open.

**Proof** This follows by items (b),c) of proposition 2.4.3 and the definition of \(\rho(k)\). \(\square\)

\(^{22}\)Bum97, exercise 2.1.8.
Chapter 3

The resolvent of $-\tilde{\Delta}_k - \lambda \mathbb{1}$

3.1 Motivation

This chapter is devoted to the solvability of the Dirichlet problem

$$\begin{cases} (-\tilde{\Delta}_k - \lambda \mathbb{1})\tilde{f} = 0 \text{ in } \tilde{F} \\ \tilde{f} = f \text{ on } \partial F \end{cases} \quad (3.1)$$

for any $f \in \mathcal{H}_k$. Depending on the regularity of $f$ (and the domain $F$) there are several approaches to attack this problem. Probably the most classical one constructs a so called Green’s function $\mathcal{G}$ for $-\tilde{\Delta}_k - \lambda \mathbb{1}$ and postulates a solution

$$\tilde{f}(z) = \int_F \mathcal{G}(z, w) f(w) d\mu(w) \quad (3.2)$$

to (3.1) requiring sufficiently smooth data and boundary. Hence we expect the resolvent $\mathcal{R}_{\lambda,k}$ of $-\tilde{\Delta}_k - \lambda \mathbb{1}$ to be of the form (3.2) whenever $\lambda < \min\{0, \frac{|k|}{2} (1 - \frac{|k|}{2}) \}$. Explicitly we presume (formally)

$$(\mathcal{R}_{\lambda,k} f)(z) = \int_F \mathcal{G}_{\lambda,k}(z, w) f(w) d\mu(w) \quad (3.3)$$

and we will construct a function $\mathcal{G}_{\lambda,k}$ such that the operator $\mathcal{R}_{\lambda,k}$ is well defined and inverts the operator $-\tilde{\Delta}_k - \lambda \mathbb{1}$. In addition $\mathcal{R}_{\lambda,k}$ will lead to the point spectrum of $-\Delta_k$ in chapter 5.

From an abstract point of view we seek to study a certain class of linear operators, namely

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$^1$Sometimes also called a fundamental solution

$^2$Eva97, chapter 2, theorem 12.
Definition 3.1.1 The expression

\[(Lf)(z) = \int_{\mathcal{F}} K(z, w)f(w)\,d\mu(w) \quad (3.4)\]

is called an integral operator and K is called the kernel of L.

Following [Iwa02, section 1.8] let's assume absolute convergence for now. Note that this does not prevent K from being singular, for instance K is often singular on the diagonal \(z = w\). Having absolute convergence we impose that the integration over \(\mathcal{F}\) is well-defined. First we observe:

Lemma 3.1.2 \(L\) is \(\Gamma\)-invariant if and only if \(K\) is \(\Gamma\)-invariant:

\[\forall \gamma \in \Gamma : (Lf)(\gamma z) = (Lf)(z) \iff K(\gamma z, \gamma w) = K(z, w) \quad (3.5)\]

Proof This follows by a change of variables, \(\Gamma\)-invariance of \(d\mu(w)\), and the definition of \(\mathcal{F}\).

Thus to construct an invariant integral operator we demand the following property:

Definition 3.1.3 Let \(k(z, w)\) be a function on \(\mathbb{H} \times \mathbb{H}\). Then \(k\) is called a point-pair invariant if \(\forall \gamma \in \Gamma : k(\gamma z, \gamma w) = k(z, w)\).

Equivalently a point-pair invariant depends solely on the hyperbolic distance

\[d(z, w) := \text{arcosh}\left(1 + \frac{|z - w|^2}{2\text{Im}(z)\text{Im}(w)}\right) \quad (3.6)\]

between \(z\) and \(w\), where \(|z - w|\) denotes the euclidean distance. Hence we may write

\[K(z, w) = K(d(z, w)) \quad (3.7)\]

and the aforementioned singularity on the diagonal \(z = w\) might occur. In particular the function \(\frac{|z-w|^2}{\text{Im}(z)\text{Im}(w)}\) itself is a point-pair invariant.

Remark 3.1.4 This discussion shows that an invariant integral operator is given by convolution.

3.2 Goal

So what should \(G_{\lambda, k}\) satisfy? Let's look at the simple case \(v \equiv 1\) and \(\Gamma = \{\pm 1\}\), the latter gives \(\mathcal{F} = \mathbb{H}\). Denote Green's function in this case by \(g_{\lambda, k}\). We propose the following properties for \(g_{\lambda, k}\):
3.2. Goal

1. (transformation law):
\[
g_{\lambda,k}(\gamma z, \gamma w) = j_{\gamma}(z,k)g_{\lambda,k}(z,w)j_{\gamma}(w,k)^{-1} \tag{3.8}
\]
This follows from lemma 1.3.1: \( g_{\lambda,k} \) has to commute with \( \cdot | \gamma \), because \( -\Delta_k \) does.

2. (eigenfunction condition):
\[
(-\Delta_k - \lambda \mathbb{1})g_{\lambda,k}(\cdot, w) = 0 \text{ in } \mathcal{H} \setminus \{w\} \tag{3.9}
\]
This holds due to the very definition of the resolvent, which should produce a solution to (3.1), after interchanging \((-\Delta_k - \lambda \mathbb{1})g_{\lambda,k}\) with the integration over \( \mathcal{F} \) (which has to be justified as well).

3. (singularity condition): \( g_{\lambda,k}(w,z) \) will have the same type of singularity as the function \( \frac{1}{2\pi} \log |z - w| \) as \( w \to z \), or more precise
\[
g_{\lambda,k}(z,w) + \frac{1}{2\pi} \log |z - w| \tag{3.10}
\]
should be continuous as \( w \to z \). This is suggested by the fundamental solution of Poissons equation in two dimensions, which is of the form\(^3\)
\[
\frac{1}{2\pi} \log |\cdot|.
\]

In the general case of a discrete non co-compact subgroup \( \Gamma < \text{SL}_2(\mathbb{R}) \) (with fundamental domain of finite volume) we use the method of averaging over \( \Gamma \) to obtain an invariant function of weight 0. To achieve arbitrary weight \( k \in \mathbb{R} \) we insert the factor \( v(\gamma)j_{\gamma}(\cdot; k) \) according to the definition of the slash-operator in equation (1.15)\(^4\). Additionally recall the temporary factor 2 occurring in the proof of proposition 2.2.9b), where we used the “folding”- and “unfolding”-technique for the first time. Compensating for this factor too we postulate:
\[
\mathcal{G}_{\lambda,k}(z,w) = \frac{1}{2} \sum_{\gamma \in \Gamma} v(\gamma)j_{\gamma}(w,k)g_{\lambda,k}(z,\gamma w) \quad \forall z \not\equiv w \mod \Gamma \tag{3.11}
\]
Note that \( z \not\equiv w \mod \Gamma \) holds precisely in \( \mathcal{F} \). Summing up our postulation of \( \mathcal{G}_{\lambda,k} \) roughly justifies
\[
\int_{\mathcal{H}} \mathcal{G}_{\lambda,k}(z,w)f(w)\mathrm{d}\mu(w) = \int_{\mathcal{F}} \mathcal{G}_{\lambda,k}(z,w)f(w)\mathrm{d}\mu(w) \tag{3.12}
\]
or in other words, we generalized \( g_{\lambda,k}(z,w) \) “suitably” to \( \mathcal{G}_{\lambda,k}(z,w) \). A full proof of this identity is included in the proof of theorem 3.2.1 at the end of

\(^3\text{Eva97, pp. 22-23.}\)
\(^4\text{This will be made more precise in the proof of proposition 3.3.2.}\)
The resolvent of $-\tilde{\Delta}_k - \lambda \mathbb{1}$

Furthermore we will show that $\lambda < \min \{0, \frac{|k|}{2} (1 - \frac{|k|}{2}) \}$ can be utilized to guarantee

\[ g_{\lambda,k}(z,w) \in O\left(e^{-(1+\epsilon)d(z,w)}\right) \quad \text{as} \quad d(z,w) \to \infty \quad (3.13) \]

for some $\epsilon > 0$ and this will indeed produce absolute convergence of (3.11) as stipulated at the beginning of section 3.1.

Let us formulate our goal:

**Theorem 3.2.1** Let $\lambda < \min \{0, \frac{|k|}{2} (1 - \frac{|k|}{2}) \}$. Then the resolvent $\mathcal{R}_{\lambda,k}$ of $-\tilde{\Delta}_k - \lambda \mathbb{1}$ is of the form

\[ \mathcal{R}_{\lambda,k} : \mathcal{H}_k \to \tilde{\mathcal{D}}_k \]

\[ f \mapsto \mathcal{R}_{\lambda,k} f = \int_{\mathcal{F}} G_{\lambda,k}(\cdot, w) f(w) d\mu(w) \quad (3.14) \]

where the kernel $G_{\lambda,k}$ is given by (3.11). Moreover $(\mathcal{R}_{\lambda,k} f)(z)$ is a continuous function on $\mathcal{H}$.

**Remark 3.2.2** Invoking theorem 2.2.4 we deduce that $\mathcal{R}_{\lambda,k}$ is self-adjoint as well. In addition $\mathcal{R}_{\lambda,k}$ is bounded in $\lambda$, see proposition A.3.4. Thus there is hope for a spectral theorem, which will be the content of chapter 5 and 6, where we will also investigate invariant integral operators in greater detail.

The goal of the present chapter is to construct the kernel $G_{\lambda,k}$.

### 3.3 Construction of $G_{\lambda,k}$

This section is based on [Roe67, \S 7] and we divide the proof of theorem 3.2.1 into four intermediate steps, which will be indicated by subsections.

#### 3.3.1 Construction of $g_{\lambda,k}$

**Proposition 3.3.1** There exists a function $g_{\lambda,k} : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ with the properties (3.8), (3.9) and (3.10). Furthermore $g_{\lambda,k}$ is real valued and is of the form

\[ g_{\lambda,k}(z,w) = \left(\frac{z - w}{w - z}\right)^{-\frac{1}{2}} \left( h_1 \left( \frac{|z - w|^2}{\text{Im}(z) \text{Im}(w)} \right) + \log \left( \frac{|z - w|^2}{\text{Im}(z) \text{Im}(w)} \right) \right) h_2 \left( \frac{|z - w|^2}{\text{Im}(z) \text{Im}(w)} \right) \quad (3.15) \]

for some holomorphic functions $h_1, h_2$ in an open neighbourhood of $\mathcal{R} \geq 0$ with $h_2(0) = -\frac{1}{4\pi}$. 

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3.3. Construction of $G_{\lambda,k}$

**Proof** The construction of $g_{\lambda,k}$ is once more divided into four steps:

**Step 1 - Translation of the desired properties:** First note that the automorphy factor can be written as 
\[ j_\gamma(z; k) = \left( \frac{cz+d}{\bar{cz} + \bar{d}} \right)^k = \left( \frac{cz+d}{\bar{cz} + \bar{d}} \right)^{\frac{1}{2}} \]
from which it follows
\[ \left( \frac{z-w}{w-z} \right)^{\frac{1}{2}} = j_\gamma(z; k) \left( \frac{\gamma z - \gamma w}{\gamma w - \gamma z} \right)^{\frac{1}{2}} j_\gamma(w; k)^{-1} \]  
(3.16)
by a short calculation. Combining with the transformation law (3.8) for $g_{\lambda,k}$ the function
\[ g_{\lambda,k}(z, w) := \left( \frac{z-w}{w-z} \right)^{\frac{1}{2}} g_{\lambda,k}(z, w) \]  
(3.17)
is a point-pair invariant for $\Gamma = SL(2, \mathbb{R})$. Moreover recall that $SL(2, \mathbb{R})$ acts transitively on $\mathbb{H}$. Thus we may fix one point in $\mathbb{H}$, say $w = i$, and it suffices to determine $g_{\lambda,k}(\cdot, i)$. Consequently we put
\[ -\Delta^\phi_k = \left( \frac{z+i}{i-z} \right)^{\frac{1}{2}} \circ \Delta_k \circ \left( \frac{z+i}{i-z} \right)^{-\frac{1}{2}} \]  
(3.18)
and (3.9) becomes
\[ (\Delta^\phi_k + \lambda 1)g_{\lambda,k}(\cdot, i) = 0 \text{ in } \mathbb{H} \setminus \{i\} \]  
(3.19)
Hence $\Delta^\phi_k$ is invariant under rotations with fixed point $i$ and $\Delta_k$ commutes with $\cdot |_k \gamma$ in virtue of lemma 1.3.1. This invariance suggests to change coordinates to the unit disk:
\[ \mathbb{H} \rightarrow D \rightarrow D \]
\[ z \mapsto z-i \mapsto \sqrt{\frac{r-1}{r}} e^{i\varphi}, \text{ for } r \geq 1, \varphi \in \mathbb{R} \]  
(3.20)
With this choice of the radius we have
\[ r = \frac{|z-i|^2}{4y} + 1 = \frac{\cosh(d(z,i)) + 1}{2} \]  
(3.21)
and we will see the justification for this particular choice in short time. Note that the so transformed function $g_{\lambda,k}^\phi$ is independent from $\varphi$, since $g_{\lambda,k}^\phi$ is a point-pair invariant. To simplify the notation we set for $r > 1$:
\[ \bar{r} := \sqrt{\frac{r-1}{r}}, \quad \rho(z) := \bar{r} e^{i\varphi} \]
\[ \rho^{-1}(\bar{r}e^{i\varphi}) = i \left( \frac{1 + e^{i\varphi}}{1 - e^{i\varphi}} \right), \quad h(r) := g_{\lambda,k}^\phi(\rho^{-1}(re^{i0}), i) \]
(3.22)

---

5The left map is called the Cayley transform.
6Solve the equation $\frac{|z-i|}{|z+i|} = \sqrt{\frac{r-1}{r}}$ and recall (3.6).
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taking without loss of generality $\varphi = 0$ in the definition of $h$. Our growth condition (3.13) becomes

$$h(r) \in O(r^{-(1+\varepsilon)}) \text{ as } r \to \infty$$

(3.23)

for some $\varepsilon > 0$. By property (3.10)

$$h(r) + \frac{1}{4\pi} \log (r-1)$$

(3.24)

should be continuous for $r \geq 1$. Finally we write (3.19) in terms of $h$ getting

$$r(r-1)h''(r) + (2r-1)h'(r) + \left(\frac{k^2}{4r} + \lambda\right)h(r) = 0$$

(3.25)

This is the hypergeometric differential equation and explains our choice (3.21) of $r$.

**Step 2 - Investigation of the weak singularity at $r = 1$:** First we deal with the assertion that (3.24) should be continuous on $r \geq 1$: We cite the following fact from usual ODE-theory, see for instance [Tes12, Theorem 4.5]:

**Frobenius method**: Consider the homogeneous, weakly singular and linear ODE of second order $y''(x) + \frac{l(x)y'(x)}{x-x_0} + \frac{g(x)y(x)}{(x-x_0)^p} = 0$ and its associated indicial equation $\rho(\rho-1) + p(x_0)\rho + q(x_0) = 0$ with solutions $\rho_1, \rho_2$. Then the solution $y(x)$ is a linear combination of $y_1 = (x-x_0)^{\rho_1}(1 + \sum_{j=1}^{\infty} c_j(x-x_0)^j)$ and

a) $y_2(x) = (x-x_0)^{\rho_2}(1 + \sum_{j=1}^{\infty} d_j(x-x_0)^j)$ if $\rho_1 - \rho_2 \notin \mathbb{Z}$.

b) $y_2(x) = C \log(x-x_0)y_1(x) + (x-x_0)^{\rho_2}(1 + \sum_{j=1}^{\infty} d_j(x-x_0)^j)$ if $\rho_1 - \rho_2 \in \mathbb{Z}_{>0}$.

c) $y_2(x) = \log(x-x_0)y_1(x) + (x-x_0)^{\rho_2}\sum_{j=1}^{\infty} d_j(x-x_0)^j$ if $\rho_1 = \rho_2$.

Translated to our setting: Insert $y := h, x_0 = 1, p(r) := \frac{r^2}{4r^4} + \lambda$ above. The indicial equation of (3.25) reduces to $\rho(\rho-1) + \rho = 0$, which has one root $\rho = 0$ with multiplicity two. Hence we are in case c) above and find

$$h(r) = C_1h_1(r) + C_2h_2(r),$$

where

$$h_1(r) = 1 + \sum_{j=1}^{\infty} c_j(r-1)^j \quad \text{and} \quad h_2(r) = \log(r-1)h_1(r) + \sum_{j=1}^{\infty} d_j(r-1)^j$$

Observe that $C_2 \neq 0$. Else $h$ would be a regular solution of (3.25) at $r = 1$ and we would have a $C^2$-solution $g$ of (3.9). Additionally $\int_{\mathbb{H}} |g|^2 d\mu < \infty$.

---

$^7$A generalization of Fuchs’ theorem.
Step 3 - Solution of (3.25), investigation of the weak singularity at $r = \infty$:

Again to clarify the following expressions we set $\alpha := \frac{1}{2} - \sqrt{\frac{1}{4} - \lambda}$ and $\beta := \frac{1}{2} + \sqrt{\frac{1}{4} - \lambda}$. The two linearly independent solutions of (3.25) are in terms of the standard Gauss hypergeometric function $\, _2F_1$ given by

$$h(r) = C_1(-r)^{\frac{1}{2}} \, _2F_1 \left( \alpha + \frac{k}{2}, \beta + \frac{k}{2}, 1 + k; r \right) + C_2(-r)^{-\frac{1}{2}} \, _2F_1 \left( \alpha - \frac{k}{2}, \beta - \frac{k}{2}, 1 - k; r \right)$$  \hspace{1cm} (3.26)

We have to fix two issues now: Shift the solution from $r < 1$, which is the domain of convergence for the standard hypergeometric function, to $r > 1$ and ensure (3.23). The transformation rule for the domain of definition can be found in [AS72, p. 559] and yields:

$$h(r) = C_1 \hat{C}_1(-r)^{\frac{1}{2}} (1 - r)^{-\left(\alpha + \frac{1}{2}\right)} \, _2F_1 \left( \alpha + \frac{k}{2}, -\beta + \frac{k}{2}, 1 + \beta - 1; \frac{1}{1-r} \right) + C_1 \hat{C}_2(-r)^{\frac{1}{2}} (1 - r)^{-\left(\beta + \frac{1}{2}\right)} \, _2F_1 \left( \beta + \frac{k}{2}, -\alpha + \frac{k}{2}, 1 + \alpha - 1; \frac{1}{1-r} \right) + C_2 \hat{C}_1(-r)^{-\frac{1}{2}} (1 - r)^{\left(\alpha - \frac{1}{2}\right)} \, _2F_1 \left( \alpha - \frac{k}{2}, -\beta - \frac{k}{2}, 1 + \beta - 1; \frac{1}{1-r} \right) + C_2 \hat{C}_2(-r)^{-\frac{1}{2}} (1 - r)^{\left(\beta - \frac{1}{2}\right)} \, _2F_1 \left( \beta - \frac{k}{2}, -\alpha - \frac{k}{2}, 1 + \alpha - 1; \frac{1}{1-r} \right)$$  \hspace{1cm} (3.27)

where the transformation constants $\hat{C}_1, \hat{C}_2$ involve solely ratios of Gamma-functions depending on the parameters of $\, _2F_1$. Recall our standing assumption $\lambda < \min \{0, \frac{|k|}{2} (1 - \frac{|k|}{2}) \} \leq 0$. Thus $\sqrt{\frac{1}{4} - \lambda} > 0$, $\alpha < 0 < \beta$ and since $\frac{r + \gamma - 1}{r - 1} \in O(1)$ our growth condition (3.23) rules out the first and third term in (3.27). Next note that $\beta = 1 - \alpha$ and $\beta - \alpha + 1 = 2\beta$. Combining with\(^8\)

$$\, _2F_1(a, b, c; z) = (1 - z)^{-b-a} \, _2F_1(c-a, c-b, c; z)$$

applied to the last term in (3.27) we infer

$$h(r) = (C_1 + C_2) \hat{C}_2(-r)^{-\beta} \left( \frac{r}{r-1} \right)^{\frac{k}{2}} \, _2F_1 \left( \beta + \frac{k}{2}, \beta + \frac{k}{2}, 2\beta; \frac{1}{1-r} \right)$$  \hspace{1cm} (3.28)

\(^8\)AS72, p. 559.
Finally we compute $\tilde{C}_2$ from [AS72, p. 559] and demanding (3.24) we arrive at the unique solution:

$$h(r) = \frac{\Gamma(\beta + \frac{k}{2})\Gamma(\beta - \frac{k}{2})}{4\pi \Gamma(2\beta)}(r - 1)^{-\beta}\left(\frac{r}{r - 1}\right)^{\frac{1}{2}}F_1\left(\beta + \frac{k}{2}, \beta - \frac{k}{2}; 2\beta; \frac{1}{1 - r}\right)$$

(3.29)

where $\Gamma(\cdot)$ temporarily denotes the Gamma-function.

**Step 4 - Summary:** Using (3.17), (3.21) and (3.22) we ultimately obtain

$$g_{\lambda, k}(z, w) = \left(\frac{z - w}{w - z}\right)^{-\frac{1}{2}}h\left(\frac{|z - w|^2}{4\text{Im}(z)\text{Im}(w)} + 1\right)$$

(3.30)

Combining this with Frobenius method from the previous step yields precisely (3.15). Finally note that we produced a unique solution $h(r)$ to (3.25) with the properties (3.23) and (3.24). But $\text{Re}(h(r))$ is another solution to (3.25) with the properties (3.23) and (3.24), hence $h(r) \in \mathbb{R}$ and this implies

$$g_{\lambda, k}(z, w) = g_{\lambda, k}(w, z)$$

(3.31)

This completes the proof of proposition 3.3.1.

□

### 3.3.2 Basic properties of $G_{\lambda, k}$

**Proposition 3.3.2** The sum (3.11) defining $G_{\lambda, k}$ converges absolutely and is a continuous function in $z$ and $w$. Additionally $G_{\lambda, k}(\cdot, w)$ enjoys the transformation law $(G_{\lambda, k}(\cdot, w))|_{k \gamma} = v(\gamma)G_{\lambda, k}(\cdot, w)$, $|G_{\lambda, k}(z, w)|$ is symmetric in its arguments and $\forall \gamma_1, \gamma_2 \in \Gamma : |G_{\lambda, k}(\gamma_1 z, \gamma_2 w)| = |G_{\lambda, k}(z, w)|$.

**Proof** Let $z \neq w \mod \Gamma$, $\log_+(x) := \min(0, \log(x))$ be the non-positive part of the (real) logarithm and set

$$s(z, w) := -\frac{1}{4\pi}\left(\frac{z - w}{w - z}\right)^{-\frac{1}{2}}\log_+\left(\frac{|z - w|^2}{\text{Im}(z)\text{Im}(w)}\right)$$

(3.32)

Clearly $s(z, w) = \overline{s(w, z)}$. Recall equation (3.16) and that $\frac{|z - w|^2}{\text{Im}(z)\text{Im}(w)}$ is a point-pair invariant. Thus $s$ satisfies the transformation law (3.8), i.e.:

$$s(\gamma z, \gamma w) = j_{\gamma}(z, k)s(z, w)j_{\gamma}(w, k)^{-1}$$

Thanks to proposition 3.3.1\(^9\) and the continuity of $\log_+$ at $x = 1$ the function

$$t: \mathbb{H} \times \mathbb{H} \rightarrow \mathbb{C}$$

$$t(z, w) \mapsto g_{\lambda, k}(z, w) - s(z, w)$$

(3.33)

\(^9\)Note that $t$ contains the non-negative part of the logarithm. Apply (3.15).
Construction of $G_{\lambda, k}$

is continuous. By its definition $t(z, w) = \overline{t(w, z)}$ and $t$ enjoys the transformation law (3.8) too. We split $G_{\lambda, k}$ in the same fashion as $g_{\lambda, k}$ above and let

$$S(z, w) := \frac{1}{2} \sum_{\gamma \in \Gamma} v(\gamma) j_{\gamma}(w; k)s(z, \gamma w) \quad (3.34)$$

$$T(z, w) := \frac{1}{2} \sum_{\gamma \in \Gamma} v(\gamma) j_{\gamma}(w; k)t(z, \gamma w) \quad (3.35)$$

giving $G_{\lambda, k} = S + T$ according to (3.11) and (3.33). Recall lastly that $\Gamma$ is a discrete subgroup of $\text{SL}_2(\mathbb{R})$ and thus acts discontinuously on $\mathbb{H}$, in other words we have

$$\forall z \in \mathbb{H} \exists U \subseteq \mathbb{H} \text{ open such that } z \in U \text{ and } \{ \gamma \in \Gamma \mid \gamma U \cap U \neq \emptyset \} \text{ is finite.}$$

We investigate $S$ and $T$ separately:

**Claim:** $S$ is continuous on $\mathbb{H} \times \mathbb{H}$.

**Proof:** We show that $S$ converges uniformly on $\mathbb{H} \times \mathbb{H}$. We have

$$|v(\gamma)| = |j_{\gamma}(w; k)| = \left| \frac{|z - \gamma w|^2}{\text{Im}(z)\text{Im}(\gamma w)} \right| = 1 \quad (3.36)$$

and we can separate $z$ and $\gamma w$ by an open set for almost every $\gamma \in \Gamma$. Phrased differently we can bound the factor $\left| \log \left( \frac{|z - \gamma w|^2}{\text{Im}(z)\text{Im}(\gamma w)} \right) \right|$ by a constant not depending on both variables, which gives uniformity in at least one variable for almost every $\gamma \in \Gamma$. Then use the symmetry $S(z, w) = S(w, z)$ to obtain uniformity in the other variable. Sum up the bounds for the finitely many remaining elements $\gamma \in \Gamma$ and use that uniform limits of continuous functions are continuous.

**Claim:** $T$ is continuous on $\mathbb{H} \times \mathbb{H}$.

**Proof:** Let $K \subseteq \mathbb{H}$ be compact. Denote by $\overline{B}_r(z) \subseteq \mathbb{H}$ the closed hyperbolic ball\(^\text{10}\) of radius $r$ around $z$. Once more invoking discontinuity of $\Gamma$ we infer

$$\forall 0 \leq n \in \mathbb{N} \exists \Xi_n \subseteq \Gamma \text{ finite, s.t. } \forall \gamma \in \Gamma \setminus \Xi_n \forall z, w \in K : d(z, \gamma w) \geq n$$

and in particular choosing $n = 1$ there are at most finitely many elements $\gamma \in \Gamma$, say $N \in \mathbb{N}$, with\(^\text{11}\) $B_1(w) \cap B_1(\gamma w) \neq \emptyset$. Moreover note that $t$ satisfies the same growth condition (3.13) as $g_{\lambda, k}$ and that the hyperbolic volume of

\(^{10}\)The hyperbolic distance $d$ is explicitly given by (3.6).

\(^{11}\) $B_1(z_1)$ denotes the open hyperbolic ball of radius 1 around $w$. 

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$\mathcal{B}_r(z)$ is bounded by $O(e^r)$ as $r \to \infty$. Collecting all those properties we can estimate for every $n \geq 0$:

$$
\sum_{\gamma \in \Sigma_n} |t(z, \gamma w)| \leq \sum_{j=n}^{\infty} \sum_{j \leq d(z, \gamma w) < j+1} |t(z, \gamma w)| \leq C_1 \sum_{j=n}^{\infty} e^{-j(1+\epsilon)} \sum_{j \leq d(z, \gamma w) < j+1} 1
$$

$$
\leq C_1 \sum_{j=n}^{\infty} e^{-j(1+\epsilon)} \sum_{d(z, \gamma w) < j+1} 1 \leq C_1 \sum_{j=n}^{\infty} e^{-j(1+\epsilon)} \sum_{j \leq d(z, \gamma w) < j+1} \frac{\mu(\mathcal{B}_{j+2}(z))}{\mu(\mathcal{B}_j(z))} \leq C_1 C_2 \sum_{j=n}^{\infty} e^{-j} \tag{3.37}
$$

where we used $\mathcal{B}_1(\gamma w) \subseteq \mathcal{B}_{j+2}(z)$ in (*) and $\mu(\cdot)$ denotes the hyperbolic volume. The right hand side is clearly finite and since $\Sigma_n$ is finite as well we obtain compact convergence of $T$. Thus continuity of $t$ implies continuity of $T$ again.

We are now able to verify the remaining properties of $G_{\lambda,k}$:

$$
\left((G_{\lambda,k}(\cdot, w))|_{k\hat{\gamma}}\right)(z) = \left(j_{\hat{\gamma}}(z; k) \right)^{-1} G_{\lambda,k}(\gamma z, w)
$$

$$
= \left(j_{\hat{\gamma}}(z; k) \right)^{-1} \frac{1}{2} \sum_{\gamma \in \Gamma} v(\gamma) j_{\hat{\gamma}}(w; k) g_{\lambda,k}(\gamma z, \gamma w)
$$

$$
= \left(j_{\hat{\gamma}}(z; k) \right)^{-1} \frac{1}{2} \sum_{\gamma \in \Gamma} \left(\sigma_k(\gamma, \hat{\gamma}) v(\gamma) v(\hat{\gamma}) \right)^{-1} g_{\lambda,k}(\gamma z, \gamma \hat{\gamma} w)
$$

$$
= \left(j_{\hat{\gamma}}(z; k) \right)^{-1} \frac{1}{2} \sum_{\gamma \in \Gamma} \left(\sigma_k(\gamma, \hat{\gamma}) v(\gamma) v(\hat{\gamma}) \right)^{-1} g_{\lambda,k}(\gamma z, \gamma \hat{\gamma} w)
$$

$$
= v(\hat{\gamma}) \frac{1}{2} \sum_{\gamma \in \Gamma} v(\gamma) j_{\hat{\gamma}}(w; k) g_{\lambda,k}(z, \gamma \hat{\gamma} w) = v(\hat{\gamma}) G_{\lambda,k}(z, w) \tag{3.31}
$$

where we used the chain rule (1.18) and $v(\gamma_1 \gamma_2) = \sigma_k(\gamma_1, \gamma_2) v(\gamma_1) v(\gamma_2)$ to change the summation order (which is justified by absolute convergence). Symmetry of $|G_{\lambda,k}|$ follows from (3.36) and symmetry of $g_{\lambda,k}$, equation (3.31). The transformation law for $G_{\lambda,k}$ can be extended to

$$
G_{\lambda,k}(\gamma_1 z, \gamma_2 w) = v(\gamma_1) j_{\gamma_1}(z; k) G_{\lambda,k}(z, w) v(\gamma_2)^{-1} j_{\gamma_2}(w; k) \tag{3.38}
$$

by the same computation as above. Compare this result with (3.8). Thus $|G_{\lambda,k}|$ is invariant under substitutions in both arguments as claimed. \qed

**Remark 3.3.3** Observe that we only utilized the two properties (3.8) and (3.31) of $g_{\lambda,k}$ in the second half of the preceding proof, which begins with the verification of the transformation law. Since both equations hold for $s$ as well, they will hold for $t$ too. Hence the last three assertions can be established for $S, T$ if we replace the pair $g_{\lambda,k}, G_{\lambda,k}$ by one of the pairs $s, S$ or $t, T$ and argue verbatim as above. Similarly proposition 3.3.4 remains valid if we replace $G_{\lambda,k}$ with $S$ or $T$ there, as the proof below will show.
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3.3.3 Hilbert space properties of $G_{\lambda,k}$

**Proposition 3.3.4** Let $z_0 \in \mathbb{H}$ and $G_{\lambda,k}$ given by (3.11). Then it holds

(a) $\|G_{\lambda,k}(z_0, \cdot)\|_2^2 < \infty$

(b) $\lim_{z \to z_0} \|G_{\lambda,k}(z, \cdot) - G_{\lambda,k}(z_0, \cdot)\|_2^2 = 0$

**Proof** Recall the definition of $S$ in (3.34), $T$ in (3.35) and that $G_{\lambda,k} = S + T$. We proceed again by *divide et impera* and show the two items both for $S$ and $T$ separately. This clearly proves the proposition directly by the triangle inequality. Let $z_0 \in \mathbb{H}$.

**Proposition 3.3.4 holds for $S$:** We split $F$ into a compact and a non-compact region as follows: Let $F = F_1 \cup F_2$, where $F_1 \subseteq \overline{B}_2(z_0) \cap F$ is the compact part and $F_2$ is the non-compact part satisfying $F_2 \cap \gamma \overline{B}_2(z_0) = \emptyset$ for almost every $\gamma \in \Gamma$ according to discontinuity of $\Gamma$. Using (3.36) we estimate:

$$\|S(z_0, \cdot)\|_2^2 \leq \left(\frac{1}{8\pi}\right)^2 \left(\int_{F_1} + \int_{F_2}\right) \left(\sum_{\gamma \in \Gamma} \left| \log\left(\frac{|z_0 - \gamma w|^2}{\text{Im}(z_0)\text{Im}(\gamma w)}\right)\right|\right)^2 d\mu(w)$$

Let us emphasize again that $z_0 \not\equiv w \mod \Gamma$ inside $\mathring{F}$. Hence the integral over $F_1$ is clearly finite by compactness of $F_1$ and continuity of the integrand. In $F_2$, it holds $d(z_0, \gamma w) \geq 2$ for almost every $\gamma \in \Gamma$, thus $\log_+ (\cdots) = 0$ for almost every $\gamma \in \Gamma$ by (3.6). We are left with finitely many integrals, which clearly converge. The second item can be verified by the same technique, recall that $s$ is continuous.

**Proposition 3.3.4 holds for $T$:** First we estimate and then “unfold” one of the sums:

$$\|T(z_0, \cdot)\|_2^2 \leq \int_F \left(\sum_{\gamma \in \Gamma} |t(z_0, \gamma w)|\right)^2 d\mu(w)$$

$$= \int_F \sum_{\gamma_1, \gamma_2 \in \Gamma} |t(z_0, \gamma_1 w)| |t(z_0, \gamma_2 w)| d\mu(w)$$

$$= 2 \int_{\mathcal{H}} |t(z_0, w)| \sum_{\gamma \in \Gamma} |t(z_0, \gamma w)| d\mu(w)$$

The right hand side is immediately bounded due to (3.13) and (3.33) if the sum is bounded. To achieve a bound for the sum we argue similar to (3.37): Choose $z_1$ such that $z_0 \in B_1(z_1)$. Once more in virtue of discontinuity of $\Gamma$ we have at most $N < \infty$ elements $\gamma \in \Gamma$ such that $B_1(z_1) \cap \gamma B_1(z_1) \neq \emptyset$. Additionally recall that $|t|$ is a point-pair invariant by remark 3.3.3 and note
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that \( d(\gamma z_0, \gamma z_1) \leq 1 \). Altogether this justifies for every \( w \in \mathcal{H} \):

\[
\sum_{\gamma \in \Gamma} |t(z_0, \gamma w)| \leq C_1 \sum_{n=0}^{\infty} e^{-n(1+\epsilon)} \sum_{d(\gamma z_0 w) < n+1} 1
\]

\[
\leq C_1 \sum_{n=0}^{\infty} e^{-n(1+\epsilon)} \sum_{d(\gamma z_1 w) < n+2} 1
\]

\[
\leq C_1 \sum_{n=0}^{\infty} e^{-n(1+\epsilon)} N \frac{\mu(B_{n+3}(w))}{\mu(B_1(z_1))}
\]

\[
\leq C_1 C_2 \sum_{n=0}^{\infty} e^{-n\epsilon} < \infty
\]

where we used \( B_1(\gamma z_1) \subseteq B_{n+3}(w) \) in (*), compare with (3.37).

It remains to prove continuity of \( T \) with respect to the Hilbertnorm \( \| \cdot \| \): Let \( \tilde{z} \in B_1(z_0) \). We argue as above, but now bound the sum \( \sum_{\gamma \in \Gamma} \) as a function of \( \tilde{z} \), where \( z_0 \) instead of \( z_1 \) is fixed. This gives

\[
\int_{\mathcal{F}} |T(\tilde{z}, w) - T(z_0, w)| \, d\mu(w) \leq \int_{\mathcal{F}} \left( \sum_{\gamma \in \Gamma} |t(\tilde{z}, \gamma w) - t(z_0, \gamma w)| \right)^2 \, d\mu(w)
\]

\[
\leq 2 \int_{\mathcal{H}} |t(\tilde{z}, w) - t(z_0, w)| \sum_{\gamma \in \Gamma} |t(\tilde{z}, \gamma w) - t(z_0, \gamma w)| \, d\mu(w)
\]

\[
\leq 2 C_3 \int_{\mathcal{H}} |t(\tilde{z}, w) - t(z_0, w)| \, d\mu(w) \xrightarrow{\tilde{z} \to z_0} 0
\]

The last step follows by (3.13) for \( t \) and continuity of \( t \). This shows proposition 3.3.4 for \( T \) and completes the proof for \( \mathcal{G}_{\lambda,k} \) too.

**Corollary 3.3.5** Consider the trivial upper bound \( \frac{1}{2} \sum_{\gamma \in \Gamma} |g_{\lambda,k}(z, w)| \) for \( \mathcal{G}_{\lambda,k}(z, w) \) by (3.36). Let \( z_0 \in \mathcal{H} \). Then \( \| \sum_{\gamma \in \Gamma} |g_{\lambda,k}(z_0, \cdot)| \| < \infty \).

**Proof** Decompose \( r := |g_{\lambda,k}| - |s| \) and argue almost verbatim as above for the first assertion, where we had \( t = g_{\lambda,k} - s \). \( \square \)

3.3.4 Proof of theorem 3.2.1

We reached the last main step, let us summarize our achievements so far:

**Proposition 3.3.6** Let \( f \in \mathcal{H}_k, z_0 \in \mathcal{H} \). Then the function

\[
(\mathcal{G}_{\lambda,k} * f)(z_0) = \int_{\mathcal{F}} \mathcal{G}_{\lambda,k}(z_0, w) f(w) d\mu(w)
\]

is well-defined, continuous and satisfies the transformation law

\[
((\mathcal{G}_{\lambda,k} * f)|_{k \gamma})(z_0) = v(\gamma)(\mathcal{G}_{\lambda,k} * f)(z_0)
\]
3.3. Construction of \( \mathcal{G}_{\lambda,k} \)

**Proof** According to proposition 3.3.4 and Hölder’s inequality we obtain \( \|(\mathcal{G}_{\lambda,k} * f)\| < \infty \) and continuity of \( (\mathcal{G}_{\lambda,k} * f) \) directly. The integral defining \( (\mathcal{G}_{\lambda,k} * f) \) is independent from the choice of \( F \) by (3.38) and satisfies \( ((\mathcal{G}_{\lambda,k} * f)|k\gamma)(z_0) = v(\gamma)(\mathcal{G}_{\lambda,k} * f)(z_0) \) for the same reason. \( \square \)

Hence it remains to prove that we have indeed constructed the kernel of the resolvent of \( -\bar{\Delta}_k - \lambda I \) during the previous three steps:

**Proof (of theorem 3.2.1)** Let \( u \in \mathcal{D}_k \) and

\[
\Psi(z_0) := \int_F \mathcal{G}_{\lambda,k}(z_0,w)(-\bar{\Delta}_k - \lambda I)u(w)d\mu(w)
\]

We seek to show \( u \equiv \Psi \). First we can reduce the claim to the case \( u \in \mathcal{D}_k^{(\infty)} \) by approximation: Recall from remark 2.2.15 that \( \bar{\Delta}_k \) is the closure of \( \Delta_k^{(\infty)} \), that is

\[
\exists (u_n)_{n \geq 1} \subseteq \mathcal{D}_k^{(\infty)} : \lim_{n \to \infty} \|u_n - u\| = 0 \quad \text{and} \quad \lim_{n \to \infty} \left\| \Delta_k^{(\infty)} u_n - \bar{\Delta}_k u \right\| = 0
\]

from which we conclude according to proposition 3.3.4

\[
\begin{align*}
\Psi(z_0) &= \lim_{n \to \infty} \int_F \mathcal{G}_{\lambda,k}(z_0,w)(-\bar{\Delta}_k - \lambda I)u_n(w)d\mu(w) \\
&= \int_F \mathcal{G}_{\lambda,k}(z_0,w)(-\bar{\Delta}_k - \lambda I)u(w)d\mu(w) \\
&= \Psi(z_0)
\end{align*}
\]

Thus assume \( u \in \mathcal{D}_k^{(\infty)} \). According to corollary 2.2.6 we may write

\[
u = \sum_{\gamma \in \Gamma} v(\gamma)^{-1}(\psi|k\gamma)
\]

for some \( \psi \in C_c^\infty(\mathcal{H}, \mathbb{C}) \). We will utilize Lebesgue’s Dominated Convergence theorem (DCT) to interchange summation over \( \Gamma \) and integration over \( F \), here we need precisely corollary 3.3.5. Furthermore recall that \( u \) satisfies the transformation law \( u|k\gamma = v(\gamma)u \) from definition 1.3.2 by definition of \( \mathcal{H}_k \) and we have \( z_0 \not\equiv w \mod \Gamma \) in \( \bar{F} \). We compute carefully:

\[
\begin{align*}
\Psi(z_0) = \frac{1}{2} \int_F \left( \sum_{\gamma \in \Gamma} v(\gamma)j_\gamma(w;k)g_{\lambda,k}(z_0,\gamma w)\right)(-\bar{\Delta}_k - \lambda I)u(w)d\mu(w) \\
&= \frac{1}{2} \sum_{\gamma \in \Gamma} \int_F \left( v(\gamma)j_\gamma(w;k)g_{\lambda,k}(z_0,\gamma w)\right)(-\bar{\Delta}_k - \lambda I)u(w)d\mu(w) \\
&= \frac{1}{2} \sum_{\gamma \in \Gamma} \int_F \left( j_\gamma(w;k)g_{\lambda,k}(z_0,\gamma w)\right)((-\bar{\Delta}_k - \lambda I)u|k\gamma)(w)d\mu(w) \\
&= \frac{1}{2} \sum_{\gamma \in \Gamma} \int_F g_{\lambda,k}(z_0,\gamma w)(-\bar{\Delta}_k - \lambda I)u(\gamma w)d\mu(w)
\end{align*}
\]
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$$\begin{align*}
\text{subst.} \quad & \frac{1}{2} \sum_{\gamma \in \Gamma} \int_{\mathbb{H}} g_{\lambda,k}(z_0, w)(-\tilde{A}_k - \lambda \mathbb{1})u(w)d\mu(w) \\
\text{unfolding} \quad & = \int_{\mathbb{H}} g_{\lambda,k}(z_0, w)(-\tilde{A}_k - \lambda \mathbb{1})u(w)d\mu(w) \quad (\text{This proves (3.12)}) \\
\frac{\partial r}{\partial \gamma} \quad & = \int_{\mathbb{H}} g_{\lambda,k}(z_0, w)(-\tilde{A}_k - \lambda \mathbb{1}) \left( \sum_{\gamma \in \Gamma} v(\gamma)^{-1}(\psi|k\gamma)(w) \right) d\mu(w) \\
\text{linearity} \quad & = \int_{\mathbb{H}} g_{\lambda,k}(z_0, w) \left( \sum_{\gamma \in \Gamma} v(\gamma)^{-1}(-\tilde{A}_k - \lambda \mathbb{1})(\psi|k\gamma)(w) \right) d\mu(w) \\
\text{DCT} \quad & = \sum_{\gamma \in \Gamma} v(\gamma)^{-1} \int_{\mathbb{H}} g_{\lambda,k}(z_0, w)(-\tilde{A}_k - \lambda \mathbb{1})(\psi|k\gamma)(w) d\mu(w)
\end{align*}$$

The second application of Lebesgue’s DCT, marked as DCT’, is justified by (3.13) together with the upper bound

$$\left| \sum_{\gamma \in \Gamma} v(\gamma)^{-1}(-\tilde{A}_k - \lambda \mathbb{1})(\psi|k\gamma)(w) \right| \leq C$$

since $\psi$ has compact support of finite type, i.e. the sum over $\Gamma$ is finite, see lemma 2.2.5. We proceed with our computation:

$$\Psi(z_0) = \sum_{\gamma \in \Gamma} v(\gamma)^{-1} \int_{\mathbb{H}} g_{\lambda,k}(z_0, w)(-\tilde{A}_k - \lambda \mathbb{1})(\psi|k\gamma)(w) d\mu(w)$$

$$= \sum_{\gamma \in \Gamma} v(\gamma)^{-1} \lim_{\varepsilon \downarrow 0} \left( \int_{\mathbb{H}\setminus B_\varepsilon(z_0)} g_{\lambda,k}(z_0, w)(-\tilde{A}_k - \lambda \mathbb{1})(\psi|k\gamma)(w) d\mu(w) \\
+ \int_{B_\varepsilon(z_0)} g_{\lambda,k}(z_0, w)(-\tilde{A}_k - \lambda \mathbb{1})(\psi|k\gamma)(w) d\mu(w) \right)$$

$$= \sum_{\gamma \in \Gamma} v(\gamma)^{-1} \lim_{\varepsilon \downarrow 0} \left( \int_{\mathbb{H}\setminus B_\varepsilon(z_0)} \left( (-\tilde{A}_k - \lambda \mathbb{1}) g_{\lambda,k}(z_0, w) \right) (\psi|k\gamma)(w) d\mu(w) \\
+ \int_{B_\varepsilon(z_0)} \left( (-\tilde{A}_k - \lambda \mathbb{1}) g_{\lambda,k}(z_0, w) \right) (\psi|k\gamma)(w) d\mu(w) \right)$$

$$= \sum_{\gamma \in \Gamma} v(\gamma)^{-1} \lim_{\varepsilon \downarrow 0} \int_{\partial B_\varepsilon(z_0)} g_{\lambda,k}(z_0, w)(\psi|k\gamma)(w) d\mu(w)$$

Finally we invoke the explicit behaviour of $g_{\lambda,k}$ as $w \to z_0$. Let us recall
proposition 3.3.1, which describes $g_{\lambda,k}$ near its singularity at $z_0 = w$:

$$g_{\lambda,k}(z_0, w) \overset{(3.15)}{=} \left( \frac{z_0 - \bar{w}}{w - z_0} \right)^{-\frac{i}{2}} \left( h_1 \left( \frac{|z_0 - w|^2}{\text{Im}(z_0)\text{Im}(w)} \right) + \log \left( \frac{|z_0 - w|^2}{\text{Im}(z_0)\text{Im}(w)} \right) h_2 \left( \frac{|z_0 - w|^2}{\text{Im}(z_0)\text{Im}(w)} \right) \right)$$

for some holomorphic functions $h_1, h_2$ near $\mathbb{R}_{\geq 0}$ with $h_2(0) = -\frac{1}{4\pi}$. Hence the term involving $h_1$ drops by regularity at $z_0$ and the term involving $h_2$ behaves exactly like Poisson’s kernel in two dimensions, compare also (3.10). More details justifying the computation of the limit $\epsilon \searrow 0$ in this case are given in [Eva97, pp. 23-25]. Thus we ultimately obtain:

$$\Psi(z_0) \overset{(3.15)}{=} \sum_{\gamma \in \Gamma} v(\gamma)^{-1}(\psi|_{\lambda}\gamma)(z_0) \overset{\text{cor. 2.2.6}}{=} u(z_0)$$

This completes the proof of theorem 3.2.1.
Eisenstein series combine two ideas we have already seen: First the function \(x + iy \mapsto y^s\) is an eigenfunction of \(-\Delta_k\) with eigenvalue \(\lambda = s(1 - s)\), recall example 1.2.5. Second we average this function over \(\Gamma\) similar to our postulation of \(\rho_{\lambda,k}\) in (3.11). We infer a function \(E\), which is invariant under \(\Gamma\) and an eigenfunction of \(-\Delta_k\) at the same time. Lastly we have to deal with the growth condition in order to construct an automorphic form. That is why we do not average over the whole group \(\Gamma\), but instead “divide” \(\Gamma\) by the stabilizer subgroup \(\Gamma_a\) of a cusp \(a\). This will be made more precise in the proof of proposition 4.1.10. Moreover the existence of the resolvent of \(-\Delta\) and its properties come in handy in section 4.2.

### 4.1 Real analytic Eisenstein series

We begin with the simpler case of weight 0 and then generalize to an arbitrary weight \(k \in \mathbb{R}\). This simplifies the proofs without changing the involved ideas.

#### 4.1.1 Weight 0

**Definition 4.1.1** Let \(a\) be a cusp of \(\Gamma\) with scaling matrix \(\gamma_a\) and stabilizer subgroup \(\Gamma_a\). Then the Eisenstein series associated to \(a\) is

\[
E_a : \mathbb{H} \times \mathbb{C} \to \mathbb{C} \\
(z, s) \mapsto \sum_{\gamma \in \Gamma_a \backslash \Gamma} v(\gamma)^{-1}\text{Im}(\gamma_a^{-1}\gamma z)^s
\]

(4.1)

Note that we sum over right cosets and multiply \(\gamma\) with \(\gamma_a^{-1}\) from the right side. This choice will be important, when we verify the automorphic properties with respect to the \(z\)-variable. But we guarantee first that \(E_a\) is well-defined and need to impose the following condition:
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Definition 4.1.2 A cusp $a$ is called singular with respect to the unitary multiplier $v$ if
\[ \forall \gamma \in \gamma_a \Gamma_\infty \gamma_a^{-1} : v(\gamma) = 1 \tag{4.2} \]

Remark 4.1.3 Recall $\Gamma_a \subseteq \gamma_a \Gamma_\infty \gamma_a^{-1}$ due to equation (1.4). We explain our choice of the stronger condition (4.2) (instead of the weaker condition $\forall \gamma \in \Gamma_a : v(\gamma) = 1$) in remark 6.7.5.

Lemma 4.1.4 If $a$ is a singular cusp with respect to the unitary multiplier $v$ then the definition of $E_a$ does not depend on the choice of the coset representative $\gamma$.

Convention: Hence we will implicitly assume that $a$ is a singular cusp with respect to the unitary multiplier $v$ when working with Eisenstein series.

Proof Let $\gamma' = \eta \gamma$ with $\eta \in \Gamma_a$. Equation (1.4) yields $\eta = \gamma_0 \tau \gamma_a^{-1}$ for some $\tau \in \Gamma_\infty$, from which we compute $\text{Im}(\gamma_a^{-1} \gamma' z) = \text{Im}(\tau \gamma_a^{-1} \gamma z) = \text{Im}(\gamma_a^{-1} \gamma z)$.

Moreover $v(\gamma') = c_0(\eta, \gamma) v(\eta) v(\gamma) = v(\eta) v(\gamma) = v(\gamma)$ shows the claim. $\square$

Now we are in position to examine the general analytic properties of $E_a(z, s)$ with respect to the $z$-variable:

Proposition 4.1.5 If $\text{Re}(s) > 1$ then the sum defining $E_a$ converges absolutely, and uniformly on compact subsets.

Proof This is a purely technical argument, for instance given in [Kub73, theorem 2.1.1] (recall that $|v(\cdot)| = 1$).

Remark 4.1.6 The proof also shows that given $\text{Re}(s) > 1$ then $E_\infty(z, s) - y^s \in O(y^{1-s})$ as $y \to \infty$ and for any cusp $a$ not equivalent to $\infty$ it holds $E_a(z, s) \in O(y^{s-1})$ as $z$ tends to that cusp. See [Kub73, p. 13].

Corollary 4.1.7 $E_a(\cdot, s)$ is an eigenfunction of $-\Delta_k$ with eigenvalue $\lambda = s(1-s)$ for every $s \in \mathbb{C}$ with $\text{Re}(s) > 1$.

Proof This follows from example 1.2.5 and the previous proposition, which allows us to exchange summation and differentiation with respect to $z$. $\square$

Corollary 4.1.8 $E_a(\cdot, s)$ is real-analytic in $z \in \mathbb{H}$ and holomorphic in $s$ for every $s \in \mathbb{C}$ with $\text{Re}(s) > 1$.

Proof This is immediate by the previous corollary and proposition 1.2.4. Holomorphicity in $s$ follows in virtue of uniform convergence on compact subsets provided by the first proposition. $\square$

Example 4.1.9 Consider the simple case $\Gamma = \text{SL}_2(\mathbb{Z})$ with only cusp $a = \infty$ and $\gamma_a = 1$. Then the associated real-analytic Eisenstein series is precisely
\[
E_\infty(z, s) \overset{(1.3)}{=} \sum_{(c,d) \in \mathbb{Z}^2 \atop \gcd(c,d) = 1} \frac{\text{Im}(z)^s}{|cz + d|^2} = \frac{1}{2\zeta(2s)} \sum_{(c,d) \in \mathbb{Z}^2 \setminus \{(0,0)\}} \frac{\text{Im}(z)^s}{|cz + d|^2}
\]
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It suffices to check that the map

\[ \Xi: \Gamma_\infty \backslash \Gamma \to \{(c, d) \in \mathbb{Z}^2 \mid \gcd(c, d) = 1\} \]

\[ \gamma = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \mapsto (c, d) \]

is a well defined bijection: \( \Xi \) is constant along the orbits \( \Gamma_\infty \gamma \), since \( \Gamma_\infty \) contains precisely the integral translations, and \( \Xi \) is surjective by Bezout’s lemma. Suppose \( \Xi(\Gamma_\infty \gamma) = \Xi(\Gamma_\infty \gamma') \). Then \( \gamma \gamma'^{-1} \in \Gamma_\infty \) and hence both representatives belong to the same orbit. For the latter equality just consider the highest common factor of \( c \) and \( d \), this is essentially converse to Bezout’s lemma.

**Proposition 4.1.10** \( E_a(\cdot, s) \) is an automorphic form of weight 0 with respect to \( z \in \mathbb{H} \) for every \( s \in \mathbb{C} \) with \( \text{Re}(s) > 1 \).

**Proof** It remains to verify the transformation law and the growth condition. Note that the shift of the summation order is justified by absolute convergence. Hence

\[ E_a(\cdot, s) |_0 \gamma = E_a(\gamma z, s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \nu(\gamma)^{-1} \text{Im}(\gamma_a^{-1} \gamma \gamma z)^s \]

\[ \gamma = \gamma^{-1} \]

\[ \nu(\gamma) \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \sigma_0(\gamma, \gamma^{-1}) \nu(\gamma)^{-1} \text{Im}(\gamma_a^{-1} \gamma z)^s = \nu(\gamma) E_a(z, s) \]

establishes the transformation law. Recall \( \gamma_a^{-1} \Gamma_a \gamma_a \subseteq \Gamma_\infty \) from (1.4) giving

\[ E_a(\cdot, s) |_0 \gamma_a = \nu(\gamma_a) \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \nu(\gamma)^{-1} \text{Im}(\gamma_a^{-1} \gamma \gamma z)^s \]

\[ = \nu(\gamma_a) \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \nu(\gamma)^{-1} \text{Im}(\gamma z)^s \]

According to equation (1.3) and the previous example we obtain

\[ E_a(\cdot, s) |_0 \gamma_a \in O(y^\kappa) \text{ as } y \to \infty \]

uniformly in \( x \) with a constant \( \kappa = \text{Re}(s) \), which is precisely demanded by the growth condition from definition 1.3.2.

4.1.2 Weight \( k \in \mathbb{R} \)

How do we generalize the previous section to an arbitrary weight \( k \in \mathbb{R} \)?

A priori we have two possibilities: Utilize the Maass operators \( R_k \) and \( L_k \) from lemma 2.2.8, which shift the weight by two up or down (without any adjustment on the index \( k \)). Or use the slash-operator, which we will do here in accordance with [Roe67, §10].

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Definition 4.1.11 Let $a$ be a cusp of $\Gamma$ with scaling matrix $\gamma_a$ and stabilizer subgroup $\Gamma_a$. Then the associated Eisenstein series to $a$ of weight $k \in \mathbb{R}$ is

$$E_a : \mathbb{H} \times \mathbb{C} \rightarrow \mathbb{C}$$

$$(z, s) \mapsto \sum_{\gamma \in \Gamma_a \backslash \Gamma} v(\gamma)^{-1} \sigma_k(\gamma_a^{-1}, \gamma)^{-1}(\text{Im}(z)^s)|k(\gamma_a^{-1} \gamma)$$ (4.4)

where the slash-operator acts on $z$, i.e.

$$E_a(z, s) = \sum_{\gamma \in \Gamma_a \backslash \Gamma} v(\gamma)^{-1} \sigma_k(\gamma_a^{-1}, \gamma)^{-1} j_{\gamma_a^{-1}, \gamma}(z; k)^{-1} \text{Im}(\gamma_a^{-1} \gamma z)^s$$

Obviously this reduces to (4.1) in the case $k = 0$.

Lemma 4.1.12 If $a$ is a singular cusp with respect to the unitary multiplier $v$ then the definition of $E_a$ does not depend on the choice of the coset representative $\gamma$.

Proof We argue in the same fashion as in the proof of lemma 4.1.4. With the notation there we see that $j_{\gamma_a^{-1}, \gamma}(z, k) = j_{\gamma_a^{-1}, \gamma}(z, k)$ using (1.18). The law for $v$ reads $v(\gamma') = \sigma_k(\eta, \gamma) v(\eta) v(\gamma) = \sigma_k(\eta, \gamma) v(\gamma)$ and we have $\sigma_k(\eta, \gamma) \sigma_k(\gamma_a^{-1}, \gamma') = \sigma_k(\gamma_a^{-1}, \gamma)$ by\footnote{An intermediate result is $\sigma_k(ada^{-1}, b) + \sigma_k(a, a^{-1} \text{dab}) = \sigma_k(a, b)$, see [Iwa97, equation (2.45)].} (1.16).

As promised, it holds:

Proposition 4.1.13 $E_a(\cdot, s)$, given by (4.4), is an automorphic form of weight $k \in \mathbb{R}$ with respect to $z \in \mathbb{H}$ for every $s \in \mathbb{C}$ with $\text{Re}(s) > 1$.

Proof All assertions from the previous section carry over verbatim to general weight $k$ by equation (3.36), $\sigma_k \in S^1$ and lemma 1.3.1, except the transformation law. In particular the series defining $E_a(z, s)$ converges absolutely enabling us to rearrange its summands. To verify the transformation law it suffices to deal with the cusp $a = \infty$ giving $\gamma_a = \mathbb{I}$, then consider $E_a(\gamma_a z, s)$ and utilize equation\footnote{Recall $\gamma_a^{-1} \Gamma_a \gamma_a \subseteq \Gamma_\infty$.} (1.4). For this reason we apply the slash-operator with respect to $\gamma_a^{-1} \gamma$ in the definition of $E_a(z, s)$. Using $\sigma_k(\mathbb{I}, \gamma) = 1 \forall \gamma$ the case $a = \infty$ simplifies to

$$(E_a|\tilde{\gamma})(z, s) = j_{\tilde{\gamma}}(z, k)^{-1} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} v(\gamma)^{-1} j_{\gamma \tilde{\gamma}}(\tilde{\gamma} z; k)^{-1} \text{Im}(\gamma \tilde{\gamma} z)^s$$

$$= \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} v(\gamma)^{-1} j_{\gamma \tilde{\gamma}}(\tilde{\gamma} z; k)^{-1} j_{\gamma}(z, k)^{-1} \text{Im}(\gamma z)^s$$

$$= v(\tilde{\gamma}) \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} v(\gamma)^{-1} \sigma_k(\gamma, \tilde{\gamma})^{-1} j_{\gamma \tilde{\gamma}}(\tilde{\gamma} z; k)^{-1} j_{\gamma}(z, k)^{-1} \text{Im}(\gamma z)^s$$

$$= v(\tilde{\gamma}) \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} v(\gamma)^{-1} j_{\gamma}(z; k)^{-1} \text{Im}(\gamma z)^s = v(\tilde{\gamma}) E_a(z, s)$$
We briefly indicate the general case. Here we need the additional factor 
\( \sigma_k(\gamma^{-1}_a, \gamma)^{-1} \) in the definition of \( E_a(z, s) \), in order to compensate for the terms appearing after shifting the summation order. To this end one needs to infer the behaviour of \( \sigma_k \). But this reduces to the law (1.16) for \( w \), which defines \( \sigma_k \). Intermediate results for \( w \) can be found in [Iwa97, pp. 40,41]. □

Following our exposition in chapter 1 we are now able to state:

**Theorem 4.1.14 (Fourier expansion of \( E_a \))** Let \( a, b \) be cusps for \( \Gamma \) and \( \text{Re}(s) > 1 \). Define \( r \) by \( s = \frac{1}{2} + ir \). Then it holds

\[
j_{\gamma_b^{-1}}(z; k)E_a(\gamma_b z, s) = \delta_{ab} y^s + \varphi_{ab}(0, s) y^{1-s} + \sum_{n \neq 0} \varphi_{ab}(n, s) W_{ir, \text{sign}(n)}^{1/2} \left( 4\pi |n| y \right) e^{2\pi inx} \tag{4.5}\]

where \( \delta_{ab} \) is the Kronecker diagonal symbol, \( W \) is the Whittaker function from theorem 1.4.1 and the coefficients are:

\[
\varphi_{ab}(n, s) = S_{ab}(0, n; c) \cdot \begin{cases} 
\pi^2 \frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)} , & \text{if } n = 0 \\
2 \frac{\pi |n|^{s-\frac{1}{2}}}{\Gamma(s)} , & \text{if } n \neq 0
\end{cases} \tag{4.6}
\]

The expression \( S_{ab}(0, n; c) \) is a special case of a Kloosterman sum, explicitly given by

\[
S_{ab}(0, n; c) = \sum_{c>0} e^{-2s} \sum_{d} e^{2\pi i \frac{n d}{c}} \tag{4.7}
\]

where the summation with respect to \( d \) is restricted to \( d \) mod \( c \) such that

\[
\begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \gamma_a^{-1} \Gamma \gamma_b
\]

**Proof** The case of weight \( k = 0 \) is developed in [Kub73, pp. 14-16] and can be carried over verbatim to the general case of weight \( k \in \mathbb{R} \), because the argument solely relies on the double coset decomposition of \( \Gamma \) and standard Fourier analysis. Compare also [Iwa02, theorem 3.4] and [Roe67, Lemma 10.2]. □

**Corollary 4.1.15** If \( s \neq \frac{1}{2} \) then \( E_a(z, s) \not\equiv 0 \).

**Proof** The constant term in the Fourier expansion does not vanish identically. □

**Remark 4.1.16** Warning: \( E_a(\cdot, s) \notin \mathcal{H}_k \), because \( E_a(\cdot, s) \) grows polynomially at the cups of \( \Gamma \). More precisely it holds

\[
E_a(\gamma_b z, s) = \delta_{ab} y^s + \varphi_{ab}(0, s) y^{1-s} + O(e^{-\epsilon y}) , \text{ as } y \to \infty \]
\[
\varphi_{ab}(n, s) \ll |n|^{|\text{Re}(s)|} + |n|^{1-|\text{Re}(s)|} , \text{ if } n \neq 0
\]
for some $\epsilon > 0$. The first line can be derived straightforward from our discussion of Fourier expansions in section 1.4 and theorem 4.1.14. The second estimate follows from the Weil bound $|S(m,n;c)| \leq d(c)\sqrt{\gcd(m,n,c)}\sqrt{c}$ for Kloosterman sums\(^3\) ($d(c)$ denotes the number of positive divisors of $c$).

Thus we are mainly interested in the constant terms of the Fourier expansion, hence we abbreviate $\phi_{ab}(s) := \phi_{ab}(0,s)$.

### 4.2 Meromorphic continuation in $s$

We turn our interest to the $s$-variable of $E_a$. Recall that we always imposed $\Re(s) > 1$ in the previous section. In this section we wish to derive a functional equation of the form $\hat{E}_a(z,1-s) = \Phi(s)E_a(z,s)$, which enables us to define $E_a$ on the whole $s$-plane with possibly countably many poles. First we have:

**Lemma 4.2.1** 4 \( \Gamma \) has only finitely many inequivalent cusps.

**Proof** This follows from the fact that $\Gamma$ is a Fuchsian group of the first kind, in particular $\Gamma$ is finitely generated. A proof of the lemma is given in [Sie43, §9].

#### 4.2.1 A first example

We illustrate the following theory by our previous example:

**Example 4.2.2** Consider $E_{\infty}$ from example 4.1.9. In this case there is indeed a meromorphic continuation in $s$: Let $\hat{E}_{\infty}(z,s) := 2\pi^{-s}\Gamma(s)\zeta(2s)E_{\infty}(z,s)$, where $\Gamma(s)$ denotes the Gamma-function here. Then $\hat{E}_{\infty}(z,s) = \hat{E}_{\infty}(z,1-s)$ giving simple poles at $s = 0,1$. We give a rough sketch:

1. \( \forall z \in \mathcal{H}, \forall t \in \mathbb{R}_{>0} \) let $\Theta(z,t) := \sum_{(m,n) \in \mathbb{Z}^2} e^{-\frac{\pi}{t}|mz+n|^2}$. Use Poisson summation to show $\Theta(z,t) = \frac{1}{t}\Theta(z,\frac{1}{t})$. Recall that the Fourier transform of $e^{-ax^2}$ is given by $\sqrt{\frac{\pi}{a}}e^{-\frac{x^2}{a}}$ for $a > 0$. Alternatively use the transformation law $\sum_{m \in \mathbb{Z}} e^{2\pi im^2}(-\frac{1}{t}) = \sqrt{-it} \sum_{m \in \mathbb{Z}} e^{2\pi im^2z}$ of the standard modular theta function for $\Gamma_0(4)$ and substitute properly.

2. By definition $\Theta(z,t) - 1 = \sum_{(0,0) \neq (m,n) \in \mathbb{Z}^2} e^{-\frac{\pi}{t}|mz+n|^2}$ and $2\zeta(2s)E_{\infty}(z,s) = \sum_{(0,0) \neq (c,d) \in \mathbb{Z}^2} \frac{\text{Im}(z)^{2s}}{|cz+d|^{2s}}$. Show that these functions are related by Mellin transformation: $\int_{0}^{\infty} (\Theta(z,t) - 1)t^s \frac{dt}{t} = \hat{E}_{\infty}(z,s)$. Recall that the Gamma-function is simply the Mellin transform of $e^{-t}$.

\(^3\)IK04, p. 280.

\(^4\)Roe66, p. 300.
3. Split the integral at $t = 1$ and note that $\int_{1}^{\infty} (\Theta(z,t) - 1)t^{s} \frac{dt}{t}$ is an entire function in $s = s$ and $s = 1 - s$. Substitute $t = \frac{1}{u}$ in $\int_{0}^{1} (\Theta(z,t) - 1)t^{s} \frac{dt}{t}$, use the first step and $\int_{1}^{\infty} \left(1 - \frac{1}{u}\right)u^{1-s} \frac{du}{u} = \frac{1}{s(1-s)} = \int_{1}^{\infty} \left(1 - \frac{1}{u}\right)u^{s} \frac{du}{u}$ to conclude.

### 4.2.2 Poles of $E_{a}$ and the Maass-Selberg formula

First we formulate the Maass-Selberg formula. This formula encodes the information about the poles and residues of the Eisenstein series $E_{a}(z,s)$ in the strip $\frac{1}{2} < \text{Re}(s) \leq 1$. The line $\text{Re}(s) = \frac{1}{2}$ will become the symmetry line of $E_{a}(z,s)$ in the subsection afterwards. Recall the Fourier expansion of an automorphic form $f$ from theorem 1.4.1 and definition 2.1.7 of a cusp sector $\mathcal{C}_{a}(Y)$, which is roughly speaking a (small) area near $a \in \mathbb{Q} \cup \{\infty\}$ inside $\mathbb{F}$. The compact part of $\mathbb{F}$ is the complement in $\mathbb{F}$ of all cusp sectors. Now we are able to introduce the idea of truncation, which ensures convergence of integrals over $\mathbb{F}$ (recall remark 4.1.16):

**Definition 4.2.3** Let $Y > 0$ and $f$ be a function, which satisfies the transformation law and the growth condition from definition 1.3.2. Denote by $f_{a}(y)$ the constant term in the Fourier expansion of $f$. Then we define the compact part of $f$ by

$$f^{Y}(z) := \begin{cases} f(z) - f_{a}(\text{Im}(\gamma^{-1}a z)) & \text{if } z \in \mathcal{C}_{a}(Y) \\ f(z) & \text{if } z \text{ is in the compact part of } \mathbb{F} \end{cases}$$

By definition $z \in \mathcal{C}_{a}(Y) \iff \text{Im}(\gamma^{-1}a z) > Y$. We formulate:

**Proposition 4.2.4 (Maass-Selberg, general case)** Let $f_{1}(z,s), f_{2}(z,s)$ be two automorphic forms with respect to $z$ and eigenvalues $\lambda_{1} = s_{1}(1-s_{1}), \lambda_{2} = s_{2}(1-s_{2})$. Suppose

$$\forall l = 1,2 \ \forall a : f_{l,a}(y) = C_{l,a}y^{s} + D_{l,a}y^{1-s}$$

i.e. $\lambda_{1} \neq \lambda_{2}$ according to example 1.2.5. Let $Y > 0$. Then it holds

$$\left\langle f_{1}^{Y}(\cdot,s_{1}), f_{2}^{Y}(\cdot,s_{2}) \right\rangle = \sum_{a} \left( \frac{C_{1,a}D_{2,a}Y^{s_{1}-s_{2}} - C_{2,a}D_{1,a}Y^{s_{2}-s_{1}}}{s_{1} - s_{2}} + \frac{C_{1,a}D_{1,a}Y^{s_{1}+s_{2}-1} - C_{2,a}D_{2,a}Y^{1-s_{1}-s_{2}}}{s_{1} + s_{2} - 1} \right)$$

Recall that we sum over a maximal set of inequivalent cusps for $\Gamma$.  

---

5. See definition 2.1.7
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**Proof** The idea is to apply Green’s formula to the compact part of $\mathcal{F}$, where $\mathcal{F}$ is chosen such that the resulting boundary terms on equivalent side segments cancel out. The remaining integrals along horocycles of height $Y$ for each cusp are computed using the Fourier expansions. Only the constant terms survive the integration in the fashion claimed above. A detailed proof is given in [Kub73, theorem 2.3.1] or in [Iwa02, pp. 92-94]. □

We apply this result to the truncated Eisenstein series $6$

$$E^Y_a(z,s) := \begin{cases} E_a(z,s) - \delta_{ab} \text{Im}(\gamma_b^{-1}z)^s - \varphi_{ab}(s) \text{Im}(\gamma_b^{-1}z)^{1-s}, & \text{if } z \in \mathfrak{c}_b(Y) \\ E_a(z,s), & \text{if } z \text{ is in the compact part of } \mathcal{F} \end{cases} \quad (4.8)$$

for any fixed $Y > 0$. Observe $E^Y_a(z,s) \ll e^{-\alpha \text{Im}(z)}$, which follows from the Fourier expansion 4.1.14 and the asymptotic behaviour of the Whittaker functions, see equation (1.28). Recall that $\varphi_{ab}$ is the constant term in the Fourier expansion of $E_a$, explicitly given by theorem 4.1.14. Inserting $E^Y_a(z,s)$ into the previous proposition yields:

**Corollary 4.2.5 (Maass-Selberg for Eisenstein series)** 7 Let $s_1, s_2$ be regular points of $E_a(z,s)$ and $E_b(z,s)$ respectively, $s_1 \neq s_2, s_1 + s_2 \neq 1$. Let $Y > 0$. Then

$$\langle E^Y_a(\cdot, s_1), E^Y_b(\cdot, s_2) \rangle = \frac{\varphi_{ab}(s_2)Y^{s_1-s_2} - \varphi_{ab}(s_1)Y^{s_2-s_1}}{s_1-s_2} + \frac{\delta_{ab}Y^{s_1+s_2-1} - \Phi_a(s_1)\Phi_b(s_2)Y^{1-s_1-s_2}}{s_1+s_2-1}$$

Note that $\Phi_a(s_1)\Phi_b(s_2)$ is an inner product of two row vectors of the constant term matrix.

Observe that the additional factors needed for general weight $k$ make no difference here, since they all have absolute value 1. This formula has the following important consequence:

**Proposition 4.2.6** 8. The poles of $E_a(z,s)$ in $\text{Re}(s) > \frac{1}{2}$ are among the poles of $\varphi_{aa}(s)$. The residue of $\varphi_{aa}(s)$ at a pole is real and positive.

**Proof** We insert $a = b$, $s_1 = \sigma + it$, $s_2 = \sigma - it$, where $\sigma > \frac{1}{2}$ and $t \neq 0$, into corollary 4.2.5 and assume that $s_1 = \sigma + it$ is a regular point of $\Phi(s)$. Then

---

6Iwa02, equation (6.29).
7The formulation is taken from [Iwa02, proposition 6.8].
8Iwa02, theorem 6.9.
we have

\[ 0 \leq \left\| E_a(z, \sigma + it) \right\|^2 + \frac{Y^{1 - 2\sigma}}{2\sigma - 1} \sum_b \left| \varphi_{ab}(\sigma + it) \right|^2 \]

\[(4.9)\]

We emphasize that this expression is real-valued, hence \(^9\) \((\ast)\), and non-negative. Thus the functions \(\varphi_{ab}(s)\) have at most finitely many poles, since \(\Gamma\) has only finitely many cusps thanks to lemma 4.2.1. \(\square\)

We will see in the upcoming subsection that the functions \(\varphi_{ab}(s)\) are holomorphic in \(\text{Re}(s) \geq \frac{1}{2}\) except at the poles. We will investigate the residues of \(E_a(z, s)\) in section 6.2 further.

### 4.2.3 General case

The general case is rather involved and fills two chapters in [Kub73]. In addition to Fourier analysis and Mellin transformation, the argument is based on the fact that the resolvent \(R_{\lambda, k}\) is a holomorphic function if \(\lambda\) moves without touching the spectrum \(^{10}\) of \(-\Delta_k\). The final result then is (summarizing the preceding subsection too and including [Roe67, Satz 10.3, Lemma 10.5]):

**Theorem 4.2.7 ([Kub73, theorem 4.3.5])** Let \(a\) be a cusp of \(\Gamma\), \(\Phi(s) := (\varphi_{ab}(s))\) be the constant term matrix\(^{11}\) from the Fourier expansion in theorem 4.1.14. Then \(E_a(z, s)\) is holomorphic in the domain \(\text{Re}(s) > \frac{1}{2}\) except at a finite number of poles of \(\varphi_{aa}(s)\) on \((\frac{1}{2}, 1]\). Moreover the function \(\varphi_{aa}(s)^{-1}E_a(z, s)\) is holomorphic in \(s\) at a pole of \(\varphi_{aa}\). Finally \(E_a(z, s)\) has a unique limit as \(\text{Re}(s) \searrow \frac{1}{2}\) and \(\Phi(s_0)\) is a unitary matrix at the limit point \(s_0\) on the line \(\text{Re}(s) = \frac{1}{2}\).

We present important corollaries of this result:

**Corollary 4.2.8**\(^{12}\) The constant term matrix \(\Phi(s)\) is meromorphic on the whole \(s\)-plane and satisfies the functional equation \(\Phi(s)\Phi(1 - s) = 1\).

**Proof** First observe that \(\Phi(s)\) is a symmetric matrix: The sum over \(d\) in (4.7) is equal to the number of representatives of distinct double cosets \(\Gamma_a \backslash \Gamma_0^{-1} \Gamma_0 / \Gamma_b\) in the case \(n = 0\) and \(c\) fixed. This number does not change under inversion and we get the same number of representatives of the “inverted” coset \(\Gamma_b \backslash \Gamma_0^{-1} \Gamma_0 / \Gamma_a\).

Second we establish the functional equation on the line \(\text{Re}(s) = \frac{1}{2}\): Let \(s_0\) be

\(^9\) implicitly using \(\varphi_{aa}(\pi) = \varphi_{aa}(s)\)

\(^{10}\) See corollary A.3.6.

\(^{11}\) Sometimes also called scattering matrix.

\(^{12}\) [Kub73, theorem 4.4.1.]
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a point on that line and \( \Phi(s_0) \) be the unique limit from theorem 4.2.7. It follows \( \Phi(s_0)\Phi(1 - s_0) = \Phi(s_0)\Phi(s_0)^T = \Phi(s_0)\Phi(s_0)^T = I \), the step previous to the last one holds again in virtue of the series representation of \( \varphi_{al}(0,s) \). Third we now are in position to apply the Schwarz reflection principle\[13\] giving the desired continuation of \( \Phi \) to the whole \( s \)-plane. \( \square \)

**Theorem 4.2.9** \[14\] Let \( a_1, \ldots, a_N \) be a complete set of inequivalent cusps for \( \Gamma \), \( \vec{E}(z,s) := (E_{a_1}(z,s), \ldots, E_{a_N}(z,s))^T \) be the column vector of all associated Eisenstein series. Then \( \vec{E} \) satisfies the functional equation

\[
\vec{E}(z,s) = \Phi(s)\vec{E}(z,1-s)
\]

Thus each \( E_a \) has a meromorphic continuation to the whole \( s \)-plane.

**Proof** Let \( s_0 \) be a point on the line \( \text{Re}(s) = \frac{1}{2} \), \( \vec{E}(z,s_0) \) the unique limit provided by theorem 4.2.7 and let \( f(z) := \vec{E}(z,s_0) - \Phi(s_0)\vec{E}(z,1-s_0) \). According to the previous corollary \( f \) is well-defined and has constant Fourier coefficient

\[
f_0(y) = \lVert y^{s_0} \rVert + \Phi(s_0)y^{1-s_0} - \Phi(s_0)(\lVert y^{1-s_0} \rVert + \Phi(1-s_0)y^{s_0}) = 0
\]

Hence \( f \) is a cusp form according to proposition 1.5.2. But on the other hand almost all compact parts of \( f \) are orthogonal to a given cusp form, which is proven in \[Kub73, \text{theorem 2.3.1}\]. (Further justification of this fact will be given in the upcoming section.) Therefore \( f \) is orthogonal to all cusp forms while being itself a cusp form. This implies \( f \equiv 0 \) and the desired functional equation on the line \( \text{Re}(s) = \frac{1}{2} \). Now apply the Schwarz reflection principle again. \( \square \)

We conclude this subsection by the following result:

**Corollary 4.2.10** \[15\] \( E_a(z,s) \) has no poles on the line \( \text{Re}(s) = \frac{1}{2} \).

**Proof** Suppose by contradiction that there exists a pole \( s_0 = \frac{1}{2} + it_0 \) of \( E_a(z,s) \) of order \( m \geq 1 \). By regularity of \( \Phi(s) \) at \( s_0 \) we get

\[
\lim_{s \to s_0} (s-s_0)^m E_a(z,s) = \lim_{s \to s_0} (s-s_0)^m E_a^Y(z,s) =: f(z)
\]

But if we let \( \sigma \to \frac{1}{2} \) in equation (4.9) we derive \( \lVert E_a^Y(\cdot, \omega + it_0) \rVert \ll 1 \) for any fixed \( t_0 \in \mathbb{R} \) including 0. We may include \( t_0 = 0 \), because \( \Phi(s) \) is holomorphic and unitary on \( \text{Re}(s) = \frac{1}{2} \). Moreover \( \Phi(\mathbb{R}) \subseteq \mathbb{R} \). This bound forces \( \lVert f \rVert = 0 \), hence \( f \equiv 0 \) by continuity of \( f \), and we derive a contradiction to the definition of a pole. \( \square \)

---

\[13\] Iwa02, theorem 6.11.

\[14\] Kub73, theorem 4.4.2.

\[15\] Sal12, Satz 5.25.
Remark 4.2.11 We will see in proposition 7.2.6 that if the weight $k$ is an odd integer then $E_a(z, s)$ is holomorphic in $\text{Re}(s) \geq \frac{1}{2}$.

Another good reference of the aforementioned results is once more [Iwa02]. The double coset decomposition can be found in chapter 2, the Fourier expansion as Theorem 3.4 in chapter 3 and the continuation is established in chapter 6 including the investigation of the poles on $(\frac{1}{2}, 1]$. In particular the estimates from remark 4.1.16 apply to the meromorphically continued Eisenstein series as well\(^\text{16}\). Nevertheless both references deal only with the case of weight $k = 0$, but this is no loss of generality as remarked at the beginning of section 4.1.2. Further reference is given in [Roe67, §10].

### 4.3 Incomplete Eisenstein series

We turn our interest once more to the $z$-variable of $E_a$ and need another modification on it in order to work with $E_a$ in the Hilbert space $\mathcal{H}_k$ according to remark 4.1.16. We could truncate $E_a$ at a (possibly very large) fixed value $Y > 0$, so that $\|E_a\|$ reduces to an integral over the compact part of $\mathbb{F}$. But there is a inconvenient price to pay: $E_a^Y(z, s)$ does not satisfy the transformation law anymore (that is why we need the sum over the right cosets). Instead recall corollary 2.2.6: Any function in $D^\infty_k \subseteq \mathcal{H}_k$ is representable by some $\psi \in C^\infty_c(\mathbb{H}, \mathbb{C})$ averaged with weight $k \in \mathbb{R}$ over the whole group. We combine this idea with the structure of a real-analytic Eisenstein series:

**Definition 4.3.1** Let $\psi \in C^\infty_c(\mathbb{R}_>, \mathbb{C})$. Then the function

$$E_a(\cdot | \psi): \mathbb{H} \to \mathbb{C}
\quad z \mapsto \sum_{\gamma \in \Gamma \backslash \Gamma_a \Gamma} \nu(\gamma)^{-1} \sigma_k(\gamma_a^{-1}, \gamma)^{-1} \left( \psi(\text{Im}(z)) \right) |k(\gamma_a^{-1} \gamma)$$

is called an incomplete Eisenstein series of weight $k \in \mathbb{R}$.

Note that we do not have a dependency on $s$ anymore, compare also proposition 6.1.1. We think of the notation $z|\psi$ as $z$ “restricted to” $\psi$. Of course there is a trade off here too: $E_a(\cdot | \psi)$ fails to be an eigenfunction of $-\tilde{\Delta}$, but satisfies the transformation law exactly by the same proof as in section 4.1.2 and furthermore $E_a(\cdot | \psi)$ is bounded on $\mathbb{H}$ by construction. Hence it makes sense to work with $E_a(\cdot | \psi)$ in $\mathcal{H}_k$, which we will do now.

**Proposition 4.3.2** Let $f$ be absolutely integrable over $\mathbb{F}$, $\psi \in C^\infty_c(\mathbb{R}_>, \mathbb{C})$ and denote by $f_a(y)$ the zeroth coefficient of $f$ in its Fourier expansion with respect to the cusp $a$ as established in theorem 1.4.1. Then

$$\langle f, E_a(\cdot | \psi) \rangle = \int_0^\infty f_a(y) \psi(y) \frac{dy}{y^2}$$

\(^\text{16}\)Iwa02, p. 86.
Proof This is a straightforward computation:

\[
\langle f, E_a(\cdot | \psi) \rangle = \int_{\mathbb{F}} f(z) \sum_{\gamma \in \Gamma} \left( v(\gamma)^{-1} \sigma_k(\gamma^{-1}, \gamma^{-1} j_{\gamma^{-1}}(z; k)^{-1} \psi(\text{Im}(\gamma^{-1} z))) \right) d\mu(z)
\]

\[
= \sum_{\gamma \in \Gamma} \int_{\gamma^{-1} \mathbb{F}} f(\gamma^{-1} \gamma w) \left( v(\gamma)^{-1} \sigma_k(\gamma^{-1}, \gamma^{-1} j_{\gamma^{-1}}(\gamma^{-1} \gamma w; k)^{-1} \psi(\text{Im}(w))) \right) d\mu(w)
\]

\[
\text{f}_{|k(\gamma^{-1})=v(\gamma^{-1})f} \quad \text{and} \quad (3.36) \Rightarrow \sum_{\gamma \in \Gamma} \int_{\gamma^{-1} \mathbb{F}} f(\gamma w) \left( \psi(\text{Im}(w)) \right) d\mu(w)
\]

unfolding and (1.4)

\[
= \int_0^\infty \left( \int_0^1 f(\gamma w) dx \right) \left( \frac{\psi(y)}{y^2} \right) \frac{dy}{y^2} = \int_0^\infty f_a(y) \frac{\psi(y)}{y^2} dy
\]

where we used that \( \{ w \in \mathbb{H} | 0 < \text{Re}(w) < 1 \} \) is a fundamental domain for \( \Gamma_\infty \). \( \square \)

Remark 4.3.3 Observe that the eigenfunction condition was neither used in section 1.4 nor section 1.5. Thus the assertions made there carry over verbatim to the present setting.

This yields in fact a first indication towards a spectral decomposition of \( \mathcal{H}_k \):

Definition 4.3.4 We abbreviate the linear space of incomplete Eisenstein series of weight \( k \in \mathbb{R} \) and the space of Hilbert functions with exponential decay as follows:

\[
\mathcal{E}_k := \text{span}_{\mathbb{C}} \{ E_a(\cdot | \psi) | \text{a cusp for } \Gamma, \psi \in C_\infty^c(\mathbb{R}_{>0}, \mathbb{C}) \} \subseteq \mathcal{H}_k
\]

\[
\mathcal{H}_{k,0} := \bigcap_{\text{a cusp}} \{ f \in \mathcal{H}_k | f_a(y) = 0 \}
\]

where \( E_a \) has weight \( k \).

Corollary 4.3.5 We have the orthogonal decomposition \( \mathcal{H}_k = \tilde{\mathcal{H}}_{k,0} \oplus \mathcal{E}_k \), where the tilde denotes the closure with respect to the Hilbert norm topology.

Proof This is immediate from the previous proposition and the fact that \( \psi \) is arbitrary (apply proposition A.1.4). \( \square \)

Remark 4.3.6 Observe that if \( f \in \mathcal{H}_{k,0} \) is an eigenfunction of \( -\Delta_k \) then \( f \) is a cusp form in the terminology of definition 1.5.1 and proposition 1.5.2. This will become important in the proof of theorem 5.1.1.

\[ \text{[17] Last assertion of theorem 1.4.1!} \]

\[ \text{[18] Iwa02, equation (3.16).} \]
Chapter 5

The point spectrum of $-\tilde{\Delta}_k$

5.1 Introduction

We reach the central part of this thesis, which is a development of a spectral decomposition of the Hilbert space $\mathcal{H}_k$ introduced in chapter 2. We split this task among the next two chapters, because the involved techniques are quite different.

Definition 4.3.4 and remark 4.3.6 enables us to formulate:

**Theorem 5.1.1 (Spectral expansion, discrete contribution)** The operator $-\tilde{\Delta}_k$ has pure point spectrum in $\mathcal{H}_{k,0}$, i.e. $\mathcal{H}_{k,0}$ is spanned by cusp forms. The eigenspaces have finite dimension. If $\{u_j\}_j$ is a countable complete orthonormal system of cusp forms then any $f \in \mathcal{H}_{k,0}$ has the representation

$$f(z) = \sum_{j=1}^{\infty} \langle f, u_j \rangle u_j(z)$$

converging in the norm topology. If additionally $f \in \mathcal{H}_{k,0} \cap D_k^{(2)}$ then the series converges pointwise absolutely, and uniformly on compacta.

**Remark 5.1.2** This result implies [Roe67, Satz 8.1].

The proof relies on the *Hilbert-Schmidt theorem*, which provides a spectral decomposition of bounded compact self-adjoint operators. The precise formulation is given in theorem A.4.2. But $-\tilde{\Delta}_k$ itself is far from being neither bounded nor compact, hence we switch back to integral operators, which were introduced in section 3.1. Those operators possess much better analytic structure and in the end we will apply the Hilbert-Schmidt theorem to a “compactly adjusted” version of the resolvent $R_{k,\lambda}$, which was constructed during chapter 3. We follow [Iwa02, chapter 4].
5. The point spectrum of $-\tilde{\Delta}_k$

5.2 Invariant integral operators revisited

Recall from chapter 3 that an invariant integral operator is given by convolution with a point-pair invariant\(^1\) kernel $K$. Restricting to functions $f : \mathbb{H} \to \mathbb{C}$ satisfying the transformation law, which is guaranteed by our definition of $\mathcal{H}_k$, we may write such an operator $L$ as in (3.2):

$$(Lf)(z) := \int_{\mathbb{H}} K(z, w) f(w) d\mu(w)$$

(5.1)

Furthermore $K$ is given by the series\(^2\)

$$K(z, w) = \frac{1}{2} \sum_{\gamma \in \Gamma} v(\gamma) j(\gamma; k) k(z, \gamma w)$$

(5.2)

where $k$ is the kernel for the trivial group $\{ \pm 1 \}$ or equivalently for the whole domain $\mathbb{H}$.

**Lemma 5.2.1** Let $k(z, w)$ be a point-pair invariant on $\mathbb{H} \times \mathbb{H}$ and fix $z \in \mathbb{H}$. Then we have

$$\int_{\mathbb{H}} k(z, w) f(w) d\mu(w) = \int_{\mathbb{F}} K(z, w) f(w) d\mu(w)$$

(5.3)

**Proof** If we replace $K$ by $G_{\lambda, k}$ and $k$ by $g_{\lambda, k}$ then the lemma reduces to equation (3.12). Hence in order to prove the lemma, we may just follow precisely the first steps in the proof of theorem 3.2.1, which are based on the unfolding technique.

This justifies the more intensive study of invariant integral operators due to the following consequence:

**Proposition 5.2.2** Let $L$ be an invariant integral operator. Then $L(\mathcal{H}_{k,0}) \subseteq \mathcal{H}_{k,0}$.

**Proof** Let $f \in \mathcal{H}_{k,0}$, $a$ be a cusp with scaling matrix $\gamma_a$. Set $g := Lf$ and $\tau(x) := \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \in \Gamma_\infty$. Denote by $f_a(y)$, $g_a(y)$ the zeroth Fourier coefficient of $f$, $g$ respectively. We verify the definition of $\mathcal{H}_{k,0}$ and need to compute:

$$g_a(y) = \int_0^1 g(\gamma_a \tau(x) z) dx = \int_0^1 \int_{\mathbb{F}} K(\gamma_a \tau(x) z, w) f(w) d\mu(w) dx$$

\(^1\)That is the function depends solely on the hyperbolic distance: $K(z, w) = K(d(z, w))$

\(^2\)Recall that this is precisely the technique of averaging $k$ over $\Gamma$. Compare also with equation (3.11).
Invoking the previous lemma gives

\[ g_a(y) = \int_0^1 \int_H k(\gamma_a \tau(x)z, w) f(w) d\mu(w) dx \]
\[ = \int_H k(z, w) \left( \int_0^1 f(\gamma_a \tau(x)w) dx \right) d\mu(w) \]
\[ = \int_H k(z, w) f_a(\text{Im}(w)) d\mu(w) = 0 \]

Hence \( g \in \mathcal{H}_{k,0} \) as claimed. \( \square \)

Next let us further examine the kernel \( K \). As we have seen in chapter 3, \( K \) is typically not bounded on \( \mathbb{F} \times \mathbb{F} \), for instance on the diagonal \( z = w \). To fix this issue we invoke another idea, that we have already seen: Subtract the "problematic parts" such as the cusp sectors from \( \mathbb{F} \) or the truncated Eisenstein series in chapter 4.

**Definition 5.2.3**\(^3\) Let \( K \) be the kernel given by (5.2), \( a \) be a cusp with scaling matrix \( \gamma_a \) and \( \tau(x) := \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \).

(i) The function

\[ H_a(z, w) := \frac{1}{2} \sum_{\gamma \in \Gamma_a \backslash \Gamma} v(\gamma) f_\gamma(w; k) \int_{-\infty}^{\infty} k(z, \gamma_a \tau(x) \gamma_a^{-1} \gamma w) dx \]

on \( \mathbb{H} \times \mathbb{H} \) is called the principal part of \( K \) at the cusp \( a \).

(ii) The expression

\[ \hat{K}(z, w) := K(z, w) - \sum_a H_a(z, w) \]

is called the compact part of \( K \), where the sum is taken over all inequivalent cusps \( a \).

(iii) We denote by \( \hat{L} \) the integral operator associated to the kernel \( \hat{K} \).

**Remark 5.2.4** \( H_a(z, w) \) is an incomplete Eisenstein series in the second variable: Use the Selberg / Harish-Chandra transform developed in section 6.4:

\[ u := d(z, w) \]
\[ q(v) := \int_0^\infty \frac{k(u)}{\sqrt{u - v}} du \]
\[ g(r) := 2q \left( \sinh \left( \frac{r}{2} \right)^2 \right) \]

Then put \( \psi(t) := \sqrt{\text{Im}} g \left( \log \left( \frac{t}{2} \right) \right) \), which decays sufficiently fast\(^4\). A proof is included in lemma 6.4.7.

\(^3\)Recall equation (1.4) and the definition of the compact part of \( \mathbb{F} \).
\(^4\)Iwa02, p. 68.
We enlist their properties:

**Proposition 5.2.5** Let $z \in \mathcal{H}$, $a$ be a cusp.

(a) $H_a(z, \cdot)|_\gamma = v(\gamma)H_a(z, \cdot)$
(b) $\forall f \in \mathcal{H}_{k,0} : \langle H_a(z, \cdot), f \rangle = 0$
(c) $\forall f \in \mathcal{H}_{k,0} : \hat{L}f = Lf$

**Proof**

(a) Argue similar to $G_{\lambda,k}$ in section 3.3.2.

(b) We substitute $z \mapsto \gamma a z$ and unfold the integral over $\mathcal{F}$ as in lemma 5.2.1. By equation (1.4) this gives an integral over the fundamental domain of $\Gamma_\infty$:

$$\langle H_a(\gamma_0 z, \cdot), f \rangle = \int_0^\infty \int_{-\infty}^\infty \left( \int_{-\infty}^{\infty} k(z, \tau(t)w) dt \right) \overline{f(\gamma_0 w)} d\mu(w)$$

$$= \int_0^\infty \left( \int_{-\infty}^{\infty} k(z, t + iy) dt \right) \left( \int_{-\infty}^{\infty} \overline{f(\gamma_0(x + iy))} dx \right) \frac{dy}{y^2}$$

$$= \int_0^\infty \left( \int_{-\infty}^{\infty} k(z, t + iy) dt \right) \overline{f_a(y)} \frac{dy}{y^2} = 0$$

(c) This follows immediately by item b) and the definition of $\hat{K}$ above. □

The central building block to utilize the Hilbert-Schmidt theorem is:

**Proposition 5.2.6** Choose $\mathcal{F}$ such that its cuspidal vertices are all distinct mod $\Gamma$. Then $\hat{\mathcal{K}}(z, w)$ is bounded on $\mathcal{F} \times \mathcal{F}$.

**Remark 5.2.7** First observe that we may assume compact support and smoothness of $k(d(z, w))$, then the general case follows in virtue of an mollification argument as in the proof of lemma 2.1.5. In fact it suffices

$$k(u), k'(u) \ll \frac{1}{(u + 1)^2}$$

where $u = d(z, w)$, see [Iwa02, p. 68]. Moreover recall $|v(\cdot)| = |j(w; k)| = 1$.

This remark enables us to follow the proof given in [Iwa02, proposition 4.5], which is based on two lemmas:

**Lemma 5.2.8** For every $z, w \in \mathcal{H}$ we have uniformly

$$H_a(\gamma_a z, w) \ll 1 + \text{Im}(z)$$

**Lemma 5.2.9 (Euler-MacLaurin formula)** Let $q(t) = t - \lfloor t \rfloor - \frac{1}{2}$ and $F \in C_c\left((a, b), \mathbb{R}\right)$ be of bounded variation. Then

$$\sum_{m \in \mathbb{L}} F(m) = \int F(t) dt + \int q(t) dF(t)$$
where we write $F$ as difference of two monotone increasing functions and define the Lebesgue-Stieltjes integral above as the difference of the two Lebesgue-Stieltjes integrals with respect to those monotone increasing functions.

We prove the proposition first:

**Proof (of proposition 5.2.6)** We begin with a geometric observation by dividing the elements $\gamma \in \Gamma$ into parabolic and non-parabolic ones: If $\gamma$ is a non-parabolic motion then $d(z, \gamma w)$ becomes arbitrarily large for almost all $\gamma$ uniformly in $z, w \in F$. But since $k(d(z, w))$ is compactly supported we have

$$K(z, w) = \sum_{\gamma \text{ parabolic}} k(z, \gamma w) + O(1)$$

If we replace the assumption of compact support of $k(d(z, w))$ with the result of lemma 5.2.8 then we are able to argue for $H_a$ in the same fashion as for $K$ above. We exclude $\gamma = 1$ and observe that all other cosets in the definition of $H_a$ give a uniformly bounded contribution thanks to lemma 5.2.8. This implies

$$H_a(z, w) = \int_{-\infty}^{\infty} k(z, \gamma_a \tau(x) \gamma_a^{-1} w) \, dx + O(1)$$

We rearrange $\sum_{\gamma \text{ parabolic}} = \sum_a \sum_{\gamma \in \Gamma_a}$ and combine the two estimates yielding:

$$\hat{K}(z, w) = \sum_a \left( \sum_{\gamma \in \Gamma_a} k(z, \gamma w) - \int_{-\infty}^{\infty} k(z, \gamma_a \tau(t) \gamma_a^{-1} w) \, dt \right) + O(1)$$

Recall that according to lemma 4.2.1 the number of inequivalent cusps for $\Gamma$ is finite. Hence we have to bound $J_a(z, w)$, for which we utilize the Euler-MacLaurin formula with $F(t) := k(z, w + t)$ (justified by remark 5.2.7):

$$J_a(\gamma_a z, \gamma_a w) = \sum_{m \in \mathbb{Z}} k(z, w + m) - \int_{-\infty}^{\infty} k(z, w + t) \, dt$$

$$= \int_{-\infty}^{\infty} \varphi(t) k(z, w + t) \, dt \ll \int_{0}^{\infty} k'(u) \, du \ll 1$$

where $u = d(z, w)$. Thus $J_a(z, w)$ is bounded on $\mathbb{H} \times \mathbb{H}$ and in particular on $F \times F$, which completes the proof. 

Now it suffices to prove the two lemmas:

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5.$\gamma$ has infinite order and moves points along horocycles (circles in $\mathbb{H}$ tangent to $\mathbb{R} \cup \{\infty\}$) or equivalent $\text{tr}(\gamma) = 2$. 

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**Proof (of lemma 5.2.8)** We substitute $w \mapsto \gamma_a w$ and seek to estimate according to equation (1.4) and (3.36):

$$H_a(\gamma_a z, \gamma_a w) = \sum_{\gamma \in \Gamma_a \setminus \Gamma} \int_{\gamma a}^{\infty} k(z, x + \gamma w) dx$$

Now invoking remark 5.2.7 the ranges of integration and summation are restricted by $|z - x - \gamma w|^2 \ll \Im(z) \Im(\gamma w)$. Dividing this either by $\Im(z)$ or $\Im(\gamma w)$ we get $\Im(z) \propto \Im(\gamma w)$ and hence the integral is bounded by $O(\Im(z))$. It remains to bound the number of summands:

**Claim 6**: Let $z \in H$ and $Y > 0$. Then $\#\{\gamma \in \Gamma_{\infty} \setminus \Gamma | \Im(\gamma z) > Y\} < 1 + \frac{10}{c_{\infty} Y}$, where $c_{\infty} = \min \left\{ c > 0 | \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma \right\}$

The claim immediately gives $H_a(\gamma_a z, \gamma_a w) \ll (1 + \frac{1}{\Im(z)}) \Im(z) = 1 + \Im(z)$ as desired.

**Proof of the claim**: This is an argument purely in hyperbolic geometry: We may choose $F$ such that it consists of points in $\Gamma_{\infty} \setminus H$ of deformation less than 1 and we may assume $z \in F$. In other words letting $\gamma = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma \setminus \Gamma_{\infty}$ with $c > 0$ we then have $\gamma z \in \{ \tilde{z} | |c \tilde{z} + d| \leq 1 \}$. Combining with our assumption $\Im(\gamma z) \overset{(1.3)}{=} \frac{\Im(z)}{|cz + d|^2} > Y$ this yields

$$y > Y, c < \frac{1}{\sqrt{yy}}, |cx + d| < \sqrt{\frac{y}{Y}}$$

Note that if we have another $\gamma' = \begin{pmatrix} * & * \\ c' & d' \end{pmatrix} \in \Gamma \setminus \Gamma_{\infty}$ with $c' > 0$ then also $\gamma' \gamma^{-1} =: \begin{pmatrix} * & * \\ c'' & d'' \end{pmatrix}$ with $c'' \geq c_{\infty}$, whence $|\frac{d'}{c'} - \frac{d}{c}| \geq \frac{c_{\infty}}{c'}$, from which we estimate the number of pairs $(c, d)$ with $C < c < 2C$ by

$$1 + 8c_{\infty}^{-1}C \sqrt{\frac{y}{Y}} \leq 10c_{\infty}^{-1}C \sqrt{\frac{y}{Y}}$$

Finally we iterate the above argument for $C(n) := \frac{1}{2^n \sqrt{y}}$, $n \geq 1$ and sum up the bounds getting $\frac{10}{c_{\infty} Y}$ as bound if $\gamma \in \Gamma \setminus \Gamma_{\infty}$. Add 1 into account for $\Gamma_{\infty}$ we infer the desired bound and conclude the proof of the lemma too. \(\square\)

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6Iwa02, lemma 2.10.
5.3. Proof of theorem 5.1.1

**Proof (of the Euler-MacLaurin formula)** We start with [AS72, p. 23.1.30]: Let 
\[ h := \frac{b - a}{N}, \quad n \geq 1. \] Then there exists a constant \( C \in (0, 1) \) depending on \( d(2N)F \) such that
\[
\sum_{k=0}^{N} F(a + kh) = \frac{1}{h} \int_{a}^{b} F(t) \, dt + \frac{1}{2} \sum_{k=1}^{n-1} \frac{h^{2k-1}}{(2k)!} B_{2k} \left( \frac{d(2k-1)F}{dt(2k-1)}(b) - \frac{d(2k-1)F}{dt(2k-1)}(a) \right) 
+ \frac{F(b) - F(a)}{2} + \frac{h^{2n}B_{2n}}{(2n)!} \sum_{k=0}^{N-1} \frac{d(2N)F}{dt(2N)}(a + h(k + C))
\]
Now let \( a \to -\infty, \ b, n, N \to \infty \), whence \( h \to 1 \) and the sums on the right hand side become integrals as claimed. \( \square \)

Summarizing this section we have constructed a bounded kernel \( \hat{K} \) out of \( K \), which induces the same integral operator \( \hat{L} \) on \( \mathcal{H}_{k,0} \).

5.3 Proof of theorem 5.1.1

The main idea is to apply the results from the previous section to the resolvent \( R_{\lambda,k} \), which is an invariant integral operator with kernel \( G_{\lambda,k} \) constructed in chapter 3. But there is one obstacle: \( G_{\lambda,k} \) is singular on the diagonal \( z = w \) or equivalently at \( 0 = u =: \Delta(z,w) \). To fix this issue recall \( \lambda = s(1 - s) \) and consider for \( a > s \geq 2 \):
\[
\hat{L} := R_{s(1-s),k} - R_{a(1-a),k}
\]  
We check the conditions for \( \hat{L} \) required by the Hilbert-Schmidt theorem:

- \( \hat{L} \) has dense range in \( \mathcal{H}_{k,0} \):
  First \( \mathcal{L} \) has dense range in \( \mathcal{H}_k \) by Hilbert’s first resolvent identity\(^7\)
  \[
  \mathcal{L} = (s(1-s) - a(1-a)) R_{s(1-s),k} R_{a(1-a),k}
  \]
  Now let \( f \in \mathcal{D}^{(2)}_k \), which is dense in \( \mathcal{H}_k \) according to lemma 2.1.5. Put
  \[
  g := \frac{(\hat{\Delta}_k + a(1-a))(\hat{\Delta}_k + s(1-s))}{s(1-s) - a(1-a)} f
  \]
  By definition \( g \in \mathcal{D}^{(2)}_k \) and \( \mathcal{L}g = f \). Hence \( \hat{L}g = f \) in virtue of proposition 5.2.5. Moreover if \( f \in \mathcal{H}_{k,0} \) then clearly \( g \in \mathcal{H}_{k,0} \). Thus \( \mathcal{H}_{k,0} \cap \mathcal{D}^{(2)}_k \) is in the range of \( \hat{L} \).

\(^7\)See proposition A.3.5.
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- $\hat{\mathcal{L}}$ is continuous:
  The free-space kernel $h(u) := g_{a(1-a),k} - g_{a(1-s),k}$ satisfies
  \[ h(u), h'(u) \ll \frac{1}{(u + 1)^2} \]
due to equation (3.13) as required by remark 5.2.7. This allows us to apply proposition 5.2.6. Combining with proposition A.1.1 gives continuity of $\hat{\mathcal{L}}$, where we evolve $\hat{K}$ from $K$ and $K$ from $h$ exactly as in the previous section.

- $\hat{\mathcal{L}}$ is compact:
  By the previous item $\hat{\mathcal{L}}$ has square integrable kernel on $\mathcal{F} \times \mathcal{F}$, which suffices to show compactness. The proof is a standard but lengthy argument from functional analysis and hence postponed to section A.2 in the appendix.

- $\hat{\mathcal{L}}$ is self-adjoint:
  Recall that $-\tilde{\Delta}_k$ is the unique self-adjoint extension of $-\Delta_k$ in $\mathcal{H}_{k_0}$, see section 2.2. Since $\mathcal{R}_{\lambda,k}$ inverts $-\tilde{\Delta}_k$ according to theorem 3.2.1, $\mathcal{R}_{\lambda,k}$ has to be self-adjoint as well. Combining proposition 5.2.5 c) and the definition of $\hat{\mathcal{L}}$ we see that $\hat{\mathcal{L}}$ is self-adjoint too.
  Alternatively argue directly via the symmetry of $|G_{\lambda,k}|$: The adjoint of an integral operator $\mathcal{L}$ with kernel $K(z,w)$ is again an integral operator $\mathcal{L}^*$ with kernel $\bar{K}(z,w) := \overline{K(w,z)}$. Apply proposition 3.3.2.

This enables us to state the following:

1. The Hilbert schmidt theorem (theorem A.4.2) provides a complete countable orthonormal system \( \{u_j\}_{j \geq 1} \) of eigenfunctions of $\hat{\mathcal{L}}$ in $\mathcal{H}_{k_0}$, which are then cusp forms by remark 4.3.6.

2. The spectrum of $\hat{\mathcal{L}}$ is entirely point spectrum except for the value 0, see proposition A.4.1. The eigenvalues to $\{u_j\}$ become arbitrarily large. Indeed, the Hilbert-Schmidt theorem allows accumulation of the eigenvalues of the eigenvalues of the (modified) resolvent $\mathcal{L}$ only at 0. Hence the eigenvalues to $\{u_j\}$ may only accumulate at infinity, because the eigenvalues of $\mathcal{L}$ and $u_j$ are inverse to each other.

3. The eigenvalues $\{\lambda_j\}_{j \geq 1}$ have no accumulation point apart from possibly 0 and in particular they have finite multiplicity\(^8\). Thus the eigenspaces for non-zero eigenvalues are finite dimensional.

4. Note that 0 is contained in the spectrum of $\hat{\mathcal{L}}$, and is contained in the point spectrum too if and only if $\hat{\mathcal{L}}$ is not injective.

\(^8\)Countably many eigenvalues without accumulation point away from 0.
5.4. A first corollary

5. It remains to show pointwise absolute and locally uniform convergence: Let \( f \in \mathcal{H}_{k,0} \cap D_k^{(2)} \), set \( g := (-\tilde{\Delta}_k - \lambda I)f \). Clearly \( \|g\| < \infty \) by definition. Let \( n, N \in \mathbb{N} \), \( n < N \) and \( \lambda_j \) be the eigenvalue to the cusp form \( u_j \). Since \( \hat{K} \) is the compact part of \( G \), which is the kernel of the resolvent for \( -\tilde{\Delta}_k \), we have

\[
\sum_{j=n}^{N} |\langle f, u_j \rangle u_j| = \sum_{j=n}^{N} |\langle f, (\lambda_j - \lambda)u_j \rangle| \left| \int_F \hat{K}(\cdot, w)u_j(w) d\mu(w) \right|
\]

\[
= \sum_{j=n}^{N} |\langle g, u_j \rangle| \left| \int_F \hat{K}(z, w)u_j(w) d\mu(w) \right|
\]

\[
\leq \left( \sum_{j=n}^{N} |\langle g, u_j \rangle|^2 \right)^{1/2} \left( \int_F |\hat{K}(\cdot, w)|^2 d\mu(w) \right)^{1/2} \xrightarrow{n, N \to \infty} 0
\]

because \( \hat{K} \) is bounded and \( F \) has finite hyperbolic volume.

These observations altogether prove theorem 5.1.1.

**Remark 5.3.1** Note that if there are no singular cusps\(^9\) for \( \Gamma \) then it holds \( \hat{K} = K \) by definition, since we have no (well-defined) principal part \( H_a \) of \( K \).

Thus in particular

\[
\int_F \int_F |G_{\lambda, k}(z, w)|^2 d\mu(z) d\mu(w) < \infty
\]

and the eigenfunctions from the first item above form a complete system in \( \mathcal{H}_k \). Compare also remark 6.7.5 and [Roe67, Satz 8.2] afterwards.

5.4 A first corollary

Theorem 5.1.1 has an immediate but nevertheless remarkable consequence, which we discuss briefly here and is formulated in [Roe67, Satz 8.3]:

**Proposition 5.4.1** Fix a non-zero eigenvalue \( \lambda \in \mathbb{C} \). Then the vector space of automorphic forms with eigenvalue \( \lambda \in \mathbb{C} \) has finite dimension.

**Proof** Let \( f \) be an automorphic form with eigenvalue \( \lambda = s(1 - s) \neq 0 \) (hence \( s \) is fixed). Let \( a_1, \ldots, a_m \) be a complete system of inequivalent cusps for \( \Gamma \) according to lemma 4.2.1. Then \( f \) has a Fourier expansion at every cusp in virtue of theorem 1.4.1. Denote by \( f_{a_1}(y), \ldots, f_{a_m}(y) \) the zeroth Fourier coefficients with respect to the expansions at \( a_1, \ldots, a_m \). Define

\[
g(z, s) := f(z) - \sum_{j=1}^{m} \delta_{a_j} E_{a_j}(z, s) f_{a_j}(\text{Im}(z))
\]

\(^9\)see equation (4.2)
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Hence $g$ is an automorphic form, since $g$ is a linear combination of Eisenstein series, see proposition 4.1.13. Moreover the zeroth Fourier coefficient at any cusp $b$ vanishes by construction, thus $g$ is a cusp form to the same non-zero eigenvalue $\lambda$. It follows that $g$ is contained in one of the finite dimensional eigenspaces occurring in theorem 5.1.1. \[\square\]

Compare this proposition to the following result in the theory of modular forms:

**Proposition 5.4.2** The vectorspace of modular forms for a congruence subgroup of $\text{SL}_2(\mathbb{R})$ is finite dimensional.

Recall that $k \in \mathbb{Z}_{\geq 0}$ by corollary 2.3.3 and the modular forms of weight 0 are the constant functions by corollary 2.3.4.

A classic proof in the case of $\text{SL}_2(\mathbb{Z})$ is based on the valence formula, see for instance [Kob93, pp. 115-118]. In our setting we can just utilize example 1.3.6 immediately proving the case of $\text{SL}_2(\mathbb{Z})$ as well. The case of a general congruence subgroup is briefly indicated in [IK04, p. 356].
Chapter 6

The continuous spectrum of $-\tilde{\Delta}_k$

6.1 Introduction

We have a loose end from the previous two chapters: On the one hand there is corollary 4.3.5 stating

$$\mathcal{H}_k = \tilde{\mathcal{H}}_{k,0} \oplus \tilde{\mathcal{E}}_k$$  \hspace{1cm} (6.1)

On the other hand according to the Hilbert-Schmidt theorem and theorem 5.1.1

$$\mathcal{H}_k = \ker(\hat{L}) \oplus \text{clos} \left( \text{span}_C \{ u_j \mid j \geq 1 \} \right) = \ker(\hat{L}) \oplus \tilde{\mathcal{H}}_{k,0}$$  \hspace{1cm} (6.2)

Recall that $\mathcal{E}_k$ is the space of incomplete Eisenstein series from definition 4.3.4, the tilde denotes the closure with respect to the Hilbert norm $\| \cdot \|$, and $\hat{L}$ is the “bounded part” of $L$, which is given by equation (5.4). Remark 5.2.4 and proposition 5.2.5 indicate a proof of $\tilde{\mathcal{E}}_k = \ker(\hat{L})$. We will prove the claims from remark 5.2.4 in lemma 6.4.7 below, which then collects this loose end.

Nevertheless this is far from being neither an explicit nor a detailed description of $\mathcal{E}_k$. So we refine our previous viewpoints to investigate the space $\mathcal{E}_k$ in greater detail. Recall the Mellin transform and its inverse

$$\mathcal{M}: \{ \varphi: \mathbb{R}_{>0} \to \mathbb{R} \} \times \mathbb{C} \to \mathbb{C}$$

$$(\varphi, s) \mapsto (\mathcal{M}\varphi)(s) := \int_0^\infty \varphi(y)y^{s-1}dy$$

$$(\mathcal{M}^{-1}\varphi)(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \varphi(s)x^{-s}ds$$

where $c \in \mathbb{R}$. The Mellin transform was briefly mentioned in section 4.2 and motivates the spectral expansion of incomplete Eisenstein series by the following proposition:
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**Proposition 6.1.1** Let $\psi \in C_0^\infty(\mathbb{R}_{>0},\mathbb{R})$, $z \in \mathbb{H}$, $s \in \mathbb{C}$, $c := \text{Re}(s) > 1$ and $a$ be a cusp. Then $E_a(z|\psi)$ is related to $E_a(z,s)$ by inverse Mellin transformation, i.e. it holds

\[ E_a(z|\psi) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} (M\psi)(-s) E_a(z,s) ds \tag{6.3} \]

**Proof** Using $\text{Re}(s) > 1$ we are able to exchange summation and integration:

\[
\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} (M\psi)(-s) E_a(z,s) ds \\
= \sum_{\gamma \in \Gamma_a \backslash \Gamma} \nu(\gamma)^{-1} \sigma_k(\gamma a^{-1}, \gamma)^{-1} j_{\gamma a^{-1}, \gamma}(z; k) \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} (M\psi)(s) \text{Im}(\gamma a^{-1} \gamma)^{-s} ds \\
= \sum_{\gamma \in \Gamma_a \backslash \Gamma} \nu(\gamma)^{-1} \sigma_k(\gamma a^{-1}, \gamma)^{-1} j_{\gamma a^{-1}, \gamma}(z; k) \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} (M\psi)(s) \text{Im}(\gamma a^{-1} \gamma) \text{Im}(\gamma^{-1} \gamma z) = E_a(z|\psi)
\]

as claimed. \(\square\)

This established, we will move the contour of integration in proposition 6.1.1 to $c = \frac{1}{2}$ and apply the functional equation for the Eisenstein series stated in section 4.2. In the end this culminates to the following:

**Theorem 6.1.2 (Spectral expansion, continuous contribution)** The space $\mathcal{E}_k$ of incomplete Eisenstein series splits orthogonally into $\Delta_k$-invariant subspaces:

\[ \mathcal{E}_k = \mathcal{P}_k \bigoplus_a \mathcal{E}_{k,a} \]

where the sum is taken over all inequivalent cusps $a$ for $\Gamma$. The spectrum of $-\Delta_k$ in $\mathcal{P}_k$ is discrete, more precisely it consists of finitely many points $0 \leq \lambda_j = s_j(1-s_j) < \frac{1}{4}$.

The spectrum of $-\Delta_k$ in $\mathcal{E}_{k,a}$ is continuous and covers the interval $[\rho(k), \infty)$.

If $\{v_j\}_j$ is an orthonormal basis of $\mathcal{P}_k$ then any $f \in \mathcal{E}_k$ has the representation

\[ f(z) = \sum_j \langle f, v_j \rangle v_j(z) + \sum_a \frac{1}{4\pi} \int_{-\infty}^{\infty} \left\langle f, E_a \left( \frac{1}{2} + it \right) \right\rangle E_a \left( z, \frac{1}{2} + it \right) dt \]

which converges in the norm topology. If additionally $f \in \mathcal{E}_k \cap \mathcal{D}_k^{(\infty)}$ and $-\Delta_k f \in \mathcal{D}_k^{(\infty)}$ then this expansion converges pointwise absolutely, and uniformly on compacta.

The location of the continuous spectrum was determined in proposition 2.4.5 already. Recall

\[ \rho(k) = \max \left\{ \frac{1}{2} \left( 1 - \frac{l}{2} \right) \mid l \equiv k \mod 2 \right\} \]
and that \( \rho(k) \leq \frac{1}{4} \) with equality if \( k \) is an odd integer.

Moreover we continue to assume implicitly that \( \Gamma \) admits at least one singular cusp \(^1\) with respect to \( \nu \), else we may conclude by remark 5.3.1. We collect further remarks to theorem 6.1.2 in section 6.7.

We adapt [Kub73, §5.3] and [Iwa02, chapter 7] to our setting. Both authors work only with Eisenstein series of weight 0, but this is no constraint. The results carry over verbatim to general Eisenstein series of weight \( k \). The underlying reason was discussed in chapter 4: Both types share the same class of properties, because the additional factors required for weight \( k \) Eisenstein series all have absolute value 1, which will come in handy later. This is all we need to begin with the development of theorem 6.1.2 in the general case of weight \( k \in \mathbb{R} \).

### 6.2 The space \( \mathcal{P}_k \)

We examine the residues of \( E_a(z,s) \), which continues the discussion of the poles in subsection 4.2.2. This leads to the space \( \mathcal{P}_k \) from theorem 6.1.2.

**Proposition 6.2.1** \(^2\) The poles of \( E_a(z,s) \) are all simple and located in the interval \( \left( \frac{1}{2}, 1 \right] \). The residues are automorphic forms, square-integrable on \( \mathbb{F} \) and orthogonal to cusp forms.

**Proof** Let \( s_j \) be a pole of order \( m \geq 1 \). Then the function

\[
f(z) := \lim_{s \to s_j} (s - s_j)^m E_a(z,s)
\]

does not vanish identically and is again an automorphic form with respect to the \( z \)-variable in virtue of proposition 4.1.13. Hence according to theorem 1.4.1 it has a Fourier expansion at any cusp \( b \) and in virtue of theorem 4.1.14 it is of the form

\[
f(\gamma_b z) = \left( \lim_{s \to s_j} (s - s_j)^m \varphi_{ab}(s) \right) y^{1-s_j} \\
+ \sum_{n \neq 0} \left( \lim_{s \to s_j} (s - s_j)^m \varphi_{ab}(n,s) \right) W_{ir,\text{sign}(n)}(4\pi |n| y) e^{2\pi i n x}
\]

On the one hand if \( \text{Re}(s_j) > \frac{1}{2} \) then the constant term is square integrable with respect to \( y \) and by exponential decay of the sum we obtain \( \|f\| < \infty \). Observe that \( f \) has eigenvalue \( \lambda = \lim_{s \to s_j} s(1-s) = s_j(1-s_j) \). On the other hand if \( \text{Re}(s) > 1 \) then the Eisenstein series are holomorphic in \( s \) thanks to

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\(^1\) see equation (4.2)  
\(^2\) Iwa02, theorem 6.10.
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proposition 4.1.13. Thus invoking proposition 2.2.2 a) we deduce \(s_j \in (\frac{1}{2}, 1]\).
Furthermore if \(s_j\) were not a pole of \(\Phi_a(s)\) or if \(m > 1\) then for any cusp \(b\) we would have\(^3\) \(\lim_{y \to s_j} (s - s_j)^m \varphi_{ab}(s) = 0\). Hence \(f\) is a cusp form. But this is impossible, because \(E_a(z, s)\) is orthogonal to cusp forms\(^4\) for \(s \neq s_j\). In other words \(f\) would be orthogonal to itself while \(f \neq 0\). □

The preceding proof motivates the following definition:

**Definition 6.2.2**

\[
\mathcal{P}_k := \text{span}_\mathbb{C}\{f \in \mathcal{E}_k \mid -\tilde{\Delta}_k f = \lambda f\}
\]

We provide a short description of \(\mathcal{P}_k\):

**Proposition 6.2.3** The space \(\mathcal{P}_k\) is spanned by the residues of the Eisenstein series \(E_a(z, s)\) for the cusps \(a\).

**Proof** We have already seen that the residues are contained in \(\mathcal{P}_k\) and adapt the proof in [Kub73, theorem 5.2.3] now. Let \(f \in \mathcal{P}_k\) and \(b\) be a cusp. Denote the constant term in the Fourier expansion of \(f\) once more by \(f_b(y)\). Then as above \(f_b(y) = C y^{1-s_0}\) for some \(s_0 > \frac{1}{2}\) and a constant \(C > 0\). Hence for some \(Y > 0\) sufficiently large

\[
\int_Y^\infty |f_b(y)|^2 \frac{dy}{y^2} < \infty
\]

Let \(a_1, \ldots, a_N\) be a complete set of inequivalent cusps for \(\Gamma\) and \(n\) be the unique index such that \(b = a_n\). Recall that \(\Phi(s)\) is a symmetric matrix, which was the first half in the proof of corollary 4.2.8. This permits us to rearrange the cusps in the following fashion: For some \(1 \leq j < n\) it holds that \(s_0\) is a simple pole of \(\varphi_{a_1 a_1}, \ldots, \varphi_{a_j a_j}, E_{a_1, \ldots, a_j}\) are linear independent and \(E_{a_n}\) is either a linear combination of \(E_{a_1}, \ldots, E_{a_j}\) or else \(\varphi_{a_n a_n}(s_0) < \infty\). Hence if \(j = 0\) we are done. Else we subtract a suitable linear combination of \(E_{a_l}\) from \(f\), such that we get an element \(g \in \mathcal{P}_k\) satisfying \(g_{a_l}(y) = 0\) for all \(l \leq j\). This can be done easily by induction, compare the proof of proposition 5.4.1. The goal is to show \(g \equiv 0\). It suffices to achieve \(\forall 1 \leq l \leq N : g_{a_l}(y) = 0\). Then \(g\) is a cusp form and we can conclude by the same argument as at the end of the previous proof.

If \(l > j\) we have \(g_{a_l}(y) = C' y^{1-s_0}\) and we utilize the Maass-Selberg formula: Construct \(c_n(z, s)\) by subtracting a suitable linear combination of \(E_{a_1}, \ldots, E_{a_j}\) from \(E_{a_n}\), such that \(s_0\) is not a pole of \(c_n\) and

\[
\left\langle S^Y, c_n^Y(z, s) \right\rangle = -\sum_{l=j+1}^{N} \frac{C' Y^{s-s_0}}{s-s_0} + \frac{C'_l \varphi_{a_n a_n}(s) Y^{1-s-s_0}}{s+s_0-1}
\]

\(^3\) The term \((s-s_j)^m\) dominates the term \(\varphi_{ab}(s)\).

\(^4\) An independent proof of this fact is given below in proposition 6.3.7.
Thus if $C' \neq 0$ then letting $s \to s_0$ the right hand side diverges, which is a contradiction to theorem 4.2.4. □

**Corollary 6.2.4** $\mathcal{P}_k$ is orthogonal to $\mathcal{H}_{k,0}$.

**Proof** Expand any $f \in \mathcal{H}_{k,0}$ into cusp forms according to theorem 5.1.1 and apply the previous result to this expansion. □

Finally let $\mathcal{P}_{k,s_j} \subseteq \mathcal{P}_k$ be space spanned by the residues of all Eisenstein series at a pole $s = s_j$. Hence its dimension is bounded by the number of cusps for $\Gamma$ and we may construct an orthonormal basis of $\mathcal{P}_{k,s_j}$ by the Gram-Schmidt algorithm. Utilizing once more the Mass-Selberg formula for Eisenstein series we have the orthogonal decomposition

$$\mathcal{P}_k = \bigoplus_{\frac{1}{2} < s_j \leq 1} \mathcal{P}_{k,s_j}$$

out of which we assemble an orthonormal basis $\{v_j(z)\}$ for $\mathcal{P}_k$.

### 6.3 Decomposition of the continuous spectrum

We introduce the spaces $\mathcal{E}_{k,a}$ in theorem 6.1.2 and for this purpose we follow [Iwa02, section 7.1].

**Definition 6.3.1** Let $f \in C^\infty_c(\mathbb{R}_{>0}, \mathbb{C})$ and $a$ be a cusp. Then

$$(E_{k,a}f)(z) := \frac{1}{4\pi} \int_0^\infty f(t)E_a\left(z, \frac{1}{2} + it\right)dt$$

is called the Eisenstein transform\(^5\) of $f$. Here the subscript $k$ indicates that we take an Eisenstein series of weight $k$.

The next two lemmas show that the image of $E_{k,a}$ is contained in $\mathcal{H}_k$:

**Lemma 6.3.2** \(\forall \gamma \in \Gamma \forall f \in C^\infty_c(\mathbb{R}_{>0}, \mathbb{C}) : (E_{k,a}f)k\gamma = \nu(\gamma)E_{k,a}f\).

**Proof** $E_a(z,s)$ obeys the same transformation law with respect to $z$, see proposition 4.1.13. □

**Lemma 6.3.3** \(\forall f \in C^\infty_c(\mathbb{R}_{>0}, \mathbb{C}) : \|E_{k,a}f\| < \infty\)

**Proof** Recall remark 4.1.16:

$$E_a\left(\gamma \cdot \frac{1}{2} + it\right) = \delta_{ab}y^\frac{1}{2} + it + q_{ab}(s)y^\frac{1}{2} - it + O(e^{-\epsilon y}) \, , \text{ as } y \to \infty$$

\(^5\)Note that the letter “$E$” is not italic. This agrees with the preceding notation of an operator $L$. Nevertheless we print the operator with a bold letter to avoid confusion with the Eisenstein series $E$ while formulating the upcoming results.
and thus $E_a$ fails to be square integrable over $\mathcal{F}$. But we may integrate by parts in the definition of $E_{k,a}f$ without catching boundary terms due to compact support of $f$. We compute

$$\int y^{\frac{1}{2}+it} dt = \mp i \frac{y^{\frac{1}{2}+it}}{\log(y)}$$

proving

$$(E_{k,a}f)(\gamma_b z) \ll \frac{\sqrt{y}}{\log(y)} \quad (6.4)$$

as $y \to \infty$. The claim follows by

$$\int_\alpha^\infty \frac{1}{y \log(y)^2} dy = \frac{1}{\log(\alpha)} < \infty \quad (6.5)$$

if $\text{Re}(\alpha) > 1$ and $\text{Im}(\alpha) \neq 0$, in other words $E_{k,a}f$ is square integrable over cusp sectors of $\mathcal{F}$. Continuity of $f(t)E_a(z, \frac{1}{2} + it)$ implies square integrability over the compact part of $\mathcal{F}$.

**Definition 6.3.4** We denote the image of $E_{k,a}$ in $\mathcal{H}_k$ by $E_{k,a}$.

Those spaces are orthogonal to each other for distinct cusps:

**Proposition 6.3.5** Let $a, b$ be two cusps for $\Gamma$. Then

$$\forall f, g \in C^\infty_0(\mathbb{R}^\times, \mathbb{C}) : \langle E_{k,a}f, E_{k,b}g \rangle = \frac{\delta_{ab}}{2\pi} \int_0^\infty f(t)\overline{g(t)} dt$$

**Proof** The strategy is to utilize the Maass-Selberg formula for Eisenstein series, which was stated in corollary 4.2.5, and finally letting $Y \to \infty$. Hence we may fix $Y > 1$ and put

$$(E_{k,a}^Y)(z) := \frac{1}{4\pi} \int_0^\infty f(t)E_a^Y(z, \frac{1}{2} + it) dt$$

where $E_a^Y$ is the truncated Eisenstein series of weight $k$ defined in equation (4.8). According to equation 6.4 in the previous proof we infer

$$(E_{k,a}^Yf)(z) = (E_{k,a}f)(z) + O\left(\frac{\sqrt{y}}{\log(y)}\right)$$

and taking $\alpha = Y$ in equation 6.5 there yields

$$\left\| (E_{k,a} - E_{k,a}^Y)f \right\| \ll \frac{1}{\sqrt{\log(Y)}}$$

\footnote{in the usual sense, not as stipulated in chapter 2.}
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We collect those estimates for two cusps \(a, b\) via the Cauchy-Schwarz inequality:

\[
\langle E_{k,a} f, E_{k,b} g \rangle = \langle E_{k,a}^Y f, E_{k,b}^Y g \rangle + O\left( \frac{1}{\sqrt{\log(Y)}} \right) \\
= \frac{1}{(4\pi)^2} \int_0^\infty \int_0^\infty f(t)g(t') \left( E_{a}^Y \left( z, \frac{1}{2} + it \right), E_{b}^Y \left( z, \frac{1}{2} + it' \right) \right) \, dt \, dt' \\
+ O\left( \frac{1}{\sqrt{\log(Y)}} \right)
\]

Now we are in position to apply the Maass-Selberg formula for Eisenstein series. Recall that \(E_a(z,s)\) has no poles on the line \(\text{Re}(s) = \frac{1}{2}\) by corollary 4.2.10. Furthermore we emphasize that the involved factors \(v, j, \sigma_k\) to define general weight \(k\) Eisenstein-series have absolute value 1 and thus do not change the formula in the case of \(a = b\). More precisely since the formula computes \(\langle E_{a}^Y (\cdot, s_1), E_{b}^Y (\cdot, s_2) \rangle\) we need to check that \(E_{a}^Y (\cdot, s_2) = E_{b}^Y (\cdot, \overline{s_2})\). This holds up to factors involving \(v, j, \sigma_k\) by equation (1.3). But these factors do not change the formula in the case of \(a = b\), because then they multiply exactly with the ones from \(E_{a}^Y (\cdot, s_1)\) to 1. Hence we take \(s_1 = \frac{1}{2} + it\), \(s_2 = \frac{1}{2} - it'\) and apply Maass-Selberg formula from corollary 4.2.5. Adding and subtracting the term \(\frac{i}{t-t'} \delta_{ab} Y^{-i(t'-t)}\) there we obtain

\[
\langle E_{a}^Y \left( z, \frac{1}{2} + it \right), E_{b}^Y \left( z, \frac{1}{2} + it' \right) \rangle = \frac{i}{t-t'} \delta_{ab} (Y^{i(t'-t)} - Y^{i(t-t')}) + \Xi(t,t')
\]

where

\[
\Xi(t,t') := \frac{i}{t-t'} \Phi_{ab} \left( \frac{1}{2} + it \right) Y^{-i(t+t')} - \frac{i}{t-t'} \Phi_{ab} \left( \frac{1}{2} - it' \right) Y^{i(t+t')} + \frac{i}{t-t'} \left( \Phi_{a} \left( \frac{1}{2} + it \right) \Phi_{b} \left( \frac{1}{2} - it' \right) - \delta_{ab} \right) Y^{-i(t'-t)}
\]

collects the remaining terms. The important observation is that theorem 4.2.7 implies continuity of \(\Xi(t,t')\) in \((t,t') \in \mathbb{R}_{>0} \times \mathbb{R}_{>0}\). This enables us to recycle the argument from the previous lemma giving

\[
\langle E_{k,a}^Y f, E_{k,b}^Y g \rangle = \frac{\delta_{ab}}{(4\pi)^2} \int_0^\infty \int_0^\infty \frac{f(t)\overline{g(t')}}{i(t-t')} (Y^{i(t-t')} - Y^{i(t'-t)}) \, dt \, dt' + O\left( \frac{1}{\log(Y)} \right)
\]

To treat the innermost integral in \(t\), we write

\[
\frac{Y^{i(t-t')} - Y^{i(t'-t)}}{i(t-t')} = \frac{2}{(t-t')} \left( e^{i \log(Y)(t-t')} - e^{i \log(Y)(t'-t)} \right) = \frac{2}{t-t'} \sin \left( \log(Y)(t-t') \right)
\]

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and use that $t'$ is bounded below by compact support. Computing the Dirichlet integral

$$\int_{-\infty}^{\infty} \sin \left( \log (Y) \frac{r}{r} \right) dr = \text{sign}(\log (Y)) \pi Y > 1 \pi$$

we infer

$$2 \int_{0}^{\infty} f(t) \frac{\sin \left( \log (Y) (t - t') \right)}{t - t'} dt = 2\pi f(t') + O \left( \frac{1}{\log(Y)} \right)$$

The error term $O \left( \frac{1}{\log(Y)} \right)$ in the outer integral then can be estimated by partial integration. Finally we let $Y \to \infty$ and conclude by the claimed expression. □

Remark 6.3.6 (Aside) The previous result proves as well that

$$E_{k,a} : C^\infty_c(\mathbb{R}_{>0}, \mathbb{C}) \to \mathcal{E}_{k,a} \subseteq \mathcal{H}_k$$

is an isometry. If we extend $E_{k,a}$ to $L^2(\mathbb{R}_{>0})$ by mollification and endow $L^2(\mathbb{R}_{>0})$ with the inner product $\frac{1}{2\pi} \int_{0}^{\infty} f(t) \overline{g(t)} dt$ then we have an isometry $E_{k,a} : L^2(\mathbb{R}_{>0}) \to \mathcal{E}_{k,a} \subseteq \mathcal{H}_k$. This is a close analogue to the Plancherel theorem for the Fourier transform.

We continue with our decomposition of $\mathcal{E}$:

Proposition 6.3.7 The space $\mathcal{E}_{k,a}$ is orthogonal to $\mathcal{H}_{k,0}$.

Proof Let $f \in \mathcal{H}_{k,0}$ and $g \in C^\infty_c(\mathbb{R}_{>0}, \mathbb{C})$. It holds $\langle E_a(z | \psi), f \rangle = 0$ for every $\psi \in C^\infty_c(\mathbb{R}_{>0}, \mathbb{C})$ by corollary 4.3.5. Hence according to proposition 6.1.1

$$0 = \frac{1}{2 \pi i} \int_{c-i\infty}^{c+i\infty} (\mathcal{M} \psi)(-s) \langle E_a(z,s), f \rangle ds$$

and since this equation holds for every $\psi$ we obtain $\langle E_a(z,s), f \rangle = 0$. Thus

$$\langle E_{k,a} g, f \rangle = \frac{1}{4\pi} \int_{0}^{\infty} g(t) \left\langle E_a(z, \frac{1}{2} + it), f \right\rangle dt = 0$$

as desired. □

Proposition 6.3.8 The space $\mathcal{E}_{k,a}$ is orthogonal to $\mathcal{P}_k$.

Proof It suffices to prove that $\mathcal{E}_{k,a}$ is orthogonal to any residue of $E_b(z,s)$ at poles $s = s_j \in \left( \frac{1}{2}, 1 \right]$ in virtue of proposition 6.2.3. Observe that $-\tilde{\Lambda}_k$ acts
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on $\mathcal{E}_{k,a}$ through multiplication. Formally letting $\langle Mf \rangle(t) := \left( \frac{1}{4} + t^2 \right) f(t)$ we have the operator identity $-\tilde{\Delta}_k \mathcal{E}_{k,a} = \mathcal{E}_{k,a} M$ provided that

$$\frac{1}{4} + t^2 = \left( \frac{1}{2} + it \right) \left( \frac{1}{2} - it \right) = s(1 - s)$$

is an eigenvalue of $E_a$ (recall remark 2.2.3!). But in the case of $s = s_j \in (\frac{1}{2}, 1]$ this yields $0 \leq s_j(1 - s_j) < \frac{1}{4} \leq \frac{1}{4} + t^2$ for $t \in \mathbb{R}$, hence a pole of $E_b$ never gives an an eigenvalue of $E_a$ as desired (recall the definition of $P_k$).

**Remark 6.3.9** This proof determines the point spectrum arising from the residues of the Eisenstein series.

Furthermore we have established the orthogonal decomposition

$$P_k \bigoplus_a \mathcal{E}_{k,a} \subseteq \mathcal{E}_k$$

and it remains to prove that the left hand side is dense in $\mathcal{E}_k$, which we will do in section 6.6.

6.4 The Selberg / Harish-Chandra transformation

Throughout this section we assume without loss of generality that $k$ is a point-pair invariant kernel for $\Gamma = \{\pm 1\}$ (or equivalently for $\mathcal{F} = \mathcal{H}$) and weight 0. The proofs below can then be generalized directly to an arbitrary fundamental domain $\mathcal{F}$ and arbitrary weight $k$ by definition of $K$ in equation (5.2) and lemma 5.2.1. Recall also our motivational discussion previous to equation (3.11). Moreover we impose the regularity $k \in C^r_c(\mathbb{R}^+, \mathbb{C})$, so that the integrals below converge. This condition can be relaxed, compare for instance remark 5.2.7 or [Iwa02, p. 31].

We motivate the goal of this section with the following result:

**Proposition 6.4.1** Any eigenfunction of $\tilde{\Delta}_0$ is also an eigenfunction of all invariant integral operators. In other words, if $\left( \tilde{\Delta}_0 + \lambda I \right) f = 0$ and if $L$ is an invariant integral operator then there exists a constant $\Lambda_L(\lambda) \in \mathbb{C}$ independent from $f$ such that $Lf = \Lambda_L(\lambda) f$.

**Proof** We follow [Iwa02, pp. 28-30]. A function $F(z, w)$ is said to be radial at $w$ if $F(z, w) = F(d(z, w), w)$. For instance a point-pair invariant is radial at all points. The converse is true if $F(z, w)$ is an eigenfunction of $\tilde{\Delta}_0$ in $z$ for any $w$ with eigenvalue independent of $w$.

**Lemma 6.4.2** Let $\lambda = s(1 - s) \in \mathbb{C}$ and $w \in \mathcal{H}$. There exists a unique function $\omega(z, w)$ in $z$ with the following properties: $\omega$ is radial in $w$, $(\Delta_0 + \lambda I) \omega(z, w) = 0$ with respect to $z$, and $\omega$ satisfies the normalization condition $\omega(w, w) = 1$. 

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**Proof** The construction is essentially analogous to the construction of $g$ during the proof of proposition 3.3.1: One finds $\omega(z, w) = {}_2F_1(s, 1 - s, 1; d(z, w))$, where the normalization condition rules out the second linear independent solution to the hypergeometric differential equation. □

Now let $L$ be an invariant integral operator and $f$ be an eigenfunction of $-\Delta_0$ with eigenvalue $\lambda$. We seek to reuse lemma 6.4.2 with $\omega$ replaced by $f$. To this end we produce a radial function at $w \in \mathcal{H}$ out of $f$: Let $\Gamma_w = \{ \gamma \in \Gamma \mid \gamma w = w \}$ and $\varrho(\gamma)$ be the Haar-measure on $\Gamma_w$, which is unique by the normalization condition $\varrho(\Gamma_w) = 1$. Then set

$$f_w(z) := \int_{\Gamma_w} f(\gamma z) d\varrho(\gamma)$$

The mapping $f \mapsto f_w$ is called the mean-value operator. We check the conditions demanded by lemma 6.4.2:

$f_w(z)$ is radial at $w$: Let $z_1, z_2$ satisfy $d(z_1, w) = d(z_2, w)$. Thus $\exists \gamma_{12} \in \Gamma_w$ such that $\gamma z_1 = z_2$ and the claim follows by invariance of both $\varrho$ and $\Gamma_w$ under the substitution of $\gamma_{12}$. $f_w(z)$ is an eigenfunction of $-\Delta_0$ with eigenvalue $\lambda$, because $f$ is. Lastly we have $f_z(z) = f(z)$ by $\varrho(\Gamma_w) = 1$ and a suitable substitution again.

Thus lemma 6.4.2 yields $(f_w(w) = f(w))$

$$f_w(z) = \omega(z, w) f(w)$$

and the last step is to verify

$$(L f_z)(z) = \int_{\mathcal{H}} k(z, w) \int_{\Gamma_z} f(\gamma w) d\varrho(\gamma) d\mu(w)$$

$$= \int_{\Gamma_z} \int_{\mathcal{H}} k(z, w) f(\gamma w) d\mu(w) d\varrho(\gamma)$$

$$= \int_{\Gamma_z} \int_{\mathcal{H}} k(\gamma z, w) f(w) d\mu(w) d\varrho(\gamma)$$

$$= \int_{\Gamma_z} (L f)(\gamma z) d\varrho(\gamma) = (L f)(z)$$

where the last equation follows by definition of $\Gamma_z$ and $\varrho(\Gamma_z) = 1$. Consequently we put

$$\Lambda(z) := \int_{\mathcal{H}} k(z, w) \omega(z, w) d\mu(w)$$

and observe that this definition is constant in $z$, because $SL_2(\mathbb{R})$ acts transitively on $\mathcal{H}$ and both $k(z, w)$ and $\omega(z, w)$ are point-pair invariants. Summing up, we infer

$$(L f)(z) = (L f_z)(z) = f(z) \int_{\mathcal{H}} k(z, w) \omega(z, w) d\mu(w) = \Lambda f(z)$$

which completes the case $F = \mathcal{H}$. □
The converse statement is also true and of our main interest in this section:

**Proposition 6.4.3** If \( f \) is an eigenfunction of all invariant integral operators then \( f \) is an eigenfunction of \( \Delta_0 \).

We need another lemma for its proof:

**Lemma 6.4.4** Let \( k: \mathbb{H} \times \mathbb{H} \to \mathbb{C} \) be a smooth point-pair invariant. Then

\[
\Delta_0^{(z)} k(z, w) = \Delta_0^{(w)} k(z, w)
\]

where the superscript indicates the variable \( \Delta_0 \) acts onto.

**Proof (of the lemma)** Recall that a point-pair invariant function depends solely on the hyperbolic distance: \( k(z, w) = k(d(z, w)) \). Hence we compute

\[
\Delta_0^{(z)} d(z, w) \text{ and } \Delta_0^{(w)} d(z, w),
\]

which yields

\[
\text{Im}(w)^2 - \text{Im}(z)^2 + \frac{|z-w|^2}{2\text{Im}(z)\text{Im}(w)}
= \text{Im}(z)^2 - \text{Im}(w)^2 + \frac{|z-w|^2}{2\text{Im}(z)\text{Im}(w)}
\]

Hence establish the claim on the line \( \text{Im}(z) = \text{Im}(w) \) in \( \mathbb{H} \). Since \( d(z, w) \) is symmetric we reverse the roles of \( z \) and \( w \) to conclude. Else recall the equivalent property \( \forall \gamma \in \Gamma : k(\gamma z, \gamma w) = k(z, w) \) and \( \Gamma \)-invariance of \( \Delta_k \), see lemma 1.2.2.

**Proof (of proposition 6.4.3)** The idea is to apply \( \Delta_0 \) to \( (Lf)(z) = \Lambda_1 f(z) \) and to observe that \( -\Delta_0^{(z)} k(z, w) \) is another point-pair invariant. We integrate by parts

\[
(-L\Delta_0 f)(z) = \int_{\mathbb{H}} k(z, w)(\Delta_0^{(w)} f(w)) \, d\mu(w) = \int_{\mathbb{H}} (\Delta_0^{(w)} k(z, w)) f(w) \, d\mu(w)
\]

\[
= \int_{\mathbb{H}} (\Delta_0^{(z)} k(z, w)) f(w) \, d\mu(w) = (-\Delta_0 L f)(z) = \Lambda_1 (-\Delta_0 f)
\]

by assumption. Hence \( -\Delta_0 f(z) \) is an eigenfunction of \( L \). We also have

\[
\int_{\mathbb{H}} (-\Delta_0^{(z)} k)(z, w) f(w) \, d\mu(w) = \Lambda_2 f(z)
\]

It follows that \( f \) is an eigenfunction of \( \Delta_0 \) with eigenvalue \( \lambda := \frac{\Lambda_2}{\Lambda_1} \) (if \( \Lambda_1 = 0 \) for all \( k \) then \( f \equiv 0 \) and there is nothing to do.).

However, we wish to find explicit formulas for the map \( L \mapsto \Lambda_L \) or equivalently a transformation of the kernel \( k \) to a function depending on \( \lambda \). These are provided by the Selberg / Harish-Chandra transformation, which were briefly mentioned in remark 5.2.4.
Proposition 6.4.5 (Selberg / Harish-Chandra transformation) Let $f$ be an eigenfunction of $\Delta_0$ with eigenvalue $\lambda = s(1 - s)$, $s = \frac{1}{2} + it$. Define

$$u := d(z,w), \quad q(v) := \int_v^\infty \frac{k(u)}{\sqrt{u - v}} du$$

$$g(r) := 2q \left( \sinh \left( \frac{r}{2} \right)^2 \right), \quad h(t) := \int_{-\infty}^\infty g(r)e^{irt}dr$$

Then $\Lambda_L(\frac{1}{4} + t^2) = h(t)$ and $h$ is called the Selberg / Harish-Chandra transform of $k$.

Proof Recall that $\Lambda$ is independent from $f$, thus we may choose $f(z) = \text{Im}(z)^s$ and evaluate the equation $(Lf)(z) = \Lambda f(z)$ at $z = i$, which yields

$$\Lambda = 2 \int_0^\infty \int_0^\infty k \left( \frac{x^2 + (y - 1)^2}{4y} \right) y^s \frac{dx}{y^2}$$

Now substitute $x = 2\sqrt{y}y$ and next $y = e'$, the computation is done in [Kub73, theorem 5.3.1].

Remark 6.4.6 (Aside) Thus from an abstract point of view an invariant integral operator can be regarded as a function $h(t)$ depending only on the eigenvalue of $f$. Hence the Selberg / Harish-Chandra transformation provides an explicit function, which represents $L$ and depends only on $\Delta_0$ ("$L = h(\Delta_0)$").

Finally we are now able to prove remark 5.2.4 and thus the equality of the decompositions of $H_k$, as promised in the introduction of the present chapter.

Lemma 6.4.7 Let $k(z,w)$ be a point-pair invariant and $\tau(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ for $x \in \mathbb{R}$. Then

$$\int_{-\infty}^\infty k(z, \tau(x)w)dx = \sqrt{\text{Im}(z)\text{Im}(w)}g\left( \log (\text{Im}(z)) - \log (\text{Im}(w)) \right)$$

(6.7)

Proof Let $z = x + iy$, $y' = \text{Im}(w)$. We compute:

$$\int_{-\infty}^\infty k(z, \tau(x)w)dx = 2 \int_0^\infty k \left( \frac{x^2 + (y' - y)^2}{yy'} \right) dx$$

$$= \sqrt{yy'} \int_0^\infty k \left( \frac{(y' - y)^2}{yy'} + t \right) \frac{dt}{\sqrt{t}}$$

7 in the sense of operator calculus
8 via proposition 5.2.5
9 Kub73, theorem 5.3.2.
Now we shift the integral by \( (y - y')^2 = (\frac{y}{y'} - \frac{y'}{y})^2 =: w \) and then substitute \( w = e^u + e^{-u} - 2 \) getting

\[
\sqrt{yy'} \int_0^\infty k\left(\frac{(y - y')^2}{yy'} + t\right) \frac{dt}{\sqrt{t}} = \sqrt{yy'} \int_w^\infty \frac{k(t)}{\sqrt{t-w}} dt
\]

\[
= \sqrt{yy'} q(w) = \sqrt{yy'} g(u) = \sqrt{yy'} g(\log(y) - \log(y'))
\]

where the last step follows by definition of the hyperbolic distance. \( \square \)

**Remark 6.5.1** There is also the possibility of inverse transformation\(^{10}\):

\[
g(r) = \frac{1}{2\pi} \int_{-\infty}^\infty h(t)e^{irt} dt, \quad q(v) = \frac{1}{2} g(2 \log(\sqrt{v} + 1 + \sqrt{v}))
\]

\[
k(u) = -\frac{1}{\pi} \int_u^\infty \frac{dq(v)}{\sqrt{v-u}}, \quad u = d(z, w)
\]

### 6.5 Convergence of the continuous part

The goal of this section is to justify the shift of the integration contour in proposition 6.1.1 from a vertical line in the domain \( \text{Re}(s) > \frac{1}{2} \) to the vertical line \( \text{Re}(s) = \frac{1}{2} \). First we adapt our strategy from section 5.2, since we deal with a discrete part again, but which now comes from the residues of the Eisenstein series instead of the cusp forms.

**Proposition 6.5.1** Let \( k \in C_c^\infty(\mathbb{R}_{>0}) \) be a point-pair invariant and \( h(t) \) its Selberg/ Harish-Chandra transform. Define

\[
H(z, w) := \frac{1}{4\pi} \sum_a \int_{-\infty}^\infty h(t) E_a\left(z, \frac{1}{2} + it\right) E_a\left(w, \frac{1}{2} - it\right) dt
\]

and \( K \) by equation (5.2). Then \( \hat{K}(z, w) := K(z, w) - H(z, w) \) is bounded on \( \mathbb{F} \). Moreover if \( f \) is a cusp form then

\[
\int_{\mathbb{F}} H(z, w)f(w) d\mu(w) = 0
\]

**Remark 6.5.2** Let \( L, \hat{L}, \hat{\hat{L}} \) be the induced integral operators by the kernels \( K, H, \hat{K} \) respectively. Then in other words, the boundedness of \( \hat{K} \) implies that \( \hat{L} \) gives an explicit decomposition of the continuous spectrum of \( L \). (In similar fashion as \( \hat{L} \) decomposes the discrete spectrum.) This will be made precise in the upcoming section. According to theorem 5.1.1, the second assertion shows \( L(\mathcal{H}_{k,0}) = \hat{L}(\mathcal{H}_{k,0}) \). But be aware of the fact that \( \hat{L}, \hat{\hat{L}} \) fail to be invariant integral operators in the sense of equation (3.5), because they are not given by convolution (see remark 3.1.4). Hence we can not just apply proposition 5.2.6 directly to prove the proposition and need some preparatory work to do so.

\(^{10}\)under sufficiently strong regularity conditions, see [Iwa02, equation (1.63)].
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Nevertheless we obtain

**Corollary 6.5.3** Let $f \in \mathcal{E}_k$. Choose an orthonormal system $\{v_j\}_j \subseteq \mathcal{P}_k$. Then the discrete part $\sum_j \langle f, v_j \rangle v_j(z)$ in theorem 6.1.2 converges in the norm topology of $\mathcal{H}_k$.

**Proof** The previous proposition enables us to argue for $\hat{L}$ (induced by $\hat{K}$) in the same fashion as in section 5.3 for $L$. The Hilbert Schmidt theorem provides convergence of the discrete part in the norm topology. \[\square\]

We prove the proposition:

**Proof** We follow [Kub73, theorem 5.3.3] and prove the second assertion first. We have seen in the proof of proposition 1.5.2 that if $f$ is a cusp form then $\text{Im}(z)^s f(\gamma_a z)$ is decreasing as $\text{Im}(z) \to \infty$ for any $\kappa \in \mathbb{R}$ and for any cusp $a$. On the other hand $f$ has no constant terms in each Fourier expansion by definition, thus the general Mass-Selberg formula (proposition 4.2.4) implies $\langle E_a^\gamma (\cdot, s), f \rangle = 0$ for any $s$. Combining these observations and letting $Y \to \infty$ gives $\langle E_a(\cdot, s), f \rangle = 0$ for any $s$, which proves the second claim.

To verify the first assertion we consider two cases: First we let $z$ and $w$ tend to the same cusp $b$. Hence let $Y > 0$ and $y := \text{Im}(z) > Y$, $\tilde{y} := \text{Im}(w) > Y$. Then by definition of the truncated Eisenstein series in equation (4.8) we have

$$\sum_{a} E_a(\gamma_b z, s)E_a(\gamma_b w, \tilde{s}) = \sum_{a} \left( \delta_{ab} y^s + \varphi_{ab}(s) y^{1-s} + E_a^\gamma (\gamma_b z, s) \left( \delta_{ab} \tilde{y}^\tilde{s} + \varphi_{ab}(\tilde{s}) \tilde{y}^{1-\tilde{s}} + E_a^\tilde{\gamma} (\gamma_b w, \tilde{s}) \right) \right)$$

Since $E_a^\gamma$ decays exponentially in virtue of theorem 1.4.1, we may omit the terms involving one of the truncated Eisenstein series and it suffices to bound the the products of the constant terms. Using that $\Phi(s)$ is unitary on the line $s = \frac{1}{2} + it$ (see theorem 4.2.7) we need to examine

$$\int_{-\infty}^{\infty} h(t) \left( \sum_{a} \left( \delta_{ab} y^s + \varphi_{ab}(s) y^{1-s} \right) \left( \delta_{ab} \tilde{y}^\tilde{s} + \varphi_{ab}(\tilde{s}) \tilde{y}^{1-\tilde{s}} \right) \right) dt$$

$$= \int_{-\infty}^{\infty} h(t) \left( \sqrt{yy} \left( \left( \frac{y}{\tilde{y}} \right)^i t + \left( \frac{\tilde{y}}{y} \right)^{it} \right) + \varphi_{bb}(s)(yy)^{s} + \varphi_{bb}(\tilde{s})(\tilde{y}y)^{\tilde{s}} \right) dt$$

We regard $\int_{-\infty}^{\infty} h(t) \varphi_{bb}(s)(yy)^{i\pm it} dt$ as the Fourier transform of $h(t) \varphi_{bb}(s)$ at $\pm \log (yy)$ and note that $h(t) \varphi_{bb}(s)$ is of class $L^1$ due to compact support of $k$. Thus the Fourier transform of $h(t) \varphi_{bb}(s)$ decays to 0 as $|\log (yy)| \to \infty$, which is known as the Riemann-Lebesgue lemma. This argument bounds

$$\int_{-\infty}^{\infty} h(t) \varphi_{bb}(s)(yy)^{s} + h(t) \varphi_{bb}(\tilde{s})(\tilde{y}y)^{\tilde{s}} dt$$
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as \( y, \tilde{y} \to \infty \). Moreover utilizing the preceding section we obtain

\[
\frac{\sqrt{y\tilde{y}}}{4\pi} \int_{-\infty}^{\infty} h(t) \left( \left( \frac{y}{\tilde{y}} \right)^{it} + \left( \frac{\tilde{y}}{y} \right)^{it} \right) dt = \frac{\sqrt{y\tilde{y}}}{2\pi} \int_{-\infty}^{\infty} h(t)e^{it(\log(y)-\log(\tilde{y}))} dt
\]

and it remains to bound the terms of the form\(^{11}\)

\[
\sum_{\gamma \in \Gamma} k(z, \gamma w) - \int_{-\infty}^{\infty} k(z, \tau(x)w) dx
\]

But observe that we arrived in exactly the same situation as in the proof of proposition 5.2.6. where we defined

\( J_b(z, w) := \sum_{\gamma \in \Gamma} k(z, \gamma w) - \int_{-\infty}^{\infty} k(z, \gamma_b \tau(x) \gamma_b^{-1} w) dx \)

and bounded \( J_b(\gamma_b z, \gamma_b w) \) using the Euler-McLaurin formula. (Recall equation (1.4) and \( k(\gamma_b z, \gamma_b \tau(x) \gamma_b^{-1} \gamma_b w) = k(z, \tau(x)w) \).) This completes the case that \( z, w \) tend to the same cusp.

Finally suppose that \( z, w \) tend to two inequivalent cusps \( a, b \). But in this case there are no terms of the form \( \sqrt{y\tilde{y}} \int_{-\infty}^{\infty} h(t) \left( \left( \frac{y}{\tilde{y}} \right)^{it} + \left( \frac{\tilde{y}}{y} \right)^{it} \right) dt \), because \( \Phi(s) \) is unitary on the line \( \text{Re}(s) = \frac{1}{2} \). Thus we may conclude exactly by the same Fourier analysis argument as in the case of one cusp \( b \). \( \square \)

Working even more carefully produces the following bound:

**Lemma 6.5.4** Let \( \{ f_j \mid j \in \mathbb{N} \} \) be an orthogonal system for \( \mathcal{H}_{k,0} \oplus \mathcal{P}_k, z \in \mathbb{H}, T \geq 1 \). Define the invariant height\(^{12}\) of \( z \) by

\[
\text{ht}_\Gamma(z) := \max \left\{ \max \{ \text{Im}(\gamma a^{-1} \gamma z) \mid \gamma \in \Gamma \} \mid a \right\}
\]

Collect all points \( s_j = \frac{1}{2} + it \) and \( s = \frac{1}{2} + ir \) in the hyperbolic disc \( |s| \leq \frac{T}{2} \). Then we have

\[
\sum_{|t| < T} |f_j(z)|^2 + \sum_a \int_{-T}^T \left| E_a \left( z, \frac{1}{2} + ir \right) \right|^2 dr \ll T^2 + \text{ht}_\Gamma(z) T
\]

where the implied constant depends on the group \( \Gamma \) alone.

**Proof** Follow [Iwa02, section 7.2]. \( \square \)

\(^{11}\)Recall equation (5.2) and (3.36).

\(^{12}\)Iwa02, equation (2.42).
Observe that \( \text{ht}_F(\gamma a z) = \text{Im}(z) \) if \( z \) approaches the cusp \( a \). This enables us to utilize the Phragmen-Lindelöf principle in the following form:

**Proposition 6.5.5 (Phragmen-Lindelöf principle)** Let \( \delta_1, \delta_2 \in \mathbb{R}, \delta_1 < \delta_2 \) and \( \Omega := \{ s \in \mathbb{C} \mid \delta_1 \leq \text{Re}(s) \leq \delta_2 \} \). Let \( F \) be a holomorphic function on a domain containing \( \Omega \). Suppose we have \( |F(s)| \in \mathcal{O}\left(e^{\text{Im}(s)\epsilon}\right) \) uniformly on \( \Omega \) for some \( \epsilon > 0 \) and \( |F(s)| \in \mathcal{O}\left(|\text{Im}(s)|^\kappa\right) \) on \( \partial \Omega \) for some \( \kappa > 0 \). Then it holds \( |F(s)| \in \mathcal{O}\left(|\text{Im}(s)|^\kappa\right) \) uniformly on \( \Omega \).

**Proof** A full proof is given for instance in [Miy89, lemma 4.3.4].

The Phragmen-Lindelöf principle can be regarded as an extension of the well-known maximum modulus principle for holomorphic functions to unbounded domains (additionally requiring just a mild growth condition).

### 6.6 Proof of theorem 6.1.2

We have collected everything we need to turn our interest to the proof of our initial theorem 6.1.2, which provides a spectral decomposition of \( \mathcal{E}_k \). We begin with the decomposition of one incomplete Eisensteins series \( E_a(z|\psi) \). Recall proposition 6.1.1: If \( c = \text{Re}(s) > 1 \) then

\[
E_a(z|\psi) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} (M\psi)(-s)E_a(z,s)ds \tag{6.8}
\]

The first step is to apply the Phragmen-Lindelöf principle to \( \delta_1 = \frac{1}{2}, \delta_2 \geq 1 \) and \( F(z) := \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} (M\psi)(-s)E_a(z,s)ds \). We check the requirements: \( F \) is holomorphic, because \((M\psi)(-s)\) is entire and \( E_a(z,s) \) was meromorphically continued to the whole \( s \)-plane during section 4.2. \( F \) obeys the necessary bounds by the previous section, precisely it follows from lemma 6.5.4 that \( E_a(z,s) \) has at most polynomial growth in \( s \) on average. Henceforth we pass a finite number of simple poles \( s_j \) in the interval \((\frac{1}{2}, 1]\) with residues

\[
r_a(z,s_j) := \lim_{s \to s_j} (s - s_j)E_a(z,s)
\]

and obtain\(^\text{13}\)

\[
E_a(z|\psi) = \sum_{\frac{1}{2} < s_j < 1} \frac{\langle E_a(\cdot|\psi), r_a(\cdot, s_j) \rangle}{\|r_a(\cdot, s_j)\|^2} r_a(z, s_j) + \frac{1}{2\pi i} \int_{\frac{1}{2}+i\infty}^{1-i\infty} \frac{\langle E_a(\cdot|\psi), r_a(\cdot, s') \rangle}{\|r_a(\cdot, s')\|^2} E_a(z, s')ds' \tag{6.9}
\]

\(^\text{13}\)Iwa02, equation (7.12).
We multiply this equation by $E$ absolutely integrable over $\langle \cdot, \cdot \rangle$ with proposition 4.3.2, which computes $H$ hence we rearrange the integral the discrete part is fine: The space $P$ which yields

Theorem 4.2.9, hence

Finally we integrate this equation in $s$ on the line $\text{Re}(s) = \frac{1}{2}$ getting:

We obtain

Finally we integrate this equation in $s$ on the line $\text{Re}(s) = \frac{1}{2}$:

\[ \sum_b \frac{1}{4\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \langle E_a(\cdot, s), E_b(\cdot, s) \rangle E_b(z, s) ds \]

We multiply this equation by $E_b(z, s)$ and sum over all inequivalent cusps $b$, which yields

\[ \sum_b \langle E_a(\cdot, s), E_b(\cdot, s) \rangle E_b(z, s) = \sum_b E_b(z, s) \int_0^\infty (\delta_{ab}y^{1-s} + \varphi_{ab}(1-s)y^s) \psi(y) \frac{dy}{y^2} \]

Now we utilize the functional equation

from theorem 4.2.9, hence

We obtain

Finally we integrate this equation in $s$ on the line $\text{Re}(s) = \frac{1}{2}$:

\[ \sum_b \frac{1}{4\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \langle E_a(\cdot, s), E_b(\cdot, s) \rangle E_b(z, s) ds \]

\[ = \frac{1}{2\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{\langle E_a(\cdot, s), r_a(\cdot, s') \rangle}{\|r_a(\cdot, s')\|^2} E_a(z, s') ds' \]

\[ \sum_{b} \langle E_a(\cdot, s), E_b(\cdot, s) \rangle E_b(z, s) = E_a(z, s)(M\psi)(-s) + E_a(z, 1-s)(M\psi)(s-1) \]

\[ \text{We multiply this equation by } E_b(z, s) \text{ and sum over all inequivalent cusps } b, \text{ which yields} \]

\[ \sum_{b} \langle E_a(\cdot, s), E_b(\cdot, s) \rangle E_b(z, s) = \sum_{b} E_b(z, s) \int_0^\infty (\delta_{ab}y^{1-s} + \varphi_{ab}(1-s)y^s) \psi(y) \frac{dy}{y^2} \]

We obtain

\[ \sum_{b} \langle E_a(\cdot, s), E_b(\cdot, s) \rangle E_b(z, s) = E_a(z, s)(M\psi)(-s) + E_a(z, 1-s)(M\psi)(s-1) \]

Finally we integrate this equation in $s$ on the line $\text{Re}(s) = \frac{1}{2}$ getting:

\[ \sum_{b} \frac{1}{4\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \langle E_a(\cdot, s), E_b(\cdot, s) \rangle E_b(z, s) ds \]

\[ = \frac{1}{2\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{\langle E_a(\cdot, s), r_a(\cdot, s') \rangle}{\|r_a(\cdot, s')\|^2} E_a(z, s') ds' \]

\[ \text{Recall } E_{k,a} = E_{k,a}(C_{k}^{\infty}(R>0, C) \subseteq \mathcal{H}_k. \]

\[ \text{Iwa02, p. 102.} \]

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where we use proposition 6.1.1 and move the contour to Re(s) = \( \frac{1}{2} \) twice (as at the beginning of this section). Note that all \( E_\nu(z, \frac{1}{2} + it) \) are needed to decompose one \( E_a(z|\psi) \).

Lastly if \( f \in \mathcal{E}_k \) then \( f \) is a linear combination of some \( E_a(z|\psi) \) by definition of \( \mathcal{E}_k \). But the argument above clearly extends by linearity to \( f \). This proves theorem 6.1.2.

### 6.7 Summary and remarks

We complete this chapter with some comments on the theory developed in chapter 5 and 6, before we move to examples and applications of our spectral theory in the next chapter. Let us summarize first, what we have achieved:

**Theorem 6.7.1 (synopsis)** Let \( f \in \mathcal{H}_k \). We may assume \( f \in \mathcal{D}^{(2)}_k \subseteq \mathcal{D}_k \) by density, so that we are able to invoke our results from chapter 2, in particular the existence of the self-adjoint extension \( \tilde{-} \Delta_k \) of \( -\Delta_k \). We proved:

(a) The spectrum of \( \tilde{-} \Delta_k \) is contained in the interval \( \left[ \frac{|k|}{2} (1 - \frac{|k|}{2}), \infty \right) \). The point spectrum covers the full interval \( \left[ \frac{|k|}{2} (1 - \frac{|k|}{2}), \infty \right) \) and contains finitely many points \( 0 \leq \lambda_j = s_j(1 - s_j) < \frac{1}{4} \) arising from the residues of the Eisenstein series. The continuous spectrum is located in the interval \( \rho(k, \infty) \), where \( \rho(k) \) is defined in section 2.4.

(b) Assemble an orthonormal basis \( \{ u_j \mid j \in \mathbb{N} \} \) of \( \mathcal{H}_{k,0} \) consisting of cusp forms. Choose a complete set of inequivalent cusps \( \alpha_1, \ldots, \alpha_n \) for \( \Gamma \) and the Eisenstein series \( E_{\alpha_1}, \ldots, E_{\alpha_n} \) of weight \( k \) associated to them. Take an orthonormal basis \( v_1, \ldots, v_N \) of \( \mathcal{P}_k \) arising as the residues of \( E_{\alpha_1}, \ldots, E_{\alpha_n} \) (every \( E_{\alpha} \) has finitely many poles). Then we have the orthogonal expansion

\[
f(z) = \sum_{j=1}^{\infty} \langle f, u_j \rangle u_j(z) + \sum_{j=1}^{N} \langle f, v_j \rangle v_j(z) + \sum_{j=1}^{n} \frac{1}{4\pi} \int_{-\infty}^{\infty} \langle f, E_\alpha (\cdot, \frac{1}{2} + it) \rangle E_\alpha (z, \frac{1}{2} + it) dt
\]

which converges in the norm topology. If in addition \( \tilde{-} \Delta_k f, f \in \mathcal{D}_k^{(\infty)} \) then this expansion converges pointwise absolutely, and uniformly on compacta.

**Remark 6.7.2 (residual spectrum)** Let us consider once more the points

\[
0 \leq \lambda_j = s_j(1 - s_j) < \frac{1}{4}, \text{ where } \frac{1}{2} < s_j \leq 1 \text{ is a pole of the scattering}
\]
matrix $\Phi(s)$. Those points belong neither to the point spectrum nor to the continuous one. Hence, on the one hand, they are contained in the residual spectrum of $-\Delta_k$. On the other hand the points $\lambda_j$ are the eigenvalues to the residues of $E_{a_1}, \ldots, E_{a_n}$. Thus those points are called *residual eigenvalues*.\footnote{Less common is the terminology *exceptional eigenvalue*.}

**Remark 6.7.3** Theorem 6.1.2 implies [Roe67, Satz 12.3].

**Remark 6.7.4 (weight)** Observe that the weight $k$ is completely encoded in the choice of the Eisenstein series $E_a(z, s)$ and the cusp forms in $\mathcal{H}_{k,0}$. This fact has enabled us to generalize the results in this chapter developed for weight 0 by [Iwa02] and [Kub73]. We indicated explicitly how to treat the additional terms caused by weight $k$ except in section 6.4. The additional effort needed there to achieve general weight $k$ hides the involved ideas behind more complicated expressions. But we specified at the beginning of section 6.4, how one may reverse our reduction to weight 0 there.

**Remark 6.7.5 (multiplier, comparison to Roelcke, part one)** Recall our standing assumption from chapter 4 onwards that $\Gamma$ admits at least one singular cusp $a$, that is

$$\forall \gamma \in \gamma_a \Gamma a \gamma_a^{-1} : v(\gamma) = 1$$

One can think of this condition as $\mathbb{F}$ being “almost compact” in the sense that compactness is relaxed to “trivial acting cusps”. Moreover our definition of singularity of $a$ ensures compatibility with the corresponding notion by Roelcke:

Let $P_a := \gamma_a \tau(1) \gamma_a^{-1}$ and call $v$ regular with respect to $\Gamma$ if at each cusp $a$ the subspace $\{z \in \mathbb{C} | v(P_a)z = z\}$ has dimension zero.

(Compare [Roe66, p. 300])

**Claim:** $v$ is regular with respect to $\Gamma$ if and only if $\Gamma$ has no singular cusps.

Indeed, the space $\{z \in \mathbb{C} | v(P_a)z = z\}$ is a linear subspace of $\mathbb{C}$ and thus either equal to $\{0\}$ or to $\mathbb{C}$. (One may think of the examples $v \equiv 1$ and $v \equiv -1$.) Now observe that $P_a$ generates $\Gamma_a$ (together with $-1$, but this is no constraint due to the normalization of $v$.) and use the property $v(\gamma_1 \gamma_2) = v(\gamma_1) v(\gamma_2) v(\gamma_2)$.

In the literature singularity of $a$ is defined by the weaker property $v|_{\Gamma_a} \equiv 1$, for example in [Iwa97, p. 44].

**Remark 6.7.6 (Roadmap, comparison to Roelcke, part two)** The full spectral expansion from theorem 6.7.1 appears in [Roe66, Lemma 5.2] and in [Roe67, Satz 7.2]. The discrete part is specialized during [Roe67, §8], where Roelcke constructs a modified resolvent kernel $G^*_\lambda\kappa$ by hand achieving properties...
similar to our modification $\tilde{K}$ from chapter 5. The space of cusp forms and its orthogonal complement were briefly introduced in [Roe67, §9]. Instead of working with incomplete Eisenstein series, Roelcke collects all $E_a \left( z, \frac{1}{2} + it \right)$ and calls this system an eigenpacket. He defines this terminology on [Roe66, p. 321] and verifies his conditions in the case of $E_a \left( z, \frac{1}{2} + it \right)$ in [Roe67, Satz 12.1]. The full proofs are rather long and tedious.

**Remark 6.7.7 (prospect)** The theory for $\mathcal{H} = L^2(\Gamma \backslash \mathbb{H})$ generalizes to the space $L^2(\Gamma \backslash \text{SL}_2(\mathbb{R}))$ via the double coset decomposition $\Gamma \backslash \mathbb{H} = \text{SL}_2(\mathbb{R}) \backslash \Gamma / \text{SO}(2,\mathbb{R})$. A good reference is [Bum97, §2], his approach is based in addition on Lie theory and representation theory. An overview of his results is given in [Bum97, theorem 2.7.1].
7.1 Examples

Let us confess that we have given a sparse amount of examples so far, so let’s devote a section to some of them.

**Example 7.1.1 (empty point spectrum)** Let $\Gamma = \{ \pm 1 \}$, $k = 0$, $v = 1$. Then the spectrum of $-\Delta_0$ is purely continuous, the point spectrum is empty. There is no complete system of cusp forms in this case.

**Example 7.1.2 (empty continuous spectrum)** An element $\gamma \in \Gamma$ is called parabolic if $\gamma$ has infinite order and moves points along horocycles (circles in $\mathbb{H}$ tangent to $\mathbb{R} \cup \{\infty\}$) or equivalent $\text{tr}(\gamma) = 2$. We briefly met those motions in the proof of proposition 5.2.6.) Suppose $\Gamma$ has no parabolic elements. Then $\Gamma$ admits a compact fundamental domain, see [Iwa97, proposition 2.3]. In particular there are no cusps for $\Gamma$, since cusps are precisely the fixed points of parabolic motions of $\Gamma$. Hence the spectrum of $-\Delta_k$ is purely point spectrum, the continuous spectrum is empty. An example of such a group is $\Gamma = \text{SO}_2(\mathbb{Z})$.

**Example 7.1.3 (non-empty residual spectrum and volume of $F$)** Taking $k = 0$, $v = 1$ and using that $\mathcal{F}$ is assumed to admit finite hyperbolic volume we see that the constant functions belong to $\mathcal{H}_0$. Hence we may consider the spectral expansion of the constant function 1: Clearly $1 \in \mathcal{H}_0 \setminus \mathcal{H}_{0,0}$, which rules out any contribution to the spectral expansion of 1 from the cusp forms. Furthermore we do not have any continuous contribution, since the Eisenstein series are not square-integrable in $z$ (recall remark 4.1.16). Thus 1 is spectrally expanded in terms of one constant residue $v_0$ with eigenvalue 0, hence $s_j = 1$ and the residual spectrum is always non-empty.

In fact one can show that $v_0(z) = \mu(\mathcal{F})^{-\frac{1}{2}}$ and that the residue is given by $\mu(\mathcal{F})^{-1}$, which is the inverse of the hyperbolic volume of the fundamental domain, see [Iwa02, proposition 6.13] for a full proof.
Finally invoking proposition 2.2.9 a) drops the condition \( k = 0 \), enabling us to state [Roe67, p. 324] as secondary reference.

Moreover it holds:

**Proposition 7.1.4** If \( \Gamma \) is a congruence subgroup of \( SL_2(\mathbb{Z}) \) then the residual spectrum contains only the lowest possible eigenvalue \( \lambda = 0 \) corresponding the constant eigenfunctions.

**Proof** Follow the proof of [Iwa02, theorem 11.3]. \( \square \)

**Example 7.1.5 (The case \( \Gamma = SL_2(\mathbb{Z}) \))** Let \( \Gamma = SL_2(\mathbb{Z}), \ 0 < k \in 2\mathbb{Z}, \ v = 1 \). Then both point and continuous spectra of \( -\tilde{\Delta}_k \) are non-empty. On the one hand recall that the modular discriminant

\[
\Delta(z) = (2\pi)^{12} e^{2\pi i z} \prod_{n=1}^{\infty} \left( 1 - e^{2\pi i nz} \right)^{24}
\]

is a modular cusp form\(^1\) of weight 12. Hence \( \Delta(z) \frac{1}{e^{2\pi i z}} \) is a modular cusp form of weight \( k \) and \( y^k \Delta(z) \frac{1}{e^{2\pi i z}} \) is an automorphic cusp form of weight \( k \) in virtue of example 1.3.6. Thus \( \lambda = \frac{k}{2} \left( 1 - \frac{1}{2} \right) \) is contained in the point spectrum of \( -\tilde{\Delta}_k \). On the other hand there is the real analytic Eisenstein series

\[
E_\infty(z, s) = \sum_{(c,d) \in \mathbb{Z}^2 \atop \gcd(c,d)=1} \left( \frac{cz+d}{|cz+d|} \right)^{-k} \frac{\text{Im}(z)^s}{|cz+d|^{2s}}
\]

associated to the cusp \( a = \infty \), recall example 4.1.9 and definition 4.4. Since \( \infty \) is the only cusp, \( E_\infty(z, s) \) has just one pole, which is located at \( s = 1 \). The residue there is computed by the first Kronecker limit formula:

\[
\sum_{(0,0) \neq (c,d) \in \mathbb{Z}^2} \frac{\text{Im}(z)^s}{|cz+d|^{2s}} = \frac{\pi}{s-1} + 2\pi (C - \log(2) - \log(\sqrt{y} |\eta(z)|^2))
\]

\[
+ O(s-1)
\]

where \( C \) denotes the Euler-Mascheroni constant and \( \eta(z) = \frac{1}{\sqrt{\Delta(z)}} \) is the Dedekind \( \eta \)-function. A full proof of this formula can be found in [Sie, pp. 4-13]. Summing up, if we take \( f \in H_k \) to \( \Gamma = SL_2(\mathbb{Z}), \ 0 < k \in 2\mathbb{Z}, \ v = 1 \) then the spectral expansion from theorem 6.7.1 takes the form

\[
f(z) = \sum_{j=1}^{\infty} \langle f, u_j \rangle u_j(z) + \frac{3}{\pi} \langle f, 1 \rangle
\]

\[
+ \frac{1}{4\pi} \int_{-\infty}^{\infty} \left( f, E_\infty\left( \cdot, \frac{1}{2} \right) \right) E_\infty\left( z, \frac{1}{2} + it \right) dt
\]

\(^1\)Kob93, pp. 111,122.
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where \( \{ u_j \} \) serves as a complete orthonormal system of automorphic cusp forms to the same data. Note that \( \mu(\mathbb{F})^{-1} = \frac{3}{\pi} = \frac{\pi}{2\mathbb{Z}(2\pi)} \) at \( s = 1 \). (The factor \( \frac{1}{2\mathbb{Z}(2\pi)} \) is caused by the different normalization of \( E_{\infty} \), see example 4.1.9.)

**Example 7.1.6 (The case \( \Gamma = \Gamma(2) \))** Let \( \Gamma = \Gamma(2) \) as defined in equation 1.5. We sketch a construction of modular cusp forms for \( \Gamma(2) \) first. Consider

\[
\Theta_{00}(z) := \sum_{n \in \mathbb{Z}} e^{\pi i n^2 z} \quad \Theta_{01}(z) := \sum_{n \in \mathbb{Z}} (-1)^n e^{\pi i n^2 z} \quad \Theta_{10}(z) := \sum_{n \in \mathbb{Z}} e^{\pi i (n + \frac{1}{2})^2 z}
\]

1. \( \Theta_{00}^4, \Theta_{01}^4, \Theta_{10}^4 \) are modular forms of weight 2 for \( \Gamma(2) \): This can be derived from the fact that \( \sum_{n \in \mathbb{Z}} e^{2\pi i n^2 z} \) is a modular form of weight \( \frac{1}{2} \) for \( \Gamma_0(4) \supseteq \Gamma(2) \).

2. \( \Theta_{10}^4 \) has a simple zero at \( \infty \), \( \Theta_{01}^4 \) has a simple zero at 1, and \( \Theta_{01}^4 \) has a simple zero at 0: Insert those functions separately\(^2\) into the valence formula for \( \Gamma(2) \):

\[
\nu_\infty(f) + \nu_0(f) + \nu_1(f) + \sum_{p \in \Gamma(2) \setminus \mathbb{H}} \frac{\nu_p(f)}{|\Gamma(2)_p|} = \frac{k}{2}
\]

Here \( f \) is a non-zero modular form of even weight \( k \) for \( \Gamma(2) \) and \( \nu_p(f) \geq 0 \) denotes the order of a zero or the negative order of a pole\(^3\) of \( f \). In case of a cusp set \( \nu_0(f) = \nu_\infty(f|k\gamma_0) \) and \( \nu_1(f) = \nu_\infty(f|k\gamma_1) \) with the modular slash operator \( (f|k\gamma)(z) = (cz + d)^{-k}f(\gamma z) \). This formula evolves from the the valence formula for \( SL_2(\mathbb{Z}) \) ([Kob93, p. 115]) as follows: Recall that \( \Gamma(2) \) has index 6 in \( SL_2(\mathbb{Z}) \). Take a complete set of coset representatives\(^4\) \( \gamma_1, \ldots, \gamma_6 \) of \( \Gamma(2) \setminus SL_2(\mathbb{Z}) \) and define \( F(z) := \prod_{j=1}^6 (f|\gamma_j)(z) \) again using the modular slash-operator. By permutation of cosets \( F \) is a modular form of weight \( 6k \) for \( SL_2(\mathbb{Z}) \). Hence we insert \( F \) into the valence formula for \( SL_2(\mathbb{Z}) \), the key step is

\[
\nu_p(F) = \sum_{j=1}^6 \nu_p(f|k\gamma_j) = \sum_{j=1}^6 \nu_{|k\gamma_j} p(f) = \sum_{w \in \Gamma(2) \setminus SL_2(\mathbb{Z}) \setminus \Gamma(2)_p} \nu_w(f)
\]

where the sum runs over \( w \in \Gamma(2) \setminus SL_2(\mathbb{Z}) \). Observe that \( SL_2(\mathbb{Z})_w \) is independent of \( w \), hence choose \( w = p \) there and invoke the orbit-stabilizer theorem to conclude.

\(^2\)Inserting \( f := \Theta_{10}^4 + \Theta_{01}^4 - \Theta_{00}^4 \) yields the famous Jacobi identity: We have \( \nu_\infty(f) \geq 2 \), since \( f \in O(e^{2\pi i z^2}) \). Hence \( f \equiv 0 \) or \( f \) contradicts the valence formula.

\(^3\)thus the sum over \( p \) is finite, because \( f \) is holomorphic

\(^4\)By modularity of \( f \) we may take any orbit representative \( \gamma_j \)
3. In particular it follows by the valence formula for $\Gamma(2)$ that the space $M_k(\Gamma(2))$ of modular forms of even weight $k \geq 2$ for $\Gamma(2)$ has dimension $\frac{k}{2} + 1$ and that there are no cusp forms of weight\(^5\) 2 and 4. If $k \geq 6$ is even then $S_k(\Gamma(2)) = \Theta_{00}^4 \Theta_{01}^4 \Theta_{10}^4 M_{k-6}(\Gamma(2))$, where $S_k$ is the subspace of cusp forms in $M_k$. Thus we infer

$$\dim_{\mathbb{C}}(S_k(\Gamma(2))) + 3 = \dim_{\mathbb{C}}(M_k(\Gamma(2))) = \frac{k}{2} + 1$$

If we assume once more $0 < k \in 2\mathbb{Z}$ then we obtain automorphic cusp forms of weight $k$ for $\Gamma(2)$. For example consider the function

$$y^{\frac{k}{2}} (\Theta_{00}^4 \Theta_{01}^4 \Theta_{10}^4(z))^\frac{k}{8}$$

Conversely we have the three Eisenstein series $E_0, E_1, E_\infty$ giving non-zero continuous spectrum as well.

**Example 7.1.7 (location of poles, part 1)** Moreover we may locate poles of Eisenstein series in $\left(\frac{1}{2}, 1\right]$: Consider once more $\Theta_{00}(z)$ from the previous example, but now for the full theta-subgroup of $SL_2(\mathbb{Z})$, which is generated by $T^2 := \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. In this case we have $k = \frac{1}{2}$ and the multiplier system $v_\Theta$ defined by $v_\Theta(T^2) = e^{-i\pi/4}$, $v_\Theta(S) = 1$. Then $y^{\frac{1}{8}} \Theta_{00}(z)$ is an automorphic form with eigenvalue $\lambda = \frac{1}{8} \left(1 - \frac{1}{4}\right) = \frac{3}{16}$. We will see in the example on Hecke groups that $\infty$ is the only cusp for the full theta-subgroup, which is precisely denoted by $H(2)$ there. In particular $y^{\frac{1}{8}} \Theta_{00}(z)$ is not a cusp form according to the previous example. Summing up the Eisenstein series $E_\infty$ associated to those data has to have a simple pole at $\hat{s} = \frac{3}{4}$ (Recall from the proof of proposition 6.2.1 that a pole $\hat{s}$ of an Eisenstein series satisfies $\frac{1}{2} < \hat{s} \leq 1$ and $\lambda = \hat{s}(1 - \hat{s})$.

**Remark 7.1.8 (Selberg’s $\frac{3}{16}$-theorem)** In the case of a congruence subgroup of $SL_2(\mathbb{Z})$ Selberg proved in [Sel65] that any automorphic form for this subgroup has eigenvalue $\lambda \geq \frac{3}{16}$. He conjectured $\lambda \geq \frac{1}{4}$ or in other words that there are no residual eigenvalues in this case.

**Example 7.1.9 (The case of Hecke groups)** We give an example of our theory in the case of a non-congruence subgroup. Moreover we encounter automorphic forms of negative weight $k$.

Let $q \geq 1$ and $\Gamma = H(q) < SL_2(\mathbb{Z})$ be the Hecke group, that is $H(q)$ is

---

\(^5\)Recall that a cusp form has to vanish at every cusp $a \in \{0, 1, \infty\}$ at least of order $v_a(f) \geq 1$.\]
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spanned by the negative inversion \( \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \) and the translation \( \begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix} \) by \( q \). In this case

\[
\mathcal{F}_q := \{ z \in \mathbb{H} \mid |z| \geq 1, |\text{Re}(z)| \leq \frac{q}{2} \}
\]

is a fundamental domain for \( H(q) \). For instance one may adapt the proof for the case \( H(1) = \text{SL}_2(\mathbb{Z}) \) given in [Kob93, pp. 100-102]. Clearly \( H(q) \) is a Fuchsian group of the first kind.

Now suppose \( q > 2 \). Then there exist infinitely many linear independent modular forms for \( H(q) \) with even weight \( k \) and arbitrary unitary multiplier \( v^6 \). We briefly summarize their construction performed in [Hec36, §3]:

1. Let \( U := \{ z \in \mathbb{H} \mid |z| \geq 1, \frac{q}{2} < \text{Re}(z) < 0 \} \) be the left open half of \( \mathcal{F}_q \). According to the Riemann mapping theorem there exists a biholomorphic conformal mapping from \( U \) to \( \mathbb{H} \) (or equivalently to the open unit disk by the Cayley-transform). The Schwarz reflection principle\(^7\) provides an holomorphic extension to the lower half plane. Use a fractional linear transformation to normalize this map, such that \( \infty, i, -i \) get mapped to \( 1, 0, \infty \). Call the final map \( h \).

2. Due to bijectivity \( h \) has just one pole, which is simple and located at \( -i \). Hence \( h \) is holomorphic on \( \mathbb{H} \) and by construction bounded there as well\(^8\). Define \( H(z) := \sqrt{h(z)} \) and \( g(z) := \frac{h'(z)}{H(z)|H(z) - 1|} \). Then \( g \) is holomorphic and non-zero on \( \mathbb{H} \cup \{ i \infty \} \), since \( h'(z) \) restricted to \( \mathbb{H} \cup \{ i \infty \} \) vanishes only at \( i \) and at \( i \infty \). Indeed \( h \) has a \( q \)-expansion with first coefficient equal to 1. Summing up there exists \( \log (g(z)) \) and hence \( g(z) = \alpha \) for any \( \alpha > 0 \).

3. Check that \( f_n(z) := g(z)^\alpha (h(z) - 1)^n \) has all required properties for every integer \( n \geq 0 \):

\[
f_n(z + q) = f_n(z) \quad f_n \left( -\frac{1}{z} \right) = (-iz)^{2\alpha} f_n(z) \quad f_n(z) \in O(z^{-\alpha})
\]

The details are given in [Hec36, §3].

This construction provides:

- infinitely many linear independent modular forms\(^9\) of weight \( k = 2\alpha \) for \( \Gamma = H(q) \).

- We obtain in particular infinitely many linear independent automorphic forms \( y^k f_n(z) \) with eigenvalue \( \lambda = \frac{k}{2} \left( 1 - \frac{k}{2} \right) \) and for some multiplier \( v_k \) of absolute value 1.

---

\(^6\)Hec36, Satz 1.

\(^7\)Sal12, Satz 5.25.

\(^8\)\( H = \text{Re}(\frac{z}{i}) > 0 \)

\(^9\)Note that this does not contradict proposition 5.4.2.
• Moreover we have\textsuperscript{10} exponential decay of $h'$ as $y \to \infty$. Hence $f_n$ decays exponentially as $y \to \infty$ too, because $h(z) - 1$ has no constant term in its $q$-expansion. Thus if $k > 1$ then $\int_{\mathcal{F}} |f_n(z)|^2 y^k d\mu(z) < \infty$ and $y^k f_n(z) \in \mathcal{H}_k$ in this case.

• We may further construct automorphic forms of negative weight $k$:
Choose $\alpha = \frac{k}{2}$ and bound $|cz + d|^{-1} \leq \gamma^{-1} \frac{q}{\sqrt{2}}$ for any $\gamma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right)$ contained in $H(q)$, which is done in [Roe66, p. 326]. Then we are able to conclude as before that $y^k f_n(z)$ is an automorphic form with eigenvalue $\lambda = \frac{k}{2} (1 - \frac{k}{2})$ for every integer $n \geq 0$.

However letting $y \to 0$ we observe that $y^k f_n(z) \notin \mathcal{H}_k$. Alternatively this can be justified as follows: Calculate $L_k(y^k f_n(z)) = 0$ using the Cauchy-Riemann equations \ref{1.12} and then apply proposition 2.3.2.

### 7.2 First applications

An immediate consequences of theorem 6.7.1 is that Bessel’s inequality and Parseval’s identity continue to hold in our setting:

**Proposition 7.2.1** Let $g \in \mathcal{H}_k$ and \{\(f_j\)_j be an orthonormal sequence in $\mathcal{H}_{k,0} \oplus \mathcal{P}_k$. Then it holds

\[
\sum_j |\langle g, f_j \rangle|^2 + \sum_a \frac{1}{4\pi} \int_{-\infty}^\infty \left| \left\langle g, E_a \left( \cdot, \frac{1}{2} + it \right) \right\rangle \right|^2 dt \leq \|g\|^2
\]

Moreover, if the sequence is complete, then

\[
\|g\|^2 = \sum_j |\langle g, f_j \rangle|^2 + \sum_a \frac{1}{4\pi} \int_{-\infty}^\infty \left| \left\langle g, E_a \left( \cdot, \frac{1}{2} + it \right) \right\rangle \right|^2 dt
\]

Especially we are able to compute $\|g\|$ without using the definition of $\|\cdot\|$.

**Proof** Those results are achieved by means of orthogonality. Bessel’s inequality follows by

\[
0 \leq \left\| g - \sum_j \langle g, f_j \rangle f_j - \sum_a \frac{1}{4\pi} \int_{-\infty}^\infty \left\langle g, E_a \left( \cdot, \frac{1}{2} + it \right) \right\rangle E_a \left( \cdot, \frac{1}{2} + it \right) dt \right\|^2 = \|g\|^2 - \sum_j |\langle g, f_j \rangle|^2 - \sum_a \frac{1}{4\pi} \int_{-\infty}^\infty \left| \left\langle g, E_a \left( \cdot, \frac{1}{2} + it \right) \right\rangle \right|^2 dt
\]

In the case of completeness we insert the spectral expansion of $g$ to obtain Parseval’s identity. \hfill $\square$
7.2. First applications

The upcoming result collects the very last loose end, which is from chapter 2, where we promised an alternative proof of the following claim from proposition 2.1.6:

**Proposition 7.2.2** Let \( f \in D_{k}^{(2)} \) satisfy \( -\Delta_{k}^{(2)} f = \lambda f \) with \( \lambda \geq \frac{1}{4} \). Then \( f \) is a cusp form.

**Proof** We had shown there already that the condition \( -\Delta_{k}^{(2)} f = \lambda f \) (for an arbitrary eigenvalue) implies that \( f \) is an automorphic form. Clearly \( f \in \mathcal{H}_k \) and by corollary 4.3.5 we have either \( f \in \mathcal{H}_{k,0} \) or \( f \in \mathcal{E}_k \). Suppose \( f \in \mathcal{E}_k \). Hence \( f \in \mathcal{P}_k \) by definition 6.2.2 of \( \mathcal{P}_k \). But every residual eigenvalue is contained in the interval \( [0, \frac{1}{4}) \), which rules out \( f \in \mathcal{P}_k \). Thus \( f \in \mathcal{H}_{k,0} \) and this proves the claim according to remark 4.3.6. \( \square \)

The surplus of this section follows [Roe67, §13]. We determine the point spectrum corresponding to \( \mathcal{H}_{k,0} \) further:

**Proposition 7.2.3** Let \( k \geq 1 \) and \( f \) be an eigenfunction of \( -\tilde{\Delta}_k \) with eigenvalue \( \lambda = \frac{k}{2} \left( 1 - \frac{k}{2} \right) \). Then \( f \) is a cusp form.

**Proof** By assumption \( \lambda \leq \frac{1}{4} \). If \( k = 1 \) then \( \lambda = \frac{1}{4} \) and hence we may conclude by the same proof as for proposition 7.2.2. Suppose \( k > 1 \) giving \( \lambda < \frac{1}{4} \). Let \( f_a(y) \) denote the zeroth Fourier coefficient of \( f \) and write \( \lambda = s(1 - s) \). Thus by equation (1.25)

\[
f_a(y) = A_0 y^{\frac{k}{2}} + B_0 y^{1 - \frac{k}{2}}
\]

On the one hand the conditions \( f \in \mathcal{H}_k \) and \( k \geq 1 \) force \( A_0 = 0 \). On the other hand \( y^{\frac{k}{2}} f \) is holomorphic in virtue of proposition 2.3.1. Hence by the Cauchy-Riemann equations (1.12) it follows \( B_0 = 0 \) as desired. \( \square \)

Moreover we derive:

**Corollary 7.2.4** If \( k \geq 1 \) then for every cusp \( a \) the Eisenstein series \( E_a(z,s) \) associated to it is holomorphic in \( s = \frac{k}{2} \).

**Proof** If \( k = 1 \) then the claim follows according to proposition 4.2.10. Suppose \( k > 1 \) and that \( E_a(z,s) \) has a pole at \( s_0 = \frac{k}{2} \) with residue \( r_a(z) \). Then \( r_a(z) \) is an automorphic form by proposition 6.2.1 with eigenvalue \( \lambda = s_0(1 - s_0) \). But proposition 7.2.3 asserts that \( r_a(z) \) has to be a cusp form as well, hence orthogonality to those implies \( r_a(z) \equiv 0 \). \( \square \)

For the sake of completeness we also state:

**Proposition 7.2.5** Let \( k \leq -1 \) and \( f \) be an eigenfunction of \( -\tilde{\Delta}_k \) with eigenvalue \( \lambda = -\frac{k}{2} \left( 1 + \frac{k}{2} \right) \). Then \( f \) is a cusp form and for every cusp \( a \) the Eisenstein series \( E_a(z,s) \) associated to it is holomorphic in \( s = -\frac{k}{2} \).
Proof. Take complex conjugates in the previous two proofs. □

We briefly turn our interest to the case $k \in \mathbb{Z}$. Then there holds the following stronger result:

Proposition 7.2.6 (location of poles, part 2) If the weight $k$ is an odd integer then every Eisenstein series is holomorphic in $s$ on $\text{Re}(s) \geq \frac{1}{2}$.

Proof. Recall that every Eisenstein series $E_a(z, s)$ has no poles on the line $\text{Re}(s) = \frac{1}{2}$ by proposition 4.2.10. We proceed by induction and start with the case $k = 1$. Assume that we have a pole $s_j \in \left(\frac{1}{2}, 1\right]$ of an Eisenstein series $E_a(z, s)$ with residue $r_a(z)$. Then on the one hand $r_a(z)$ is an automorphic form according to proposition 6.2.1. Its eigenvalue $\lambda$ satisfies $\lambda < \frac{1}{4}$ as in the proof of proposition 6.3.8. But on the other hand $\lambda \geq \frac{1}{4}$ in virtue of proposition 2.4.3, which is a contradiction.

The induction step follows by application of $R_k$ or $L_k$ from section 2.2. We have seen in lemma 2.2.8 that the operators $R_k$ and $L_k$ shift the weight by 2 up and down respectively. □

Lastly we consider the special case $\lambda = 0$, which leads us to $k \in 2\mathbb{Z}$ as follows:

Proposition 7.2.7 Fix the eigenvalue $\lambda = 0$. Then it holds

(a) If $k \geq 4$ is an even integer then the operator product $R := R_{k-2}R_{k-4} \cdots R_2$ is an isomorphism from the space of automorphic forms of weight 2 and eigenvalue 0 to the space of automorphic forms of weight $k$ and eigenvalue 0.

(b) If $k \leq 4$ is an even integer then the operator product $L := L_{k-2}L_{k-4} \cdots L_2$ is an isomorphism from the space of automorphic forms of weight $-2$ and eigenvalue 0 to the space of automorphic forms of weight $k$ and eigenvalue 0.

(c) Let $k = 2$ and $f$ be an automorphic form of weight 2 and eigenvalue $\lambda = 0$. Then $f$ is an automorphic cusp form if and only if $y^{-1}f$ is a modular cusp form of weight 2.

(d) $k \equiv 0 \mod 2$

(e) If $k \neq 0$ then every eigenfunction of $-\tilde{\Delta}_k$ to $\lambda$ is a cusp form and the Eisenstein series are holomorphic in $s = 1$.

The picture is completed by corollary 2.3.4, which deals with the case $k = 0$ and $\lambda = 0$:

If $f$ satisfies the transformation law $f|_{0, \gamma} = f$ and is an eigenfunction of $-\Delta_0^{(2)}$ with eigenvalue $\lambda = 0$, then $f$ is constant.

We prove the proposition:
Proof (a) $R_k$ is obviously linear by its definition 2.2.7 and if $\lambda \neq -\frac{k}{2}(1 + \frac{k}{2})$ then $R_k$ is bijective with inverse $L_k$ according to lemma 2.2.8.

(b) Take complex conjugates in a)

(c) Apply proposition 2.3.1 and for the converse direction example 1.3.6.

(d) Suppose by contradiction $k \neq 0 \mod 2$ and let $f \neq 0$ be an eigenfunction to the eigenvalue $\lambda = 0$. Without loss of generality $k > 0$, else take complex conjugates. Let $l \in (0,2)$ be congruent to $k \mod 2$. Put

$$g := \begin{cases} f, & \text{if } l = k \\ L_l + L_{2l} + \cdots L_k f, & \text{else} \end{cases}$$

Hence by lemma 2.2.8 $g$ is an eigenfunction of weight $l$ and still to the eigenvalue 0. In virtue of proposition 2.2.9 c) we obtain

$$0 = \|L_l g\|^2 + \frac{l}{2} \left(1 - \frac{l}{2}\right) \|g\|^2$$

But since $l \in (0,2)$ we have $\frac{l}{2} \left(1 - \frac{l}{2}\right) > 0$ and $g \neq 0$ now implies the contradiction $\|L_l g\|^2 < 0$.

(e) It suffices to prove the first assertion, the second one follows by corollary 7.2.4. We have $k \neq 0$ and may assume $0 < k \in 2\mathbb{Z}$ by item d) and complex conjugation. We further reduce to the case $k = 2$ due to item a). But now we are in the situation of proposition 7.2.3, which gives the conclusion. \hfill \square

7.3 Poincaré series

Up to this section we just have established the existence of automorphic cusp forms via the Hilbert-Schmidt theorem and met those during some of our proofs. Thus we generalize the construction of Eisenstein series leading to another class of examples for automorphic cusp forms. Recall that the space of modular cusp forms to a cusp $a$ is spanned by the modular Poincaré series $P_1, P_2, P_3, \ldots$, see [Iwa97, corollary 3.5]. However note that the automorphic Poincaré series might not form a complete system of automorphic cusp forms, since those arise from their modular counterparts (compare also remark 1.3.7). But as a byproduct, this approach yields the “modular variant” of the Eisenstein series, compare remark 7.3.6.

Define

$$j_{\gamma}(z;k) := (cz + d)^k$$

for $\gamma = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma$
which is the “modular factor” from definition 1.3.4. Let \( a \) be a cusp of \( \Gamma \) with scaling matrix \( \gamma_a \) and stabilizer subgroup \( \Gamma_a \). Consider a 1-periodic, bounded, and holomorphic function \( p: \mathbb{H} \to \mathbb{C} \). We will see that the function

\[
P: \mathbb{H} \to \mathbb{C} \\
z \mapsto \sum_{\gamma \in \Gamma_a \backslash \Gamma} v(\gamma)^{-1} \sigma_k(\gamma_a^{-1}, \gamma) \gamma^{-1} j_{\gamma_a^{-1}} \gamma(z; k)^{-1} p(\gamma_a^{-1} \gamma z)
\]

(7.1)

is a modular form for \( \Gamma \) of weight \( k > 2 \) (Replace \( \text{SL}_2(\mathbb{Z}) \) with \( \Gamma \) in definition 1.3.4). To this end we seek absolute convergence of the series (7.1): By boundedness of \( p \) and equation (3.36) \( P \) is majorized by

\[
\sum_{\gamma \in \Gamma_a \backslash \Gamma} |J_{\gamma_a^{-1}} \gamma(z; k)^{-1}| \leq \Im(z)^k \sum_{\gamma \in \Gamma_a \backslash \Gamma} \Im(\gamma_a^{-1} \gamma z)^{\frac{k}{2}}
\]

which is an Eisenstein series with \( s = \frac{k}{2} \). Thus if \( k > 2 \) then we have absolute convergence according to proposition 4.1.5. Moreover it follows that \( P \) is holomorphic on \( \mathbb{H} \), since \( p \) is holomorphic and 1-periodic\(^{11}\).

The crucial part is to deal with the growth condition from definition 1.3.2 and this is verified via the Fourier expansion. Simplifying its computation we specialize to the case \( p_m(z) = e^{2\pi imz} \) for \( m \in \mathbb{Z}_{\geq 0} \). Indeed \( p_m \) is clearly holomorphic and 1-periodic. Furthermore \( p_m(z) \) is bounded on \( \mathbb{H} \) due to

\[
|p_m(z)| = e^{-2\pi m \Im(z)} \sum_{m \Im(z) \geq 0} \in (0, 1]
\]

(7.2)

Recall that a cusp form is required to decay exponentially at all cusps. Hence, roughly speaking, this equation indicates that we will obtain cusp forms in the case \( m \geq 1 \), which is our first goal. But let us establish the basic setting, before we are able to turn our interest to this observation again.

This discussion motivates the following definition:

**Definition 7.3.1** Let \( a \) be a cusp of \( \Gamma \) with scaling matrix \( \gamma_a \) and stabilizer subgroup \( \Gamma_a \). Let \( k \in \mathbb{R}_{\geq 2} \) and \( m \in \mathbb{Z}_{\geq 0} \). Then the associated Poincaré series to a of weight \( k \) is

\[
P_m: \mathbb{H} \to \mathbb{C} \\
z \mapsto \Im(z)^{\frac{k}{2}} P(z) \\
= \Im(z)^{\frac{k}{2}} \sum_{\gamma \in \Gamma_a \backslash \Gamma} v(\gamma)^{-1} \sigma_k(\gamma_a^{-1}, \gamma)^{-1} j_{\gamma_a^{-1}} \gamma(z; k)^{-1} e^{2\pi im\gamma_a^{-1} \gamma z}
\]

where we omit the dependency of \( P_m \) on the cusp \( a \) for notational convenience.

\(^{11}\)Recall that a Fourier series converges absolutely and locally uniformly if the sum over all Fourier coefficients converges absolutely.
Following our exposition of the Eisenstein series we summarize:

**Proposition 7.3.2** \( P_m(z) \) enjoys the following properties:

(a) If \( a \) is a singular cusp with respect to the unitary multiplier \( v \) then \( P_m(z) \) is well-defined, i.e. the summands do not depend on the choice of the coset representative \( \gamma \).

(b) If \( k > 2 \) then \( P_m(z) \) converges absolutely on \( \mathbb{H} \).

(c) \( P_m(z) \) satisfies the transformation law \( (P_m|k\tilde{\gamma})(z) = v(\tilde{\gamma})P_m(z) \) for every \( \tilde{\gamma} \in \Gamma \).

(d) \( P_m(z) \) is an eigenfunction of \(-\Delta_k\) to the eigenvalue \( \lambda = \frac{k}{2}(1 - \frac{k}{2}) \).

**Remark 7.3.3** The modular factor \( \tilde{j}_\gamma(z;k) \) has the same\(^{12}\) chain-rule property (equation (1.18)) as our automorphy factor \( j_\gamma(z;k) \). This fact enables us to reuse our argument from proposition 4.1.13 (and lemma 4.1.12) in the proof below.

**Proof** (a) Since \( e^{2\pi iz} \) is 1-periodic and \( \Gamma_\infty \) contains the integral translations in \( \Gamma \), we may argue exactly as in lemma 4.1.4 and lemma 4.1.12.

(b) The condition \( k > 2 \) is caused by the different automorphy factor \( \tilde{j}_{\gamma^{-1}}(z;k) \), which we discussed above.

(c) Arguing in the same fashion as in proposition 4.1.13 we see that \( P \) satisfies the modular transformation law

\[
P(\gamma z) = v(\gamma)\tilde{j}_\gamma(z;k)P(z)
\]

Another reference is [Iwa97, proposition 3.1]\(^{13}\). Hence we apply example 1.3.6 and obtain the desired automorphic transformation law.

(d) This is guaranteed by the fact that (7.1) defines a holomorphic function. Hence \( P \) satisfies the Cauchy-Riemann equations (1.12) and we deduce exactly as in example 1.3.6 that \( y^{\frac{k}{2}}P(z) \) is an eigenfunction of \(-\Delta_k\) to the eigenvalue \( \lambda = \frac{k}{2}(1 - \frac{k}{2}) \).

As briefly mentioned in our preliminary discussion we need to compute the Fourier expansion of \( P(z) \) from equation (7.1) to verify the automorphic growth condition for \( P_m \). Recall that we seek a \( q \)-expansion\(^{14}\) for \( P \) in the holomorphic case according to remark 1.4.2.

**Theorem 7.3.4** Fix a cusp \( a \) of \( \Gamma \) with scaling matrix \( \gamma_a \) and stabilizer subgroup \( \Gamma_a \) and define \( P \) by (7.1) with \( p(z) = e^{2\pi imz} \) for \( m \in \mathbb{Z}_{\geq 0} \). Let \( b \) be another cusp

\(^{12}\) Compare [Iwa97, equation (2.49)].

\(^{13}\) The slash-operator there is defined via \( \tilde{j}_\gamma(z;k) \) instead of \( j_\gamma(z;k) \).

\(^{14}\) That is an expansion with respect to non-negative powers of \( q = e^{2\pi iz} \).
with scaling matrix $\gamma_b$ and recall the Kloosterman sums from theorem 4.1.14:

$$S_{ab}(m, n; c) = \sum_{\gamma} p(\gamma)^{-1} e^{2\pi i \frac{am + bn}{c}}$$

where the sum runs over $\gamma = \left( \begin{array}{cc} a & * \\ c & d \end{array} \right) \in \Gamma_\infty \backslash \Gamma_b / \Gamma_\infty$.

(a) If $m = 0$ then we have

$$\tilde{f}_{\gamma_b^{-1}}(z; k) P(\gamma_b z) = \delta_{ab} + \sum_{n=1}^{\infty} \eta_{ab}(n) e^{2\pi i nz}$$

where

$$\eta_{ab}(n) = \left( \frac{2\pi}{i} \right)^{k \! - \! 1} \frac{n^{k-1}}{\Gamma(k)} \sum_{c>0} c^{-k} S_{ab}(0, n; c)$$

and $\Gamma(k)$ denotes the Gamma-function.

(b) If $m \geq 1$ then we have

$$\tilde{f}_{\gamma_b^{-1}}(z; k) P(\gamma_b z) = \sum_{n=1}^{\infty} \xi_{ab}(m, n) e^{2\pi i nz}$$

where

$$\xi_{ab}(m, n) = \left( \frac{n}{m} \right)^{k-1} \left( \delta_{mn} \delta_{ab} + 2\pi i^{k-1} \sum_{c>0} c^{-1} S_{ab}(m, n; c) J_{k-1} \left( \frac{4\pi \sqrt{mn}}{c} \right) \right)$$

and $J_l(x)$ denotes the Bessel-function of order $l$.

**Proof** The computation fills [Iwa97, section 3.2] completely and thus is omitted here. $\square$

**Corollary 7.3.5**

(a) $P$ satisfies the modular growth condition (remark 1.3.5) and hence is a modular form.

(b) For every $m \geq 0$ and $k > 2$, $P_m(z)$ satisfies the automorphic growth condition (definition 1.3.2) and hence is an automorphic form.

**Proof** (a) Observe that both Fourier expansions admit no negative terms. Hence both expansions are holomorphic at $e^{2\pi i z} = q = 0$.

(b) Use example 1.3.6. Alternatively analyze the growth of the Fourier coefficients via the Weil bound

$$|S(m, n; c)| \leq d(c) \sqrt{\gcd(m, n, c) / c}$$
for Kloosterman sums\textsuperscript{15} (\(d(c)\) denotes the number of positive divisors of \(c\)), which we briefly mentioned in remark 4.1.16. The Bessel function can be bounded as follows\textsuperscript{16}: 

\[
J_l(x) \ll \min \left\{ x^{-\frac{1}{2}}, x^\delta \right\} \leq x^\delta \text{ for } -\frac{1}{2} \leq \delta \leq l
\]

Then apply theorem 1.4.1. \(\square\)

Remark 7.3.6 Reducing to the case \(m = 0\) and dropping the factor \(\text{Im}(z)^{\frac{1}{2}}\) yields\textsuperscript{17} holomorphic Eisenstein series of weight \(k > 2\). An explicit outline for the case \(\Gamma = \text{SL}_2(\mathbb{Z})\) and with trivial multiplier \(\nu = 1\) is given in [DS05, exercise 1.1.4]. Then generalize in virtue of equation (1.4), (3.36) and example 4.1.9. Additionally those Eisenstein series obey the modular transformation law by the proof of proposition 7.3.2 c). The modular growth condition from definition 1.3.4 is satisfied as well according to the previous corollary.

The case \(m \geq 1\) is of our main interest in this section and we are now in position to verify that we have indeed constructed automorphic cusp forms.

Corollary 7.3.7 If \(m \geq 1\) then \(P_m\) is an automorphic cusp form.

Proof Theorem 7.3.4 shows that if \(m \geq 1\) then \(P(z)\) from (7.1) has a vanishing zeroth Fourier coefficient at every cusp. In other words \(P(z)\) defines a modular cusp form. Hence \(P_m\) has vanishing zeroth Fourier coefficient at every cusp too and we apply proposition 1.5.2 to conclude. \(\square\)

Lastly we invoke our Hilbert-space theory:

Proposition 7.3.8\textsuperscript{18} Let \(m \geq 1\), \(a\) be a cusp and \(P_m\) be the associated Poincaré series. Let \(f\) be a modular form to the same data \(\Gamma, \nu, k\). Let \(g\) be the corresponding automorphic form to those data, explicitly given by \(g(z) := y^\frac{1}{2} f(z)\). Then the orthogonal projection of \(P_m\) onto \(g\) is given by:

\[
\langle g, P_m \rangle = \frac{\Gamma(k - 1)}{(4\pi m)^{k-1}} f_a(m)
\]

where \(f_a(m)\) is given by

\[
\sum_{n=0}^{\infty} f_a(n) e^{2\pi inz}
\]

In other words \(f_a(m)\) is the \(m\)-th Fourier coefficient in the \(q\)-expansion of \(f\) with respect to the cusp \(a\):

\textsuperscript{15}IK04, p. 280.
\textsuperscript{16}Iwa97, equation (5.16).
\textsuperscript{17}IK04, p. 357.
\textsuperscript{18}Iwa97, theorem 3.3.
Proof We adapt the proof of proposition 4.3.2 and argue via the unfolding technique: Without loss of generality we reduce to the case \( a = \infty \) giving \( \gamma_a = 1 \). The general case follows by conjugating the group \( \Gamma \) (which we did for instance in chapter 4). Recall that the strip \( \{0 \leq \text{Re}(z) \leq 1\} \subseteq \mathbb{H} \) is a fundamental domain for \( \Gamma_\infty \). We compute

\[
\langle g, P_m \rangle = \int_{\mathbb{F}} f(z) \frac{\overline{P(z)} y^k}{y} d\mu(z)
\]

\[
= \int_{\mathbb{F}} f(z) \left( \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \overline{v(\gamma)^{-1} j^\gamma(z; k)}^{-1} e^{2\pi i m y z} \right) y^k d\mu(z)
\]

\[
\overset{(*)}{=} \int_{\mathbb{F}} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} f(\gamma z) e^{2\pi i m y z} \text{Im}(\gamma z)^k d\mu(z)
\]

\[
= \int_0^\infty y^k \int_0^1 f(z) e^{2\pi i (-m z)} dx \frac{dy}{y^2}
\]

\[
\overset{(**)}{=} f_\infty(m) \int_0^\infty y^{k-2} e^{-4\pi m y} dy = f_\infty(m) \frac{\Gamma(k-1)}{(4\pi m)^{k-1}}
\]

where \((*)\) holds due to the modular transformation law for \( f \) and equation (1.3). Furthermore we inserted the \( q \)-expansion of \( f \) at \( \infty \) in step \((**)\) and used orthogonality of the functions \( e^{2\pi i l z} \). \( \square \)

Corollary 7.3.9 \( \forall m \geq 1 : \|P_m\| < \infty \)

Proof Insert \( g = P_m \) and its \( q \)-expansion from theorem 7.3.4 into proposition 7.3.8. Alternatively use that if \( m \geq 1 \) then \( P_m \) is an automorphic cusp form and hence decays exponentially. \( \square \)

7.4 Spectral expansion of \( G_{\lambda,k} \)

Recall our main result from chapter 3: We constructed a kernel function \( G_{\lambda,k} : \mathbb{F} \times \mathbb{F} \to \mathbb{C} \) such that if \( \lambda < \min \{ 0, \frac{|k|}{2}, (1 - \frac{|k|}{2}) \} \) then

\[
\forall k \in \mathbb{R} \forall f \in \mathcal{H}_k : (\tilde{\Delta}_k - \lambda \mathbb{I}) \int_{\mathbb{F}} G_{\lambda,k}(z, w) f(w) d\mu(w) = f(z)
\]

In other words if \( L \) denotes the invariant integral operator defined by \( G_{\lambda,k} \) then \( L = (\tilde{\Delta}_k - \lambda \mathbb{I})^{-1} \).

Let us briefly turn our interests towards a spectral expansion of a general kernel \( K : \mathbb{F} \times \mathbb{F} \to \mathbb{C} \) given by equation (5.2):

Proposition 7.4.1 \( ^{19} \) Let \( w \in \mathbb{H} \) and \( k : \mathbb{H} \times \mathbb{H} \to \mathbb{C} \) be a point-pair invariant free space kernel, which is smooth and compactly supported. Let \( h(t) \) be the Selberg

\(^{19}\text{Iwa02, theorem 7.4.} \)
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/ Harish-Chandra transform of $k$ from proposition 6.4.5. Choose an orthonormal system $\{f_j\}_j$ for $\mathcal{H}_{k,0} \oplus \mathcal{P}_k$. Then we have

$$K(z,w) = \sum_j h(t_j)f_j(w)f_j(z) + \sum_a \frac{1}{4\pi} \int_{-\infty}^{\infty} h(t)E_a(w, \frac{1}{2} + it)E_a(z, \frac{1}{2} + it)dt$$

We emphasize again that it suffices to impose the regularity conditions $[\text{Iwa02, equation (1.63)]}$ on $h$. (Compare also our discussion at the beginning of section 6.4.) We prove the proposition:

**Proof** We compute the orthogonal projections

$$\langle K(\cdot, w), f_j \rangle_{5\text{.1}} \overset{\text{lemma }6.4.5}{=} \int_{\mathcal{H}} k(z,w)f_j(z)d\mu(z) \overset{\text{prop. }6.4.5}{=} h(t_j)f_j(w)$$

where $s_j = \frac{1}{2} + it_j$, and exactly in the same fashion

$$\left\langle K(\cdot, w), E_b(\cdot, \frac{1}{2} + it) \right\rangle = h(t)E_b(w, \frac{1}{2} + it)$$

We insert $z \mapsto K(z,w)$ into theorem 6.7.1 and conclude. □

However we are not able to apply this result directly to the pair

$$K(z,w) = \mathcal{G}_{\lambda,k}(z,w), \quad k(z,w) = g_{\lambda,k}(z,w)$$

because of their singularity on the diagonal $z = w$. To recover the regularity conditions stated in $[\text{Iwa02, equation (1.63)]}$ we rese an idea from the proof of theorem 5.1.1: In order to annihilate the singularity consider the difference

$$k(z,w) := g_{s(1-s),k}(z,w) - g_{a(1-a),k}(z,w)$$

for $\text{Re}(s) > 1$ and $a > 1$. We compute the projections of

$$K(z,w) := \mathcal{G}_{s(1-s),k}(z,w) - \mathcal{G}_{a(1-a),k}(z,w)$$

onto the eigenfunctions $f_j$:

$$\langle K(z, \cdot), f_j \rangle = \int_{\mathcal{H}} (\mathcal{G}_{s(1-s),k}(z,w) - \mathcal{G}_{a(1-a),k}(z,w))f_j(w)d\mu(w)$$

$$= \left( (\Delta_k + s(1-s))^{-1} - (\Delta_k + a(1-a))^{-1} \right)f_j(z)$$

$$= \left( (s-s_j)^{-1}(1-s-s_j)^{-1} - (a-s_j)^{-1}(1-a-s_j)^{-1} \right)f_j(z)$$

$$=: \lambda_{a,s}(s_j)$$

(7.3)

Performing exactly the same computation with $f_j$ replaced by $E_a(z,s)$ we obtain$^{20}$

$^{20}$[Iwa02, theorem 7.5.]

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Proposition 7.4.2 Let \( \text{Re}(s) > 1 \) and \( a > 1 \). Then
\[
\mathcal{G}_{s(1-s),k}(z,w) - \mathcal{G}_{a(1-a),k}(z,w) = \sum_{j} \chi_{s,a}(s_j) f_j(z) \overline{f_j(w)}
+ \sum_{s} \frac{1}{4\pi i} \int_{\text{Re}(v) = \frac{1}{2}} \chi_{s,a}(v) E_a(z,v) \overline{E_a(w,v)} dv
\]

We wish to drop the assumption \( \text{Re}(s) > 1 \). In a first step, utilizing analytic continuation of the Eisenstein series, we may extend proposition 7.4.2 to \( \text{Re}(s) > \frac{1}{2} \) without any modification. The second step is an extension to the symmetry line \( s = \frac{1}{2} + it \). Rewriting our expansion in proposition 7.4.2 we reduce to the following lemma:

Lemma 7.4.3 The function
\[
\hat{\mathcal{G}}_{s(1-s),k}(z,w) := \mathcal{G}_{s(1-s),k}(z,w) - \chi_{\frac{1}{2} + it,s}(0) \sum_{j} f_j(z) \overline{f_j(w)}
\]
has a holomorphic extension to the line \( s = \frac{1}{2} + it \).

Proof Denote by \( \Lambda \) the eigenspace of \( -\hat{A}_k \) to the eigenvalue \( \lambda = s(1-s) = \frac{1}{4} + t^2 \) and by \( \Lambda^\perp \) its orthogonal complement in \( \mathcal{H}_k \). Recall that the resolvent of \( -\hat{A}_k \) is holomorphic in the parameter \( \lambda \) outside the spectrum\(^{21}\). Thus \( \hat{\mathcal{G}}_{s(1-s),k}(z,w) \) has no poles on \( \Lambda^\perp \) and it suffices to show:

Claim: \( \hat{\mathcal{G}}_{s(1-s),k} \) represents \( \mathcal{G}_{s(1-s),k}(z,w) \) on \( \Lambda^\perp \) and vanishes identically on \( \Lambda \).

Proof: If \( g \in \Lambda^\perp \) then \( \forall f_j \in \Lambda : \langle g, f_j \rangle = 0 \) by definition. Hence
\[
\langle g, \sum_{j} f_j(\cdot) \overline{f_j(w)} \rangle = 0
\]

\[
\left( \hat{A}_k - s(1-s) \right) \mathcal{G}_{s(1-s),k}(z,w) g(w) d\mu(w) = \left( \hat{A}_k - s(1-s) \right) \langle \mathcal{G}_{\lambda,k}(z,\cdot), g \rangle = \left( \hat{A}_k - s(1-s) \right) \langle \mathcal{G}_{s(1-s),k}(z,\cdot), g \rangle = g(z)
\]

If \( g \in \Lambda \) then we assemble an orthonormal basis \( \{f_j\}_j \) of \( \Lambda \) out of all \( \{f_j\}_j \) and compute
\[
\left( \hat{A}_k - s(1-s) \right) \int_{\mathcal{F}} \mathcal{G}_{s(1-s),k}(z,w) f_{j_n}(w) d\mu(w)
= \hat{A}_k f_{j_n}(z) - s(1-s) f_{j_n}(z)
\]
and
\[
\left( \int_{\mathcal{F}} \mathcal{G}_{s(1-s),k}(z,w) f_{j_n}(w) d\mu(w) - \chi_{\frac{1}{2} + it,s}(0) \sum_{j} \langle f_{j_n}, f_j \rangle f_j(z) \right)
= f_{j_n}(z) - \hat{A}_k s(1-s) f_{j_n}(z) - \frac{1}{4} + t^2 - s(1-s) f_{j_n}(z) = 0
\]

Expanding \( g(z) = \sum_{j} \langle f, f_{j_n} \rangle f_{j_n}(z) \) proves the claim and the lemma. \( \square \)

\(^{21}\)See corollary A.3.6.
This lemma gives an explicit formula for the relation of the residue of $G_{\lambda,k}$ to an eigenvalue $\lambda$ as well:

**Corollary 7.4.4** Let $s_j = \frac{1}{2} + it_j$ and denote by $\Lambda_j$ the eigenspace of $-\tilde{\Delta}_k$ to the eigenvalue $\lambda = s_j(1 - s_j)$. If $\Lambda_j$ is not trivial then $G_{\lambda,k}$ has a simple pole at $\lambda$ with residue

$$\frac{1}{2s_j - 1} \sum_{u_m \in \Lambda_j} u_m(z) \overline{u_m(w)}$$

**Proof** By the previous lemma $\hat{G}_{\lambda,k}(z, w)$ has residue 0 at any point $s_j = \frac{1}{2} + it_j$. Furthermore it holds

$$\lim_{s \to s_j} \frac{(s - s_j)(2s_j - 1)}{s_j(1 - s_j) - s(1 - s)} = 1$$

Rearranging completes the proof.  \[\square\]
This final chapter presents another application of our spectral theory. Our primary reference here is the first half of the article [Har10].

8.1 Prologue

We follow the exposition of integral binary quadratic forms given in [Zag81, pp. 57-62].

Definition 8.1.1 An integral binary quadratic form is a homogeneous polynomial

\[ Q(x, y) := ax^2 + bxy + cy^2 \in \mathbb{Z}[x, y] \]

of degree 2 with (fixed) coefficients \(a, b, c \in \mathbb{Z}\) and two variables \(x, y\).

Consider \( \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) \) and \( \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + \beta y \\ \gamma x + \delta y \end{pmatrix} \).

We calculate

\[
Q(ax + \beta y, \gamma x + \delta y) = \left( a\alpha^2 + b\alpha \gamma + c\gamma^2 \right) x^2 + \left( a\beta^2 + b\beta \delta + c\delta^2 \right) y^2 \\
+ \left( 2ax\beta + b(a\delta + \beta \gamma) + 2c\gamma \delta \right) xy
\]

This motivates the following

Definition 8.1.2 Two forms \(Q(x, y) = ax^2 + bxy + cy^2\) and \(Q'(x, y) = a'x^2 + b'xy + c'y^2\) are called equivalent if \(\exists \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \text{SL}_2(\mathbb{Z})\) such that \(a' = \hat{a}, b' = \hat{b}, c' = \hat{c}\).
Lemma 8.1.3 This relation is indeed an equivalence relation.

Proof This follows from the fact that SL$_2$(Z) is a group. For instance the inverse transformation is given by

$$x = \delta \hat{x} - \beta \hat{y} \quad \text{and} \quad y = -\gamma \hat{x} + \alpha \hat{y}$$

which yields the symmetry of our relation.

We would like to count the equivalence classes and hence need an invariant of the equivalence relation described above.

Definition 8.1.4 Let $Q(x, y)$ be an integral binary quadratic form. Then the number $d := b^2 - 4ac$ is called discriminant of the form $Q$.

Lemma 8.1.5 Let $Q$, $Q'$ be two equivalent forms. Then their discriminants agree:

$$\hat{b}^2 - 4\hat{a}\hat{c} = b^2 - 4ac.$$ 

Proof This is checked by a straightforward calculation.

The possible discriminants are the integers congruent to 0 or 1 mod 4 and for each such number $d$ there is an integral binary quadratic form $Q(x, y)$ with discriminant $d$, namely take

$$Q(x, y) := \begin{cases} 
\frac{x^2 - \frac{d}{4}y^2}{2} & \text{if } d \equiv 0 \text{ mod } 4 \\
\frac{x^2 + xy + \frac{1-d}{4}y^2}{2} & \text{if } d \equiv 1 \text{ mod } 4 
\end{cases}$$

Thus a better question is: How many equivalence classes of forms of a given discriminant $d$ do we have?

Proposition 8.1.6 (Lagrange) Let $0 \neq d \in \mathbb{Z}$. Then there are only finitely many equivalence classes of integral binary quadratic forms of discriminant $d$.

Proof We prove the proposition the case that $d$ is square-free. If $d = m^2$, $m > 0$ is a square then there are precisely $m$ equivalence classes$^1$.

The idea is to show the following

Claim: Every form $Q(x, y) = ax^2 + bxy + cy^2$ is equivalent to a form $Q'(x, y) := a'x^2 + b'xy + c'y^2$ with the properties

$$|b'| \leq |a'| \leq |c'|, \quad b'^2 - 4a'c' = d$$

Now observe that these conditions are satisfied by at most finitely many triples $(a', b', c') \in \mathbb{Z}^3$. Indeed,

$$|d| = |b'^2 - 4a'c'| \geq 4|a'c'| - b'^2 \geq 3b'^2$$

$^1$This is an exercise in Zagier’s book, see [Zag81, p. 72].
and hence there are only \(\ll \sqrt{|d|}\) choices for \(b'\). Since the product \(a'c'\) is determined by \(b'\) (via \(d\)), we have for each choice of \(b'\) at most finitely many choices for \(a'\) and \(c'\).

Thus it remains to prove the claim. Consider the substitutions
\[
T(x, y) := (x - y, y), \quad S(x, y) := (-y, x)
\]
inducing the transformations
\[
T(a, b, c) = (a, b - 2a, c + a - b), \quad S(a, b, c) = (c, -b, a)
\]
on forms. Applying \(T\) or \(T^{-1}\) for finitely many times we achieve \(|b'| \leq |a|\). If \(|a| \leq |c'|\) then we are done. Else we apply \(S\) yielding some \((a'', b'', c'')\) with \(|a''| < |a|\) and then we start over with this form. We cannot apply \(S\) infinitely often, because \(|a|\) decreases at each step. Note that the discriminant is an invariant of \(S\) and \(T\). This proves the claim and the proposition. \(\square\)

8.2 The setting

Standing assumptions:

- \(Q\) is not a product of linear factors in \(\mathbb{Z}[x, y]\). Equivalently \(d\) is square-free.

- \(Q\) is positive definite, i.e. \(a, c > 0\).

- \(d\) is a fundamental discriminant, that is \(d\) cannot be written as \(d' e^2\) for some smaller discriminant \(d'\) and integer \(e\).

Altogether we infer that \(Q\) is a primitive form: \(\gcd(a, b, c) = 1\).

Definition 8.2.1 The number of equivalence classes of integral binary quadratic forms of a given discriminant \(d\) is called the class number and denoted by \(h(d)\).

Lemma 8.2.2 It holds
\[
h(d) \ll \epsilon |d|^{\frac{1}{2} + \epsilon}
\]

Proof This is included in the previous proof of proposition 8.1.6. \(\square\)

Next we embed \(Q(\sqrt{d})\) into \(\mathbb{C}\) such that \(\sqrt{d} > 0\) for \(d > 0\) and \(-i\sqrt{d} > 0\) for \(d < 0\). The conjugation map is given by \(q_1 + q_2 \sqrt{d} := q_1 - q_2 \sqrt{d}\) for any \(q_1, q_2 \in Q\). Letting
\[
z := \frac{-b + \sqrt{d}}{2a}, \quad z := \frac{-b - \sqrt{d}}{2a}
\]
8. Duke’s theorem

yields

\[ Q(x, y) = ax^2 + bxy + cy^2 = a(x - zy)(x - \overline{z}y) \]

We consider once more the substitutions \( S, T \) from the proof of proposition 8.1.6 and observe\(^2\) that the action of \( SL_2(\mathbb{Z}) \) on \( z \) and \( \overline{z} \) is the usual one given by fractional linear transformations:

\[ T(z) = z + 1, \quad S(z) = -\frac{1}{z} \]

Summing up we consider the standard action of \( SL_2(\mathbb{Z}) \) on certain conjugate pairs of points of \( Q(\sqrt{d}) \) embedded in \( \mathbb{C} \). If \( d < 0 \) then \( z \in \mathbb{H} \) and we obtain \( h(d) \) points on \( \mathbb{F} := SL_2(\mathbb{Z}) \setminus \mathbb{H} \):

**Definition 8.2.3** Those\(^3\) \( h(d) \) points on \( \mathbb{F} \) are called Heegner points of discriminant \( d < 0 \).

If \( d > 0 \) then \( z, \overline{z} \in \mathbb{R} \) and we consider the (hyperbolic) geodesics \( G_{z, \overline{z}} \) connecting them. That is \( G_{z, \overline{z}} \) is a semicircle perpendicular to the real-axis at \( z \) and \( \overline{z} \). We obtain \( h(d) \) geodesics on \( \mathbb{F} \).

**Proposition 8.2.4** Let \( G_{z, \overline{z}} \subset \mathbb{H} \) be a hyperbolic geodesic connecting \( z, \overline{z} \) and \( \tilde{G}_{z, \overline{z}} \subset \mathbb{F} \) be its projection onto \( \mathbb{F} \). Then \( \tilde{G}_{z, \overline{z}} \subset \mathbb{F} \) is closed and has length \( 2\log(\lambda_d) \), where \( \lambda_d > 1 \) is the smallest totally positive unit\(^4\) of the ring of integers \( \mathcal{O}_d \) of \( Q(\sqrt{d}) \).

**Proof** This is a purely technical argument, which is performed in [Har10, pp. 379,380]. \( \square \)

**Proposition 8.2.5** It holds

\[
|d|^{1 - \epsilon} \ll \epsilon h(d) \ll \epsilon |d|^{1 + \epsilon}, \quad \text{if } d < 0,
\]

\[
d^{1 - \epsilon} \ll \epsilon h(d) \log(\lambda_d) \ll \epsilon d^{1 + \epsilon}, \quad \text{if } d > 0
\]

where \( \lambda_d \) is the same number as in the previous proposition.

**Proof** The upper bounds are a consequence of Dirichlet’s class number formula, which computes \( h(d) \) in both cases \( d < 0 \) and \( d > 0 \). The formula takes the equations (15) to (18) in [Dav80, chapter 6]. The lower bounds are established by combining Dirichlet’s formula with Siegel’s theorem, which is stated at the beginning of [Dav80, chapter 21]. \( \square \)

\(^2\)Note that the corresponding matrices \[
\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}
\]
generate \( SL_2(\mathbb{Z}) \). A proof of this fact can be found for instance in [Miy89, theorem 4.1.1].

\(^3\)corresponding to the equivalence classes of integral binary quadratic forms as described above.

\(^4\)A totally positive unit is a unit of \( \mathcal{O}_d \), which is positive under both embeddings of \( Q(\sqrt{d}) \) into \( \mathbb{R} \). Recall that any unit has norm \( N(q_1 + q_2 \sqrt{d}) = (q_1 + q_2 \sqrt{d})(q_1 - q_2 \sqrt{d}) = \pm 1 \).
8.3 The theorem and a proof sketch

Definition 8.3.1 The set of Heegner points or closed geodesics in \( \mathbb{F} \) respectively is denoted by \( \Lambda_d \).

Proposition 8.2.5 motivates the question, whether \( \Lambda_d \) becomes equidistributed in \( \mathbb{F} \) as \( d \to -\infty \) or \( d \to \infty \) respectively. Duke’s theorem provides an answer to this question in the following stronger form:

Theorem 8.3.2 (Duke) \( \exists \delta > 0 \) such that for every \( g \in C_c^\infty(\mathbb{F}, \mathbb{C}) \):

\[
\frac{1}{h(d)} \sum_{z \in \Lambda_d} g(z) - \int_{\mathbb{F}} g(z) d\mu(z) \ll_{g} |d|^{-\delta}, \text{ if } d < 0
\]

\[
\frac{1}{h(d)2 \log(\lambda_d)} \sum_{G \in \Lambda_d} \int_{G} g(z) ds(z) - \int_{\mathbb{F}} g(z) d\mu(z) \ll_{g} d^{-\delta}, \text{ if } d > 0
\]

A full proof would exceed the limits of this section, but our spectral machinery enables us to attack this result in the following fashion:

Proof sketch: Fix the weight \( k = 0 \) and the multiplier \( v \equiv 1 \). Then we may extend \( g \) to \( \mathbb{H} \) by

\[ g(\gamma z) = g(z), \quad z \in \mathbb{F}, \gamma \in SL_2(\mathbb{Z}) \]

Hence \( g \) satisfies the transformation law and clearly \( g \in \mathcal{H}_k \) due to compact support. According to example 7.1.5 we obtain

\[
g(z) = \sum_{j=1}^\infty \langle g, u_j \rangle u_j(z) + \frac{3}{\pi} \langle g, 1 \rangle
\]

\[+ \frac{1}{4\pi} \int_{-\infty}^{\infty} \left\langle g, E_\infty\left(\cdot, \frac{1}{2} + it\right) \right\rangle E_\infty\left(z, \frac{1}{2} + it\right) dt\]

where \( \{u_j\} \) serves as a complete orthonormal system of automorphic cusp forms to the same data and \( E_\infty(z,s) \) is given by

\[
E_\infty(z,s) = \sum_{\gamma \in \Gamma_\infty \setminus \Gamma} \text{Im}(\gamma z)^s \sum_{(c,d) \in \mathbb{Z}^2 \atop \gcd(c,d) = 1} \text{Im}(z)^s \frac{1}{|cz+d|^2s}
\]

Lemma 8.3.3 Let \( g \in C_c^\infty(\mathbb{H}, \mathbb{C}) \) and \( \kappa > 0 \). Then it holds

\[
\langle g, u_j \rangle \ll_{g,k} (1 + |t_j|)^{-\kappa}, \quad \left\langle g, E_\infty\left(z, \frac{1}{2} + it\right) \right\rangle \ll_{g,k} (1 + |t|)^{-\kappa}
\]

where \( t_j, t \) are the spectral parameters\(^6\) of \( u_j, E_\infty \) respectively.

\(^5\)Recall equation (1.1) for the definition of \( ds(z) \).

\(^6\)That is \( \lambda = \frac{1}{4} + t^2 \)
8. Duke’s theorem

**Proof** On the one hand we have
\[
\left(\frac{1}{4} + t^2\right) \langle g, u_j \rangle = \langle g, -\Delta_0 u_j \rangle = \langle -\Delta_0 g, u_j \rangle \leq \| -\Delta_0 g \| \| u_j \| = \| -\Delta_0 g \|
\]
which yields the first bound. On the other hand the computation from the proof of proposition 4.3.2 simplifies to\(^7\)
\[
\langle g, E_\infty(z, \frac{1}{2} + it) \rangle = \int_F \sum_{\gamma \in \Gamma \setminus \Gamma} \text{Im}(\gamma z)^{\frac{1}{2}+it} d\mu(z)
\]
\[
= \sum_{\gamma \in \Gamma \setminus \Gamma} \int_{\gamma F} g(\gamma^{-1} w) \text{Im}(w)^{\frac{1}{2}-it} d\mu(w) = \int_0^\infty \left( \int_0^1 g(w) dw \right) \frac{y^{\frac{1}{2}-it}}{y^2} dy
\]
The second bound now becomes a property of the Mellin transform\(^8\):
\[
(M\varphi)(s) \ll \varphi(1 + |s|)^{-\kappa}
\]
for any \(\kappa > 0\), which is achieved upon repeated partial integration. □

This discussion and proposition 8.2.5 reduce Duke’s theorem to the following claim:

**Lemma 8.3.4** \(\exists \delta, \tilde{\kappa} > 0\) such that if \(g = u_j\) or \(g = E_\infty(z, s)\) then
\[
\sum_{z \in \Lambda_d} g(z) \ll \tilde{g} (1 + |t|)^{\delta} |d|^{\frac{1}{2}-\delta}
\]
\[
\sum_{G \in \Lambda_d} \int_G g(z) ds(z) \ll \tilde{g} (1 + |t|)^{\delta} |d|^{\frac{1}{2}-\delta}
\]
where \(t\) is determined by \(\Delta_0 g(z) = (\frac{1}{4} + t^2) g(z)\).

We sketch how one might establish this lemma. Recall that in both cases \(g\) is an automorphic form and hence has the following Fourier expansion\(^9\)
\[
g(x + iy) = C_1 y^{\frac{1}{2} + it} + C_2 y^{\frac{1}{2}-it} + \sum_{n \neq 0} g_n(y) W_{n,0}(4\pi |n| y) e^{2\pi inx}
\]
In the case of \(g = u_j\) clearly \(C_1 = C_2 = 0\) and in the case of \(g = E_\infty(z, s)\) we have seen that \(C_1 = |C_2| = 1\). Now we invoke some basic Hecke theory, which will reduce our attention to the first Fourier coefficient:

**Lemma 8.3.5** The Fourier coefficients \(g_n(y)\) are proportional to the Hecke eigenvalues of \(g\).

\(^7\)Note that \((y^s) = y^s\), since \(y \in \mathbb{R}_{>0}\).

\(^8\)Iwa02, equation (7.3).

\(^9\)This is theorem 1.4.1 with trivial transformation law and scaling matrix.
8.3. The theorem and a proof sketch

**Proof** The space of cusp forms admits an orthogonal basis of simultaneous eigenfunctions of all the Hecke operators. Moreover the Eisenstein series $E_\infty(z, s)$ is an eigenfunction of all the Hecke operators as well. Denoting the $n$-th Hecke eigenvalue by $a_g(n)$ this justifies

$$g_n(y) = g_1(y)a_g(n), \ \forall n \in \mathbb{N}$$

Further reference to those claims is provided by [Iwa02, section 8.5] and [Iwa97, section 6.4]. Lastly note that the involution $z \mapsto -z$ is an eigenfunction for the Hecke algebra too\(^{10}\), which yields $g_n(y) = \pm g_{-n}(y)$ and hence extends the relation $g_n(y) = g_1(y)a_g(n)$ to all $n \in \mathbb{Z}$.

In particular note that $g_1(y) \neq 0$ and $a_g(1) = 1$.

The first Fourier coefficient of $g$ admits the following bound:

**Lemma 8.3.6**

$$|g_1(y)| \ll \varepsilon \left(1 + |t|\right)^{\varepsilon} e^{\frac{\pi}{2}|t|}$$

**Proof** In the case of a cusp form combine equations [Iwa02, (8.1), (8.5), (8.43)] (which is included in [Iwa02, theorem 8.3]). In the case of an Eisenstein series combine equations [Iwa02, (8.2), (8.6), (8.43)]. A full article on this topic was written by [HL94].

Thus we would like to relate the sums $\sum_{z \in \Lambda_d} \cdots$ from lemma 8.3.4 to $g_1(y)$. This is indeed possible:

**Lemma 8.3.7** Let $\Lambda(s, \Pi)$ be the completed automorphic $L$-function associated to an automorphic representation\(^{11}\) $\Pi$ and let $\left(\frac{d}{\cdot}\right)$ be the Legendre symbol. Then

$$\left| \sum_{z \in \Lambda_d} \cdots \right|^2 = c_d |d|^2 |g_1(y)|^2 \Lambda\left(\frac{1}{2}, \varepsilon\right) \Lambda\left(\frac{1}{2}, \varepsilon \otimes \left(\frac{d}{\cdot}\right)\right)$$

where $c_d$ takes only finitely many different values.

**Proof** See [Har10, equation (17)] for a list of references.

It remains to bound the occuring factors $\Lambda(s, \Pi)$: The finite part $L(s, \Pi)$ of $\Lambda(s, \Pi)$ is defined in terms of Hecke eigenvalues. The infinite part of $\Lambda(s, \Pi)$ is a product of exponential and gamma factors, whose contribution in the previous lemma can be bounded by $\ll (1 + |t|) e^{-\pi|t|}$ using Stirling’s approximation\(^{12}\).

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\(^{10}\)HL94, p. 161.

\(^{11}\)A formal definition of those objects is quite involved and we do not require it for our purposes here.

\(^{12}\)Har10, p. 382.
We insert the bound for $|g_1(y)|$ together with these observations into the previous lemma. Hence lemma 8.3.4 reduces to the following subconvex bounds:

**Lemma 8.3.8**  
(a) If $g$ is a cusp form then there are constants $\hat{\delta}, \hat{\kappa} > 0$ such that
$$L\left(\frac{1}{2}, g \otimes \left(\frac{d}{-} \right)\right) \ll (1 + |t|)^{\hat{\kappa}} |d|^\frac{1}{2} - \hat{\delta}$$

(b) If $g$ is an Eisenstein series then there are constants $\hat{\delta}, \hat{\kappa} > 0$ such that
$$L\left(\frac{1}{2} + it, \left(\frac{d}{-} \right)\right) \ll (1 + |t|)^{\hat{\kappa}} |d|^\frac{1}{2} - \hat{\delta}$$

**Proof** Both bounds were proven by Conrey-Iwaniec for any $\delta < \frac{1}{6}$. Precisely compare [CI00, corollary 1.5]. □

**Proof (of lemma 8.3.4)** Going backwards and counting the exponents we obtain
$$\left| \sum_{z \in \Lambda_d} g(z) \right|^2 \ll_g |d|^\frac{1}{2} + \frac{1}{2} - \hat{\delta} (1 + |t|)^{2\hat{\kappa} + 1}$$
$$\left| \sum_{z \in \Lambda_d} \int_G g(z) ds(z) \right|^2 \ll_g |d|^\frac{1}{2} + \frac{1}{2} - \hat{\delta} (1 + |t|)^{2\hat{\kappa} + 1}$$

Hence
$$\sum_{z \in \Lambda_d} g(z) \ll_g |d|^\frac{1}{2} + \frac{1}{2} - \hat{\delta} (1 + |t|)^{\hat{\kappa}}$$
$$\sum_{z \in \Lambda_d} \int_G g(z) ds(z) \ll_g |d|^\frac{1}{2} + \frac{1}{2} - \hat{\delta} (1 + |t|)^{\hat{\kappa}}$$
as desired. □

**Proof (of Duke’s theorem)** Insert the bounds from lemma 8.3.3 and proposition 8.2.5 into lemma 8.3.4. Then conclude by the the spectral expansion of $g$. □
Appendix A

Functional Analysis

Throughout this appendix let \((\mathcal{H}, \langle \cdot, \cdot \rangle)\) be a Hilbert space, \(L\) be a linear operator on \(\mathcal{H}\) with domain of definition \(D(L) \subseteq \mathcal{H}\) and normed target space \(Y\). Denote by \(\mathcal{H}^*, L^*\) their respective topological duals (see also section 2.2). If \(V \subseteq \mathcal{H}\) is a subspace then \(\text{clos}(V)\) is the usual closure of \(V\) in \(\mathcal{H}\) with respect to norm topology (which is induced by \(\|\cdot\| := \sqrt{\langle \cdot, \cdot \rangle}\)).

We follow [Str] unless stated otherwise. Further reference is given by the books [SB18] and [Rud73].

A.1 Generalities

**Proposition A.1.1** \(L\) is continuous if and only if

\[
\|L\| := \sup \{ \|Lx\| \mid \|x\| \leq 1 \} < \infty
\]

**Proof** [Str, Satz 2.2.1] \(\square\)

**Definition A.1.2** (i) \(L\) is called closed if

\[
\text{graph}(L) = \{ (x, Lx) \mid x \in D(L) \}
\]

is closed in the product space \(\mathcal{H} \times Y\), where \(Y\) is the normed target space of \(L\).

(ii) \(L\) is called closable if the closure of \(\text{graph}(L)\) is linear, that is

\[
(0, y) \in \text{clos}(\text{graph}(L)) \Rightarrow y = 0
\]

In this case the operator \(\tilde{L}\) with \(D(\tilde{L}) = \text{clos}(\text{graph}(L))\) is called the closure of \(L\). In other words \(\tilde{L}\) is the smallest extension of \(L\) in the sense of set inclusions of graphs.
Proposition A.1.3 It holds \( D(L) \subseteq D(\tilde{L}) \subseteq \text{clos}(D(L)) \) and
\[
D(\tilde{L}) = \{ x \in \mathcal{H} \mid \exists (x_j)_j \subseteq D(L) \ \exists y \in \mathcal{H} : \]

Proof [Str, Bemerkung 3.4.2]

Proposition A.1.4 ("Fundamental lemma in the calculus of variations") Let \( \Omega \subseteq \mathbb{R}^n \) be open, \( f \in L^1_{\text{loc}}(\Omega) \). Assume
\[
\forall \varphi \in C_0^\infty(\Omega) : \int_\Omega f \varphi \, dx = 0
\]
Then \( f \equiv 0 \) almost everywhere in \( \Omega \).

Proof [Str, Satz 3.4.3]

Definition A.1.5 Let \( X \) be a normed space. Then \( X \) is called reflexive if \( X^{**} \equiv X \).

Proposition A.1.6 Every Hilbert space is reflexive.

Proof [Str, Beispiel 5.1.1 ii)]

Definition A.1.7 Let \( (v_j)_j \subseteq \mathcal{H} \) be a collection of non-vanishing elements. Then \( (v_j)_j \) is called
(i) orthogonal system if \( \forall j \neq l : \langle v_j, v_l \rangle = 0 \).
(ii) orthonormal system if it is an orthogonal system and \( \forall j : \langle v_j, v_j \rangle = 1 \).
(iii) complete if \( \forall f \in \mathcal{H} \ \forall j : \langle f, v_j \rangle = 0 \Rightarrow f = 0 \). Equivalently \( (v_j)_j \) is complete if \( \text{span}\{v_j \mid j\} \) is dense in \( \mathcal{H} \).

A.2 Compact operators

Definition A.2.1 Let \( L \) be continuous. Then \( L \) is called compact if
\[
\text{clos}\left(L(B_1(0))\right)
\]
is compact.

Definition A.2.2 Let \( X \) be a normed space and \( (x_j)_j \subseteq X \) be a sequence. Then \( (x_j)_j \) converges weakly to \( x \in X \) if \( \forall H \in X^* : H(x_j) \xrightarrow{i \to \infty} H(x) \).

Lemma A.2.3 Let \( L \) be continuous.
(a) If \( L \) is compact then \( L \) maps weakly convergent sequences onto norm convergent sequences.
\( (b) \) If \( L \) is defined on a Hilbert space and maps weakly convergent sequences onto norm convergent sequences then \( L \) is compact.

The proof of \( b) \) needs the following result:

**Proposition A.2.4 (Eberlein-Šmulian)** Let \( X \) be a reflexive normed space and \( (x_j)_j \subseteq X \) be a bounded sequence. Then \( (x_j)_j \) admits a weakly convergent subsequence.

**Proof** [Str, Satz 5.3.2] \( \square \)

**Remark A.2.5** In other words \( B := \text{clos}(B_1(0)) \subseteq X \) is weakly compact. Compare: \( B \) is weak\(^*\)-compact in any separable space (This is the Banach-Alaoglu theorem, [Str, Satz 5.3.1]) and compact with respect to the norm topology if and only if \( X \) is finite dimensional ([Str, Satz 2.1.4]).

**Proof (of the lemma)** (a) [Str, Lemma 6.2.2]

(b) Recall that Hilbert spaces are reflexive and observe that any sequence in \( \text{clos}(L(B_1(0))) \) is bounded due to the inequality \( ||Lx|| \leq ||L|| ||x|| \). Thus the result of Eberlein-Šmulian provides a a weakly convergent subsequence, which gets mapped by \( L \) onto a norm convergent sequence. Hence \( \text{clos}(L(B_1(0))) \) is sequentially compact, which is equivalent to compactness in metric spaces. \( \square \)

**Lemma A.2.6** Weakly convergent sequences are bounded.

**Proof** [Str, Satz 4.6.1] \( \square \)

**Definition A.2.7** Let \( \Omega \subseteq \mathbb{R}^n \) be open and assume \( L \) is an integral operator with kernel function \( K : \Omega \times \Omega \rightarrow \mathbb{R} \). Then \( L \) is called Hilbert-Schmidt operator if \( K \in L^2(\Omega \times \Omega) \).

**Proposition A.2.8** Hilbert-Schmidt operators are compact.

**Proof** Let \( (f_j)_j \subset D(L) \) be a weakly convergent sequence with weak limit \( f \). Then \( (f_j)_j \) is bounded. Moreover by Fubini’s theorem \( ||K(\cdot, x)||_{L^2(\Omega)} < \infty \) for almost every \( x \in \Omega \). Hence \( L \) is continuous, formally

\[
\lim_{j \to \infty} (L f_j)(x) = \lim_{j \to \infty} \int_{\Omega} K(x, y) f_j(y) dy = (L f)(x)
\]

To show that \( L f_j \) is norm convergent we use the Cauchy-Schwarz inequality and our assumption:

\[
||L f_n||(x) \leq ||f||_{L^2(\Omega)} \int_{\Omega} |K(x, y)|^2 dy < \infty
\]
Thus we are allowed to apply Lebesgue’s dominated convergence theorem giving
\[
\lim_{j \to \infty} \int_{\Omega} |(Lf_j)(x)|^2 \, dx = \int_{\Omega} |(Lf)(x)|^2 \, dx
\]
In other words \( \|L f_j\|_{L^2(\Omega)} \xrightarrow{j \to \infty} \|L f\|_{L^2(\Omega)} \). Lastly note that the inner product is continuous (fixing one of its arguments). This yields
\[
\|L f_j - L f\|^2 = \|L f_j\|^2 + \|L f\|^2 - 2 \langle L f_j, L f \rangle \xrightarrow{j \to \infty} 0
\]
Summing up \( L \) maps weakly convergent sequences onto norm convergent sequences. The claim follows. \( \square \)

### A.3 Spectral calculus

Let \( X \) be a Banach space over \( \mathbb{C} \) and \( L: D(L) \subseteq X \to X \) linear.

**Definition A.3.1**

(i) The set
\[
\rho(L) := \{ \lambda \in \mathbb{C} \mid (L - \lambda \mathbb{1}) \text{ is bijective with linear and continuous inverse} \}
\]
is called resolvent set of \( L \).

(ii) The map
\[
\mathcal{R}: \rho(L) \to X^*
\]
\[
\lambda \mapsto \mathcal{R}_\lambda := (L - \lambda \mathbb{1})^{-1}
\]
is called the resolvent of \( L \).

**Remark A.3.2** Note that Struwe uses the opposite sign convention for the resolvent.

**Definition A.3.3** The complement \( \sigma(L) := \mathbb{C} \setminus \rho(L) \) is called the spectrum of \( L \). Let \( L \) be closed. Then \( \sigma(L) \) is partitioned into the following subsets:

(i) The set
\[
\sigma_p(L) := \{ \lambda \in \mathbb{C} \mid (L - \lambda \mathbb{1}) \text{ is not injective} \} \subseteq \sigma(L)
\]
is called the point spectrum of \( L \). It consists of the eigenvalues of \( L \) with eigenspaces \( \text{Ker}(L - \lambda \mathbb{1}) \neq \{0\} \).

(ii) The set
\[
\sigma_c(L) := \{ \lambda \in \sigma(L) \mid (L - \lambda \mathbb{1}) \text{ is injective and has dense image} \}
\]
is called continuous spectrum of \( L \).
The set \( \sigma_r(L) := \sigma(L) \setminus (\sigma_p(L) \cup \sigma_c(L)) \) is called the residual spectrum of \( L \).

**Proposition A.3.4** Let \( z_0 \in \rho(L) \). Then

\[
\{ z \in \mathbb{C} \mid |z - z_0| < \frac{1}{\| R_{z_0} \|} \} \subseteq \rho(L)
\]

In other words \( \rho(L) \) is open and \( \sigma(L) \) is closed. Moreover

\[
\forall z \in \rho(L) : \| R_z \| \geq \frac{1}{\text{dist}(z, \sigma(L))}
\]

Lastly the map \( z \to R_z \) is continuous on \( \rho(L) \).

**Proof** [Str, Satz 6.5.1]

**Lemma A.3.5 (Hilbert’s first resolvent identity)**

\[
\forall \lambda, \nu \in \rho(L) : R_\lambda - R_\nu = (\nu - \lambda) R_\nu R_\lambda
\]

**Proof** [Str, Satz 6.5.2 ii]

**Corollary A.3.6** The map \( \lambda \to R_\lambda \) is holomorphic on \( \rho(L) \) with derivative

\[
\frac{dR_\lambda}{d\lambda} = -R_\lambda^2
\]

**Proof** Combine the previous two assertions.

**Proposition A.3.7** Let \( D(L) \subseteq D(L^*) \) be a symmetric operator on a Hilbert space. Then the following are equivalent:

(a) \( D(L) = D(L^*) \), i.e. \( L \) is self-adjoint.

(b) \( \sigma(L) \subseteq \mathbb{R} \).

(c) \( \exists z_1, z_2 \in \rho(L) : \text{Im}(z_1) < 0 < \text{Im}(z_2) \).

**Proof** [Str, Satz 6.6.2]

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**A.4 The Hilbert-Schmidt theorem**

**Proposition A.4.1** Let \( L \) be a continuous, compact and self-adjoint operator on a Hilbert space \( \mathcal{H} \). Then

\[
\sigma(L) \setminus \{0\} = \sigma_p(L) \setminus \{0\}
\]

**Proof** [Str, Beispiel 6.5.2]
**Theorem A.4.2 (Hilbert-Schmidt)** Let \( 0 \neq L \) be a continuous, compact and self-adjoint operator on a Hilbert space \( \mathcal{H} \). Then it holds:

(a) \( L \) has almost countably many eigenvalues \( (\lambda_j)_j \subseteq \mathbb{R} \setminus \{0\} \), which may accumulate only at 0. Moreover \( L \) admits an orthonormal collection \( (e_j)_j \) of eigenvectors corresponding to the eigenvalues \( \lambda_j \), such that

\[
\forall x \in \mathcal{H} : Lx = \sum_j \lambda_j \langle x, e_j \rangle e_j
\]

(b) We have an orthogonal decomposition

\[
\mathcal{H} = \ker(L) \oplus \overline{\text{span}\{e_j|j\}}
\]

**Proof** [Str, Struwe 6.7.2] \( \square \)
Bibliography


[Bibliography]


Glossary

Chapter 1

H  The upper half plane
μ  Hyperbolic measure
Γ  Fuchsian group of the first kind
F  Fundamental domain
a, b  Cusps
γa  Scaling matrix
Γa  Stabilizer subgroup
1  Identity element or operator
k  weight

Δk  Hyperbolic Laplacian of weight k
λ  Eigenvalue
jγ(z; k)  automorphy factor
\gamma \slash \gamma  slash-operator
σk  auxiliary function for \upsilon
\upsilon  unitary multiplier
fe(y)  constant Fourier coefficient of f
W  Whittaker function

Chapter 2

\mathcal{H}_k  L^2(\mathbb{H}\backslash \Gamma) admitting the transformation law of weight k
\langle \cdot, \cdot \rangle  Petersson inner product
\mathcal{D}_k^{(2)}  see definition 2.1.4
\mathcal{D}_k^{(\infty)}  see definition 2.1.4
\mathfrak{C}(Y)  Cusp sector
L  General linear operator (in this chapter only)

Rk  Maass operator increasing the weight by 2
Lk  Maass operator decreasing the weight by 2
\tilde{\Delta}_k  Self-adjoint extension of \Delta_k
\mathcal{D}_k  Domain of definition of \tilde{\Delta}_k
ρ(k)  Border of the continuous spectrum
chapter 3

\( \mathcal{R}_{\lambda,k} \) Resolvent operator of \( \hat{\Delta}_k \)

\( \mathcal{G}_{\lambda,k} \) Green’s function on \( \mathcal{F} \), i.e. kernel of \( \mathcal{R}_{\lambda,k} \)

\( \mathcal{g}_{\lambda,k} \) Green’s function on \( \mathcal{H} \), i.e. point-pair invariant on \( \mathcal{H} \)

\( d(z,w) \) Hyperbolic distance

\( L \) Invariant integral operator

\( K \) Automorphic kernel

\( k \) free space kernel

chapter 4

\( E_a(z,s) \) Eisenstein series of weight \( k \)

\( E_a^\text{tr}(z,s) \) truncated Eisenstein series of weight \( k \)

\( E_a(z|\psi) \) incomplete Eisenstein series of weight \( k \)

\( \mathcal{H}_{k,0} \) Hilbert functions with vanishing constant Fourier coefficient

\( \mathcal{E}_k \) subspace of incomplete Eisenstein series

\( \delta_{ab} \) Kronecker delta function

\( \varphi_{ab}(s) \) constant Fourier coefficient of \( E_a(z,s) \) expanded at \( b \)

\( \varphi_{ab}(n,s) \) \( n \)-th Fourier coefficient of \( E_a(z,s) \) expanded at \( b \)

\( \Phi(s) \) constant term matrix (or scattering matrix) of \( E_a(z,s) \)

\( S_{ab}(m,n;c) \) Kloosterman sum

chapter 5

\( u_j \) orthonormal cusp forms

\( \tau(x) \) matrix representing translation by \( x \)

\( L \) modified resolvent

\( K \) bounded part of \( K \)

\( L \) bounded part of \( L \)

\( \hat{L} \) bounded part of \( \hat{L} \)

chapter 6

\( \mathcal{M} \) Mellin transform

\( \mathcal{P}_k \) Space of residues of \( E_a(z,s) \)

\( v_j \) orthonormal residues

\( h(t) \) Selberg / Harish-Chandra transform of \( k \)

\( \Lambda_L(\lambda) \) eigenvalue of \( L \)

\( E_{k,a} \) Eisenstein transform

\( \mathcal{E}_{k,a} \) \( E_{k,a}(C_c^\infty(\mathbb{R}_{>0},\mathbb{C})) \)

\( r_a(z,s_j) \) Residue of \( E_a(z,s) \) at \( s_j \)
chapter 7

$P$ holomorphic Poincaré series

$P_n$ automorphic Poincaré series

$\tilde{f}_j(z;k)$ modular factor

$f_j$ orthonormal system for $\mathcal{H}_{k,0} \oplus \mathcal{P}_k$

$\chi_{s,a}(v)$ see equation (7.3)

chapter 8

$Q(x,y)$ integral binary quadratic form

$d$ discriminant of $B$

$h(d)$ class number

$\lambda_d$ See proposition 8.2.4

$\Lambda_d$ Set of Heegner points or closed geodesics in $\text{SL}_2(\mathbb{Z})\backslash \mathbb{H}$ respectively
Eigenständigkeitserklärung


Die Dozentinnen und Dozenten können auch für andere bei ihnen verfasste schriftliche Arbeiten eine Eigenständigkeitserklärung verlangen.

Ich bestätige, die vorliegende Arbeit selbständig und in eigenen Worten verfasst zu haben. Davon ausgenommen sind sprachliche und inhaltliche Korrekturvorschläge durch die Betreuer und Betreuerinnen der Arbeit.

**Titel der Arbeit** (in Druckschrift):

Spectral Theory of Automorphic Forms on the Hyperbolic Plane

**Verfasst von** (in Druckschrift):

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Name(n):
Mono

Vorname(n):
Andreas

Ich bestätige mit meiner Unterschrift:
- Ich habe keine im Merkblatt „Zitier-Konigae“ beschriebene Form des Plagiats begangen.
- Ich habe alle Methoden, Daten und Arbeitsabläufe wahrheitsgetreu dokumentiert.
- Ich habe keine Daten manipuliert.
- Ich habe alle Personen erwähnt, welche die Arbeit wesentlich unterstützt haben.

Ich nehme zur Kenntnis, dass die Arbeit mit elektronischen Hilfsmitteln auf Plagiate überprüft werden kann.

**Ort, Datum**

Zürich, den 20.06.2019

Unterschrift(en)

Bei Gruppenarbeiten sind die Namen aller Verfasserinnen und Verfasser erforderlich. Durch die Unterschriften bürgen sie gemeinsam für den gesamten Inhalt dieser schriftlichen Arbeit.