Numerical Methods to Quantify the Model Risk of Basket Default Swaps

A. Schröter* and P. Heider†

October 20, 2010

Abstract

The valuation of basket default swaps depends crucially on the joint default probability of the underlying assets in the basket. It is known that this probability can be modeled by means of a copula function which links the marginal default probabilities to a joint probability. The valuation bears risk due to the uncertainty of the copula, the relation of the assets to each other and the marginal distributions which we call together the model risk. To value basket default swaps and to compute model risk parameters we present an efficient numerical approach based on importance sampling and applicable to different classes of copula models. Our numerical findings show that the choice of the underlying copula model influences strongly the risk profile of the basket and should be tailored advisedly.

Key words: model-risk, basket default swap, importance sampling, credit derivatives sensitives, Archimedean copula

*Mathematisches Institut der Universität zu Köln, Weyertal 86-90, 50931 Köln, aschroet@math.uni-koeln.de
†Mathematisches Institut der Universität zu Köln, Weyertal 86-90, 50931 Köln, pheider@math.uni-koeln.de

1 Introduction

Over the past years the popularity of credit derivative securities has grown considerably. The most prominent derivative in this class is the credit default swap (CDS) whose contingent payment is triggered by a default event of a single specified reference entity. A large subclass of credit derivatives are securities which are based on default events of multiple reference entities. Popular securities in this subclass are collateralized debt obligations (CDOs) and basket default swaps (BDS). In this paper we focus mainly on BDS, yet ideas and algorithms can easily be adapted to other basket-type securities.

Briefly, a m-th to default swap (mBDS) is a contract between two parties – the *protection buyer* and the *protection seller* whose contingent payoff is triggered by a cumulative default event of an agreed upon basket of reference entities. The buyer pays at regular intervals a protection fee to the seller until either m-th entities of the basket have defaulted or the maturity of the contract is reached. Conversely, in case of m default events before maturity the seller pays the loss rate of the m-th asset to the protection buyer at time of the m-th default.

To price basket securities adequately the joint probability distribution of default times must be known. Specific baskets of obligors are less frequently traded, while credit derivatives on a single reference entity – e.g. a 5-year CDS – are traded rather liquidly. Therefore, the marginal distribution of default time for a single obligor can be implied by market data reasonably well, [8], whereas the joint distribution usually cannot be deduced from market data. As remedy Li suggested in his seminal work, [18], to link the marginal distribution to a joint distribution by means of a copula function.

Of course, the default times of multiple obligors are correlated and these relations amongst the obligors in the basket must be reflected in the joint distribution. Depending on the chosen copula a certain correlation structure is induced and the copula should be chosen such that the induced correlation structure matches the observed respectively assumed correlations of the market.

A common choice is a Gaussian-type copula, [18, 22]. The Gaussian copula became very popular because it allows analytical and semi-analytical pricing formulas for some basket derivatives, e.g. [1, 15, 13, 16]. However, a Gaussian copula can only induce a linear correlation structure, [10, 18, 9] which might be a harsh restriction. Moreover, at least $\frac{n(n-1)}{2}$ parameters have to be specified and calibrated which renders a reasonable calibration almost impossible due to an typically illiquid market situation. Therefore, other classes of copula functions have been discussed recently, [23, 22]. In this paper we will concentrate on Archimedean-type copula function which we will briefly introduce in Sect. 3.2.

To describe the correlation amongst the obligors in the basket reasonable well we choose *Kendall's* τ as concordance measure, [21, 9]. Kendall's τ allows for nonlinear correlation and is more suitable to represent the interrelations within the basket. We will define a concordance structure for the given basket based on Kendall's τ measure. It is possible that different copula function imply the same *concordance structure* and consequently different prices of the derivative security are possible.

We subsume under the term *model risk* all uncertainty in the price of a basket derivative which is due to the choice of copula. Hence, the model risk is mainly driven by the uncertainty of choosing a suitable copula to a given concordance structure, by the uncertainty of the concordance structure which represents the interrelations in the basket and by the uncertainty from the marginal distribution which is independent from the choice of the copula.

In this paper we will present efficient numerical methods to evaluate the model risk. To value the model risk due to the choice of copula we compute prices induced by different copula models implying the same concordance structure. This gives a lower bound for the possible price spread. The model risk due to uncertainty of the concordance structure itself is valued by computing the price sensitivity with respect to changes in the concordance structure. Analogously, the model risk due to uncertainty of marginal distributions is valued by parameterizing the marginal distribution and computation of the corresponding sensitivities of the basket price.

For Gaussian copula based joint probabilities Joshi (cp.[17]) suggested an efficient numerical method to value sensitivities of mBDS, which was further improved by Chen (cp. [5]). They use Monte-Carlo simulation together with a clever importance sampling approach to reduce the variance. In this paper we follow their ideas and use importance sampling for an efficient implementation of Monte-Carlo simulation for Archimedean copula based joint probabilities. For the computation of the sensitivities we discuss the *likelihood ratio* method and the *finite difference* method, [2, 12], applied to our framework for Archimedean copulae and importance sampling.

We observe that the copula has strong impact on the price of a basket derivative. Prices implied by two different copulae might differ substantially even if the concordance structure is the same. Therefore, the choice of the copula function determines the risk - profile of the BDS and be considered as part of the modeling.

The paper is organized as follows. First, we set up the cash flows of a mBDS and introduce the necessary notation for the rest of the paper in Sect. 2. We define in Sect. 2.4 the *concordance structure* of a basket which describes the interdependence of default times in the basket. In Sect. 3 we briefly introduce Gaussian and Archimedean copulae and give explicit formulae to compute the concordance structure in a multi-dimensional setting. Sect. 4 introduces variance reduction by importance sampling for Gaussian and Archimedean based probabilities. Then, we present importance sampling by Joshi-Kainth and Chen-Glasserman for Gaussian based copula models in Sect. 4.1. Following, we extend their approach to Archimedean based models in Sect. 4.2 and give numerical examples to demonstrate the performance of our approach. The variance reduction allows for an efficient numerical valuation on which the subsequent model risk evaluation is based. We detail on the model risk and its numerical computation in Sect. 5 and present numerical studies. Finally, we summarize our results in Sect. 6.

2 Basket Default Swaps and Model Risk

We consider a basket derivative of n defaultable assets,

$$S := \{S_1, ..., S_n\}\,,\ n \in \mathbb{N}, \tag{2.1}$$

which are usually corporate bonds or loans. We denote by N_j the nominal amount of S_j and by R_j the recovery rate of S_j. Hence, in case of default of asset S_j the basket suffers a loss of $(1 - R_j)N_j$. Further, we assume that the short-rate $r(t)$ is known and a deterministic function. The total nominal of the basket is $N = \sum\limits_{j=1}^{n} N_j$.

A m-th to default swap ($m \in \{1, ..., n\}$) protects its buyer against the m-th default in the basket between time $t = 0$ (today) and maturity $t = T > 0$ of the mBDS.

We denote by τ_j the random default time of asset S_j and set $\tau_j = \infty$ if S_j never defaults. The random variables $\tau_1, \ldots, \tau_n$ are positive and have a joint probability distribution $f(t_1, \ldots, t_n)$. For a tuple $\boldsymbol{t} = (t_1, \ldots, t_n)$ let $\iota_j(\boldsymbol{t}) \in \{1, \ldots, n\}$ be such that $t_{\iota_1(\boldsymbol{t})} \leq \ldots \leq t_{\iota_n(\boldsymbol{t})}$. If the tuple is clear from the context we omit the reference to it and just write ι_j for the index of the j-th smallest component of $\boldsymbol{t}$. Hence, the m-th default in the basket is τ_{ι_m} with $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_n)$ and the basket suffers a loss of $(1 - R_{\iota_m})N_{\iota_m}$ at time τ_{ι_m} of the m-th default.

2.1 Contingent payments

We distinguish two streams of payments of a mBDS, the *premium leg* which is paid by the buyer at regular intervals to the protection seller, and the *protection leg* which is paid by the seller in case of the m-th default.

Premium leg: We assume an annual premium payment of $s \cdot N$, where s is a spread upon which buyer and seller agreed in advanced. The premium is paid regularly only if less than m defaults have occurred, in case of the m-th default accured interest are paid till the time of default and then the payments stop. Assuming continuous compounding the present value of the mBDS premium leg for a given tupel $\boldsymbol{t} = (t_1, \ldots, t_n)$ of default times[1] is

$$\mathcal{Z}_{pre}^{(m)}(t_1, \ldots, t_n) := \begin{cases} \sum\limits_{t=1}^{\lfloor t_{\iota_m} \rfloor} sN \exp^{-r(t)t} + (t_{\iota_m} - \lfloor t_{\iota_m} \rfloor)\, sN \exp^{-r(t_{\iota_m})t_{\iota_m}} & t_{\iota_m} \leq T \\ \sum_{t=1}^{T} sN \exp^{-r(t)t} & \text{otherwise} \end{cases}. \tag{2.2}$$

Protection leg: The protection leg is paid only in case of the m-th default and compensates the buyer for the loss suffered due to the m-th default. Assuming continuous compounding the present value of the mBDS protection leg for a given tupel $\boldsymbol{t} = (t_1, \ldots, t_n)$ of default times is

$$\mathcal{Z}_{pro}^{(m)}(t_1, \ldots, t_n) := \begin{cases} (1 - R_{\iota_m})\, N_m \exp^{-r(t_{\iota_m})t_{\iota_m}} & t_{\iota_m} \leq T \\ 0 & \text{otherwise} \end{cases}. \tag{2.3}$$

Adaptions in Eqs. (2.2) and (2.3) for semi- or quarter-annual payments and simply compounding can easily be made.

[1] we distinguish explicitly in notation between random vector $\boldsymbol{\tau}$ and a realization $\boldsymbol{t}$ of $\boldsymbol{\tau}$

Definition 2.1. The value of a mBDS as seen from the seller is the expected difference between premium and protection leg over all possible outcomes of random default times $(\tau_1, \ldots, \tau_n)$,

$$V^{(m)} := \mathbb{E}\left(\mathcal{Z}^{(m)}_{pre}(\tau_1, \ldots, \tau_n) - \mathcal{Z}^{(m)}_{pro}(\tau_1, \ldots, \tau_n)\right). \tag{2.4}$$

The mBDS is signed at time $t = 0$ by both parties only if the spread s is chosen such that $V^{(m)} = 0$.

Definition 2.2. The mBDS fair spread $s_{m:n}$ is defined by

$$s_{m:n} := \frac{\mathbb{E}(\mathcal{Z}^{(m)}_{pro})}{\mathbb{E}(\tilde{\mathcal{Z}}^{(m)}_{pre})} \tag{2.5}$$

with

$$\tilde{\mathcal{Z}}^{(m)}_{pre}(t_1, \ldots, t_n) := \begin{cases} \sum\limits_{t=1}^{\lfloor t_{\iota_m} \rfloor} N \exp^{-r(t)t} + (t_{\iota_m} - \lfloor t_{\iota_m} \rfloor)\, N \exp^{-r(t_{\iota_m})t_{\iota_m}} & t_{\iota_m} \leq T \\ \sum_{t=1}^{T} N \exp^{-r(t)t} & \text{otherwise} \end{cases} > 0. \tag{2.6}$$

The expectations in Eqs. (2.4) and (2.5) can be expressed explicitly by means of the joint distribution density f of default times, we get

$$V^{(m)} = \int\limits_{\mathbb{R}^n} \left(\mathcal{Z}^{(m)}_{pre}(t_1, \ldots, t_n) - \mathcal{Z}^{(m)}_{pro}(t_1, \ldots, t_n)\right) f(t_1, \ldots, t_n) d\boldsymbol{t}$$

$$s_{m:n} = \frac{\int\limits_{\mathbb{R}^n} \mathcal{Z}^{(m)}_{pro}(t_1, \ldots, t_n) f(t_1, \ldots, t_n) d\boldsymbol{t}}{\int\limits_{\mathbb{R}^n} \tilde{\mathcal{Z}}^{(m)}_{pre}(t_1, \ldots, t_n) f(t_1, \ldots, t_n) d\boldsymbol{t}}.$$

2.2 Marginal default distributions

While the joint density distribution f of default times $(\tau_1, \ldots, \tau_n)$ cannot be estimated robustly by market data, the marginal density $f_i(t_i)$ of the random variable τ_i is typically extracted from data of the more liquid CDS and bond market. The market prices are used to build and calibrate a *hazard rate function* $h_i(t_i)$ and one assumes the following functional relation for marginal distribution $F_i(t_i)$ and density $f_i(t_i)$,

[8]

$$F_i(t_i) := \mathbb{P}(\tau_i \le t_i) = 1 - \exp\left(-\int\limits_0^{t_i} h_i(u)du\right) \tag{2.7}$$

$$f_i(t_i) = h_i(t_i)\exp\left(-\int\limits_0^{t_i} h_i(u)du\right). \tag{2.8}$$

2.3 Joint default distribution

The marginal distributions $F_i(t_i)$ are linked to a joint distribution function $F(t_1, \ldots, t_n)$ by a copula function $C(u_1, \ldots, u_n)$ such that the joint distribution is consistent with the observed marginal distributions. We define,

$$F(t_1, \ldots, t_n) = \mathbb{P}(\tau_1 \le t_1, \ldots, \tau_n \le t_n) := C(F_1(t_1), \ldots, F_n(t_n)) \tag{2.9}$$

with copula C which still has to be specified yet. This approach is justified by the famous Sklar's theorem, [25], which states under mild assumptions on regularity that each joint distribution F with marginals F_i is uniquely determined by a copula as in Eq. (2.9) and, vica versa that the function defined in Eq. (2.9) is a distribution with marginals F_i for any copula C.

In this sense, choosing a copula is a crucial part of modeling the basket and induces enormous risk on the valuing process. The question of *choosing the right copula* is a widely discussed topic.

Finally, the density of the joint distribution is

$$f(t_1, \ldots, t_n) = \frac{\partial^n F}{\partial t_1 \cdots \partial t_n} = \frac{\partial^n C}{\partial F_1 \cdots \partial F_n} \cdot \frac{\partial F_1}{\partial t_1} \cdots \frac{\partial F_n}{\partial t_n}. \tag{2.10}$$

2.4 Concordance and correlation

Let $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_n)$ be a vector of random default times which are distributed according to F in Eq. (2.9). The chosen copula in (2.9) is reflected in the correlation structure of the random vector $\boldsymbol{\tau}$. A typical way to measure the correlation between components is to use the linear correlation coefficients

$$\rho(\tau_i, \tau_j) := \frac{\mathrm{Cov}(\tau_i, \tau_j)}{\sqrt{\mathrm{Var}(\tau_i)}\sqrt{\mathrm{Var}(\tau_j)}} = \frac{\mathbb{E}\left((\tau_i - \mathbb{E}(\tau_i)) \cdot (\tau_j - \mathbb{E}(\tau_j))\right.}{\sqrt{\mathrm{Var}(\tau_i)}\sqrt{\mathrm{Var}(\tau_j)}}. \tag{2.11}$$

However, only linear relations between τ_i and τ_j can be grasped properly by $\rho(\tau_i, \tau_j)$. A more suitable approach is so called *Kendall's* τ^K which measures the concordant and discordant pairs of two random variables. For a pair (X, Y) of random variables it is defined by

$$\tau^K(X, Y) := \mathbb{P}((X - \tilde{X})(Y - \tilde{Y}) > 0) - \mathbb{P}((X - \tilde{X})(Y - \tilde{Y}) < 0), \tag{2.12}$$

where $(\tilde{X}, \tilde{Y})$ is an independent copy of (X, Y), [21]. The concordance measure τ^K is defined for every pair of continuous random variable, it is symmetric $\tau^K(X, Y) = \tau^K(Y, X)$ and satisfies $|\tau^K(X, Y)| \leq 1$, $\tau^K(X, X) = 1$, $\tau^K(X, -X) = -1$, and $\tau^K(X, Y) = 0$, if X, Y are independent, [21].

Definition 2.3. Let $F_i(t)$, $i = 1, \ldots, n$ be given marginal distributions and let C be a copula. Further, let $X = (X_1, \ldots, X_n)$ be a random vector with joint distribution $F(x_1, \ldots, x_n) = C(F_1(x_1), \ldots, F_n(x_n))$ as given in Eq. (2.9). Kendall's rank coefficients matrix $\mathcal{T}(C)$ is defined by

$$\mathcal{T}(C; F_1, \ldots, F_n) := \left(\tau^K(X_i, X_j)\right) \in \mathbb{R}^{n \times n} \tag{2.13}$$

and is just written as $\mathcal{T}(C)$ if the F_i are clear from the context. For any matrix $M \in \mathbb{R}^{n \times n}$ let $\mathcal{T}^{-1}(M)$ denote the set of copulas which are mapped by (2.13) on M. The matrix M must be symmetric, positive, semi-definite and has sup-norm $\|M\|_\infty \leq 1$ by the properties of Kendall's τ^K.

2.5 Model risk

We render the term model risk more precisely. We distinguish between three sources of model risk:

(i) Modeling the basket involves specifying the matrix $M = (\tau^K(\tau_i, \tau_j))$ and marginals F_i. For valuing the security a copula $C \in \mathcal{T}^{-1}(M)$ has to be chosen to specify the joint distribution. The uncertainty in this choice and consequently in the value of the security is part of the model risk.

(ii) The matrix M itself bears an enormous model risk as it determines the joint distribution as well.

(iii) Moreover, the mapping $\mathcal{T}(\cdot; F_1, \dots, F_n)$ depends on the marginals F_i and hence does the copula-set $\mathcal{T}^{-1}(M)$. In our intensity-based default model approach the marginal F_i is determined by the hazard rate functions $h_i(t)$. Therefore, the hazard rates have also to be accounted for the model risk.

3 Copula models

To understand the model risk in full detail it is necessary to compute the value of the basket derivative for all possible $C \in \mathcal{T}^{-1}(M)$ which is not feasible. However, it is possible to compute values for copulae $C \in \mathcal{C}(M) \subseteq \mathcal{T}^{-1}(M)$ in a finite subset $\mathcal{C}(M)$. The subset $\mathcal{C}(M)$ should be carefully chosen to approximate $\mathcal{T}^{-1}(M)$ well enough to mirror the risk behaviour of $\mathcal{T}^{-1}(M)$. Moreover, computational aspects have to be considered and efficient pricing algorithms must be provided.

Suitable, well-understood and widely used copulae in finance and risk management are in the class of Gaussian and Archimedean copulae, e.g. [6, 3]. We briefly repeat the most important facts of Gaussian and Archimedean copulae.

3.1 Gauss copula

The Gauss copula is based upon the Gaussian distribution and is defined by

$$C_R^{Gau}(u_1, ..., u_n) := \Phi^n_{\mathbf{0},R}(\Phi^{-1}_{0,1}(u_1), ..., \Phi^{-1}_{0,1}(u_n)) \tag{3.1}$$

where $\Phi^n_{\mathbf{0},R}$ is the multivariate Gaussian distribution with mean $\mathbf{0}$ and *correlation matrix* $R \in \mathbb{R}^{n \times n}$. Further, $\Phi^{-1}_{0,1}$ is the inverse of the uni-dimensional Gaussian distribution with mean 0 and standard deviation 1. From the correlation matrix $R = (R_{ij})$ one can compute easily Kendall's rank coefficients, [4],

$$\mathcal{T}(C_R) = \left(\frac{2}{\pi} \arcsin(R_{ij})\right)_{i,j=1,\dots,n}. \tag{3.2}$$

3.2 Archimedean copula

Let ϕ be a continuous, strictly decreasing function from $[0,1]$ to $[0,\infty]$ such that $\phi(0)=\infty$, $\phi(1)=0$ and the inverse ϕ^{-1} is completely monotone on $[0,\infty)$. Then, one can prove that $C:[0,1]^n\to[0,1]$ defined by

$$C_\phi(u_1,\dots,u_n) := \phi^{-1}(\phi(u_1)+\ldots+\phi(u_n)) \tag{3.3}$$

is a copula, [21]. The function ϕ is called the generator of C. Often one finds a family ϕ_θ of generator functions which define a family C_{ϕ_θ} of copula functions.

The associated Kendall's rank matrix $\mathcal{T}(C_\phi)$ of an Archimedean copula with generator ϕ is *homogeneous*, i.e. it has ones on the diagonal and all other entries are given by, [21],

$$\mathcal{T}(C_\phi)_{ij} = 1 + 4\int_0^1 \frac{\phi(t)}{\phi'(t)}\mathrm{d}t, \quad i\neq j. \tag{3.4}$$

Example 3.1. A well–known Archimedean copula is Clayton copula, [23, 19], which is generated by

$$\forall t\in[0,1] : \phi_\theta^{Clay}(t) := t^{-\theta}-1,\ \theta\in(0,\infty). \tag{3.5}$$

Thus, the Clayton class of copulae is defined by

$$\forall(u_1,...,u_n)\in[0,1]^n : C_\theta^{Clay}(u_1,...,u_n) := \phi_\theta^{Clay^{-1}}(\phi_\theta^{Clay}(u_1)+...+\phi_\theta^{Clay}(u_n)). \tag{3.6}$$

In the following we will use Clayton copula mostly in our numerical studies. Sampling algorithms for Clayton copula and other Archimedean copulae can be found e.g. in [20, 14].

4 Efficient valuation by importance sampling

Due to the complexity of basket securities analytical closed-form solutions for the pricing problem are only possible in special situations. For general model assumptions the expectations in Eq. (2.4) and Eq. (2.5) have to be computed by Monte-Carlo simulations.

For this one draws k different realizations $\left(t_1^{[i]}, ..., t_n^{[i]}\right),\ i \in \{1, ..., k\}$ of default times jointly distributed to the model implied distribution F and takes the expectation of the discounted payoff,

$$\mathbb{E}(\mathcal{Z}) = \frac{1}{k}\sum_{i=1}^{k} \mathcal{Z}\left(t_1^{[i]}, ..., t_n^{[i]}\right). \tag{4.1}$$

Unfortunately, Monte-Carlo simulation is not very efficient in many market-typical situations. The variance of the computed expectation might be large and decays too slowly for increasing k to give reliable results. To make Monte-Carlo more efficient it is necessary to reduce the variance significantly.

Before we discuss algorithms to reduce the variance we give an example which outlines the importance to study the set $\mathcal{T}^{-1}(M)$ and the implied prices.

Example 4.1. Consider the following basket, see Exhibit 6 in [18]. The basket consists of 5 assets each with nominal 1 and flat hazard rates $h_i(t_i) \equiv 0.1$. The recovery rates are 0, thus in case of a default a total loss is suffered in the particular position. The maturity of the contract is 2 years and we assume that the risk-free short-rate is constant 0.1 during this period.

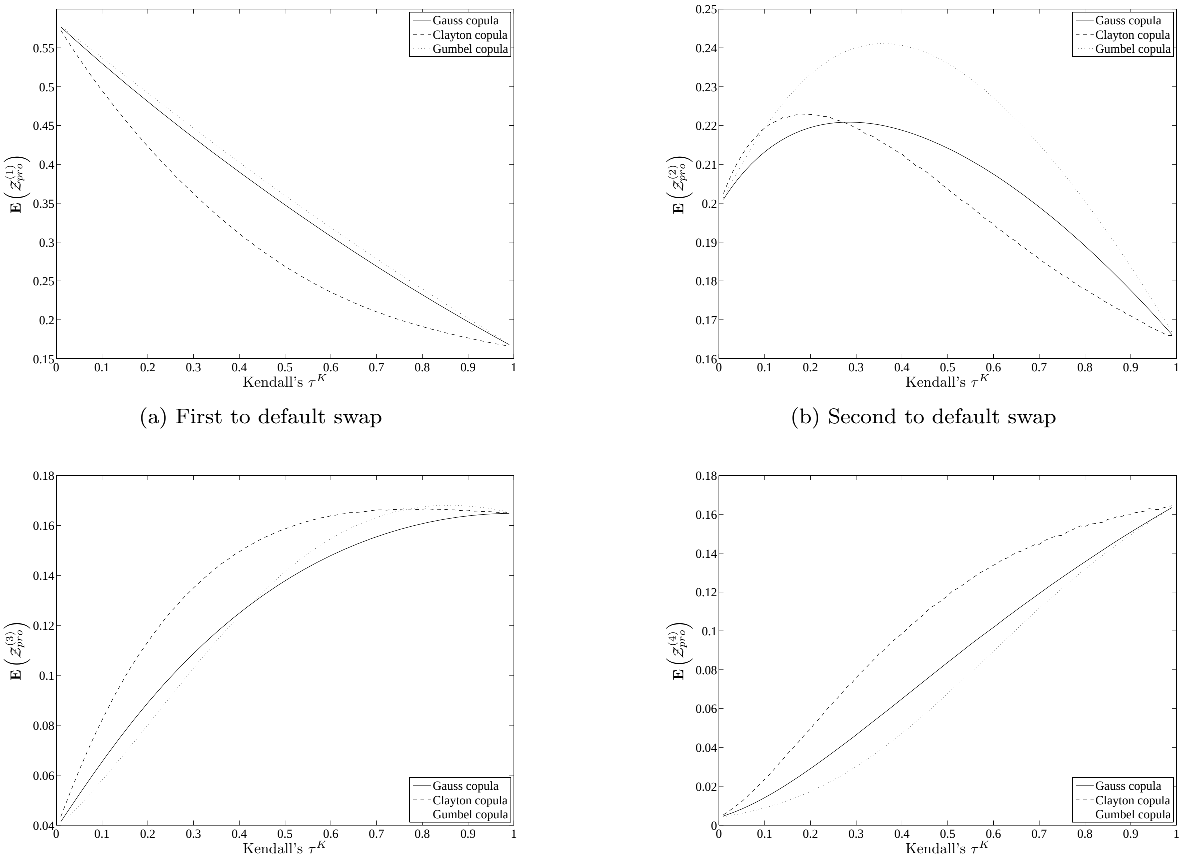
We assume a *homogeneous* concordance coefficient matrix

$$M(\tau^K) = \begin{pmatrix} 1 & \tau^K & \tau^K & \tau^K & \tau^K \\ \tau^K & 1 & \tau^K & \tau^K & \tau^K \\ \tau^K & \tau^K & 1 & \tau^K & \tau^K \\ \tau^K & \tau^K & \tau^K & 1 & \tau^K \\ \tau^K & \tau^K & \tau^K & \tau^K & 1 \end{pmatrix} \tag{4.2}$$

with parameter τ^K and compare different copulae $C \in \mathcal{T}^{-1}(M(\tau^K))$ for different values of τ^K. As copulae we chose a Clayton, a Gaussian and a Gumbel copula[2]. For $\tau^K \to 0$ and $\tau^K \to 1$ the value $\mathbb{E}(\mathcal{Z}_{pro})$ of the protection leg can be computed analytically, for other τ^K we have plotted $\mathbb{E}(\mathcal{Z}_{pro})$ in Fig. 1. We considered a first, a second and a third to default swap on the mentioned basket.

Observe that $\mathbb{E}(\mathcal{Z}_{pro})$ can vary substantially for fixed τ^K although all copula imply the same concordance structure in the basket. Moreover, the quantitative behaviour of each considered copula model changes for differing m. Depending on the copula defaults in the basket are more likely which is reflected in

[2]generator: $\phi_\theta(t) = (-\log t)^\theta, \theta \geq 1$



(a) First to default swap

(b) Second to default swap

(c) Third to default swap

(d) Fourth to default swap

Figure 1: The model risk is the uncertainty of the value due to the choice of copula to given concordance structure of the basket. Depending on the copula defaults in the basket a more likely which is reflected in the expected protection payment.

the expected protection payment. The price uncertainty due to the chosen copula vanishes only for the extreme cases $\tau^K = 0$ or $\tau^K = 1$. The choice of the copula determines the risk profile of the derivative and should be chosen advisedly and with respect to regulatory requirements.

4.1 Variance reduction for Gaussian densities by Joshi-Kainth and Chen-Glasserman

In this subsection we will introduce a technique for Gaussian implied default densities which reduces the variance of Monte-Carlo simulation by valuing artificially generated defaults.

The main source of increased variance for the calculation of $V^{(m)}$ or $s_{m:n}$ is the simulation of the protection leg $\mathbb{E}\left(\mathcal{Z}_{pro}^{(m)}\right)$. To see this, let $\boldsymbol{t}^{[i]} = \left(t_1^{[i]}, ..., t_n^{[i]}\right)$ be the realization of default times simulated in the i^{th} run of the Monte-Carlo loop. Obviously, only for $t_{\iota_m} < T$ we have $\mathcal{Z}_{pro}^{(m)}(\boldsymbol{t}^{[i]}) > 0$ and otherwise $\mathcal{Z}_{pro}^{(m)}(\boldsymbol{t}^{[i]}) = 0$. Typically, the number of runs with less than m defaults, i.e $\mathcal{Z}_{pro}^{(m)}(\boldsymbol{t}^{[i]}) = 0$ is much larger than the number of runs which triggered a protection payment, i. e. $\mathcal{Z}_{pro}^{(m)}(\boldsymbol{t}^{[i]}) > 0$. Consequently, the values of $\mathcal{Z}_{pro}^{(m)}(\boldsymbol{t}^{[i]}),\ i \in \{1, \ldots, k\}$ are very volatile and $\mathbb{E}\left(\mathcal{Z}_{pro}^{(m)}\right)$ has large variance.

Due to this fact it is essential to improve the calculation of $\mathbb{E}\left(\mathcal{Z}_{pro}^{(m)}\right)$ by applying variance reduction. A good working method for Gaussian copula implied joint default probabilities was developed by [17] and improved by [5]. The idea of this technique is a modification of the default probabilities, which shifts the defaults in *areas of importance* so that at least m defaults are forced in every Monte-Carlo run. Hence, the value $\mathcal{Z}_{pro}^{(m)}(\boldsymbol{t}^{[i]})$ of the protection leg will be strictly greater than zero for every $i \in \{1, \ldots, k\}$.

For the reader's convenience we briefly sum up variance reduction for Gaussian copula models. Let C_R^{Gau} be the model copula with correlation matrix R. We will generate realizations of default times $\boldsymbol{t}^{[i]} = \left(t_1^{[i]}, \ldots, t_n^{[i]}\right)$ with $t_{\iota_m}^{[i]} \leq T$ and correction factors $CF^{[i]}$ so that

$$\mathbb{E}(\mathcal{Z}_{pro}^{(m)}) = \lim_{k \to \infty} \frac{1}{k} \sum_{i=1}^{k} \mathcal{Z}_{pro}^{(m)} \left(t_1^{[i]}, \ldots, t_n^{[i]}\right) \cdot CF^{[i]}. \tag{4.3}$$

Following Joshi and Kainth let $AA^{tr} = R$ be the Cholesky decomposition of the correlation matrix R and let the tuple

$$(l_1, \ldots, l_n) := (F_1(T), \ldots, F_n(T)) \tag{4.4}$$

denote the *standardized border* of the mBDS.

For the i-th Monte-Carlo run we proceed as follows:
First, we draw a uniformly distributed tuple $(v_1, \ldots, v_n)$ and set $CF^{[i]} := 1$.
We will compute numbers z_j, x_j and $u_j^{[i]}$ for $j = 1, \ldots, n$. Recursively, assume that we have already considered $j-1$ of n assets ($j-1 \in \{1, \ldots, n-1\}$) and have observed k defaults within these $j-1$ assets. Further, we assume that we have computed the numbers $z_l, x_l, u_l^{[i]}$ for $l = 1, \ldots, j-1$.
If $k \geq m$ we have $t_{\iota_m}^{[i]} \leq T$ and we have already forced a contingent payment for this Monte-Carlo run.

In this case we set by the standard Gaussian copula algorithm, [18],

$$z_j := \Phi_{0,1}^{-1}(v_j), \quad x_j := (a_{j,1}, \ldots, a_{j,j}) \cdot (z_1, \ldots, z_{j-1}, z_j)^{tr}, \text{ and } \quad u_j^{[i]} := \Phi_{0,1}(x_j). \tag{4.5}$$

Otherwise, $k < m$ and we do the following: First, we calculate the conditional probability p of default of the j-th asset,

$$p = P\left(u_j^{[i]} \leq l_j | u_j^{[i]}, \ldots, u_{j-1}^{[i]}\right) = \Phi_{0,1}\left(\frac{\Phi_{0,1}^{-1}(l_j) - \sum_{l=1}^{j-1} z_l a_{j,l}}{a_{j,j}}\right), \tag{4.6}$$

given $u_l^{[i]}$, $l \in \{1, \ldots, j-1\}$. Secondly, we create a *synthetic* probability q of default. Joshi and Kainth suggested the use of

$$q := \frac{m-k}{n-j} \tag{4.7}$$

whereas Glasserman and Chen modified this choice to

$$q := \max\left\{\frac{m-k}{n-(j-1)}, p\right\}. \tag{4.8}$$

The probability q forces at least m default within n assets and guarantees the validity of $\boldsymbol{t}_{\iota_m}^{[i]} \leq T$. Thus, in the next step we have to test whether v_j lies in the synthetic area of default or not. If so, we have to shift v_j to the real default area, which implies that the corresponding $u_j^{[i]}$ will not be greater then l_j. Hence, if $v_j \leq q$ we modify v_j

$$v_j \leftarrow p \cdot \frac{v_j}{q} \tag{4.9}$$

and multiply the correction factor $CF^{[i]}$ by p/q. Otherwise, $v_j > q$ and we modify v_j by

$$v_j \leftarrow p \cdot \frac{(1-p)(v_j - q)}{1-q} \tag{4.10}$$

and multiply the correction factor $CF^{[i]}$ by $1 - p/1 - q$. Finally, we compute $z_j, x_j, u_j^{[i]}$ as in Eq. (4.5).

Summarizing, for each Monte-Carlo run i we get a correction factors $CF^{[i]}$, a copula output $\left(u_1^{[i]}, \ldots, u_n^{[i]}\right)$ and corresponding default times $\left(t_1^{[i]}, \ldots, t_n^{[i]}\right) := \left(F_1^{-1}\left(u_1^{[i]}\right), \ldots, F_n^{-1}\left(u_n^{[i]}\right)\right)$ which satisfy assumption $t_{\iota_m}^{[i]} \leq T$ as well as Eq. (4.3).

Remark. Although the effect of variance reduction to the simulation of $\mathbb{E}\left(\mathcal{Z}_{pro}^{(m)}\right)$ dominates within the calculation of $V^{(m)}$ or $s_{m:n}$ it is also possible to apply variance reduction to $\mathbb{E}\left(\mathcal{Z}_{pre}^{(m)}\right)$ or $\mathbb{E}\left(\tilde{\mathcal{Z}}_{pre}^{(m)}\right)$.

Equivalent to the simulation of $\mathbb{E}\left(\mathcal{Z}_{pro}^{(m)}\right)$ we have to handle the problem, that most of the runs within the Monte-Carlo simulation deliver the same value, which is no longer zero but $\sum\limits_{t=1}^{T} sN \exp^{-r(t)t}$ in this new context. Hence we have to *invert* the premium leg to

$$\mathcal{Z}_{pre}^{(m),*} := \sum_{t=1}^{T} sN \exp^{-r(t)t} - \mathcal{Z}_{pre}^{(m)}, \tag{4.11}$$

with the property

$$\mathcal{Z}_{pre}^{(m),*}(t_1, \ldots, t_n) \begin{cases} = 0 & t_{\iota_m} > T \\ \\ > 0 & t_{\iota_m} \leq T. \end{cases} \tag{4.12}$$

Now, we can apply above importance sampling to $\mathcal{Z}_{pre}^{(m),*}$ and will receive a value strictly greater than zero as well as a correction factor in every run. Finally, we get

$$\mathbb{E}\left(\mathcal{Z}_{pre}^{(m)}\right) = \sum_{t=1}^{T} sN \exp^{-r(t)t} - \mathbb{E}\left(\mathcal{Z}_{pre}^{(m,*)}\right). \tag{4.13}$$

Of course this technique is also suitable to the simulation of $\mathbb{E}\left(\tilde{\mathcal{Z}}_{pre}^{(m)}\right)$.

4.2 Variance reduction for Archimedean densities

We will generalize and employ the above variance reduction technique to Archimedean densities.

For given marginals $F_1, \ldots, F_n$ let $f_{Arc}(t_1, \ldots, t_n)$ be the joint density of default times $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_n)$ implied by an Archimedean copula and $f_{Gau}(t_1, \ldots, t_n)$ the density implied by a Gaussian copula with correlation matrix R which has to be specified yet. The idea is to use default times $\boldsymbol{\tau}^* = (\tau_1^*, \ldots, \tau_n^*)$ with Gaussian copula implied density to compute the expectation $\mathbb{E}(\mathcal{Z}(\tau_1, \ldots, \tau_n))$.

We have,

$$
\begin{aligned}
\mathbb{E}(\mathcal{Z}(\tau_1,\ldots,\tau_n)) &= \int\limits_{\mathbb{R}^n} \mathcal{Z}\left(t_1,\ldots,t_n\right)\cdot f_{Arc}\left(t_1,\ldots,t_n\right)\mathrm{d}\boldsymbol{t} \\
&= \int\limits_{\mathbb{R}^n} \mathcal{Z}\left(t_1,\ldots,t_n\right)\cdot \frac{f_{Arc}\left(t_1,\ldots,t_n\right)}{f_{Gau}\left(t_1,\ldots,t_n\right)} f_{Gau}\left(t_1,\ldots,t_n\right)\mathrm{d}\boldsymbol{t} \\
&=: \int\limits_{\mathbb{R}^n} \mathcal{Z}^*\left(t_1,\ldots,t_n\right)\cdot f_{Gau}\left(t_1,\ldots,t_n\right)\mathrm{d}\boldsymbol{t} = \mathbb{E}(\mathcal{Z}^*(\tau_1^*,\ldots,\tau_n^*)). \qquad (4.14)
\end{aligned}
$$

The expectation in Eq. (4.14) can be computed efficiently by variance reduction as in the previous section. A complete algorithm is given in Algo. 1. However, the performance depends strongly on the approximating density f_{Gau}. Because we have not specified the correlation matrix R yet we can use R to optimize the performance of the variance reduction technique.

4.2.1 Density functions

First, we compute f_{Gau} and f_{Arc} explicitly. By Eq. (3.1) we get for the density of a Gaussian copula,

$$
c_R^{Gau}\left(u_1,...,u_n\right) := \frac{\partial^n \left[C_R^{Gau}\left(u_1,...,u_n\right)\right]}{\partial u_1\cdots\partial u_n} = \frac{1}{\sqrt{|\det R|}}\exp\left(-\frac{1}{2}\eta^{tr}\left(R^{-1}-I_n\right)\eta\right) \qquad (4.15)
$$

with $\eta^T = \left(\Phi_{0,1}^{-1}\left(u_1\right),...,\Phi_{0,1}^{-1}\left(u_n\right)\right)$ and I_n the n dimensional identity matrix. Thus, Eq. (2.9) and the chain rule yield

$$
f_{Gau}\left(t_1,\ldots,t_n\right) = \frac{\partial^n\left[C_R^{Gau}\left(F_1(t_1),\ldots,F_n(t_n)\right)\right]}{\partial t_1\cdots\partial t_n} = c_R^{Gau}\left(F_1(t_1),\ldots,F_n(t_n)\right)\cdot\prod_{i=1}^n \frac{\mathrm{d}F_i(t_i)}{\mathrm{d}t_i} \qquad (4.16)
$$

Analogously, we get from Eq. (3.3) the density of an Archimedean copula generated by ϕ,

$$
c_\phi^{Arc}(u_1,\ldots,u_n) = \frac{\partial^n\left[C_\phi\left(u_1,\ldots,u_n\right)\right]}{\partial u_1\cdots\partial u_n} = \frac{d^n[\phi_\theta^{-1}]}{dt^n}(\phi_\theta(u_1)+\ldots+\phi_\theta(u_n))\cdot\prod_{i=1}^n\frac{\mathrm{d}\left[\phi(u_i)\right]}{\mathrm{d}u_i}, \qquad (4.17)
$$

and further

$$
f_{Arc}(t_1,\ldots,t_n) = c_\phi^{Arc}(F_1(t_1),\ldots,F_n(u_n))\cdot\prod_{i=1}^n\frac{\mathrm{d}F_i\left(t_i\right)}{\mathrm{d}t_i}. \qquad (4.18)
$$

Algorithm 1 Calculation $\mathbb{E}_{Arc}(\mathcal{Z})$ via variance reduction

Initialize $\Lambda \leftarrow 0$ and the Cholesky composition $AA^T = R$.
Calculate $(l_1, \ldots, l_n) \leftarrow (F_1(T), \ldots, F_n(T))$
for $y = 1$ to N **do**
  Initialize $k \leftarrow 0$ and $CF \leftarrow 1$.
  Sample $(v_1, \ldots, v_n) \sim \mathcal{U}[0,1]$.
  for $i = 1$ to n **do**
    if $k < m$ **then**
      $p \leftarrow \Phi_{0,1}\left(\frac{\Phi_{0,1}^{-1}(l_i) - \sum_{j=1}^{i-1} z_j a_{i,j}}{a_{i,i}}\right)$
      $q \leftarrow \max\left\{\frac{m-k}{n-(i-1)}, p\right\}$
      if $v_i \leq q$ **then**
        $v_i \leftarrow p \cdot v_i / q$
        $CF \leftarrow CF \cdot p/q$
        $k \leftarrow k + 1$
      else
        $v_i \leftarrow p \cdot \frac{(1-p)(v_i - q)}{1-q}$
        $CF \leftarrow CF \cdot (1-p)/(1-q)$
      end if
    end if
    $z_i \leftarrow \Phi_{0,1}^{-1}(v_i)$
    $x_i \leftarrow \sum_{j=1}^{j=i} a_{i,j} \cdot z_i$
    $u_i \leftarrow \Phi_{0,1}(x_i)$
    $t_i \leftarrow F_i^{-1}(u_i)$
  end for
  $\Lambda \leftarrow \Lambda + \mathcal{Z}(t_1, \ldots, t_n) \cdot CF \cdot \frac{f_{Arc}(t_1,\ldots,t_n)}{f_{Gau}(t_1,\ldots,t_n)}$
end for
return Λ/N

The density f_{Gau} depends implicitly on the matrix R. To avoid an extreme and volatile quotient $\frac{f_{Arc}}{f_{Gau}}$ in Eq. (4.14) we determine R and thus f_{Gau} by the condition

$$\min_{R\in\mathcal{R}} \int_{\mathbb{R}^n} \left| \frac{f_{Arc}(t_1,\dots,t_n)}{f_{Gau}(t_1,\dots,t_n)} - 1 \right| \cdot f_{Gau}(t_1,\dots,t_n)\mathrm{d}\boldsymbol{t} = \min_{R\in\mathcal{R}} \|f_{Arc} - f_{Gau}\|_{L^1} \tag{4.19}$$

with $\mathcal{R} = \{R \in \mathbb{R}^{n\times n} | \|R\|_\infty \leq 1, R\,\mathrm{sym.}, \mathrm{pos.\text{-}semi\ definite}\}$. This guarantees that the functional behaviour of f_{Arc} is reasonable well captured by f_{Gau}.

By means of Eqs. (4.16) and (4.18) we find

$$\begin{aligned} \|f_{Arc} - f_{Gau}\|_{L^1} &= \int_{\mathbb{R}^n} |c_\phi^{Arc}(F_1(t_1),\dots,F_n(u_n)) - c_R^{Gau}\,(F_1(t_1),\dots,F_n(t_n))\,| \cdot \prod_{i=1}^n \frac{\mathrm{d}F_i\,(t_i)}{\mathrm{d}t_i}\mathrm{d}\boldsymbol{t} \\ &= \int_{[0,1]^n} |c_\phi^{Arc}(u_1,\dots,u_n) - c_R^{Gau}\,(u_1,\dots,u_n)\,|\mathrm{d}\boldsymbol{u}, \end{aligned}$$

where we use the co-ordinate transformation $u_i := F_i(t_i)$ to compute the integral. Hence, the norm $\|f_{Arc} - f_{Gau}\|_{L^1}$ is independent from the marginal distributions. Unfortunately, the integrand might be very involved and the integral cannot be computed analytically in most practical relevant situations so that one has to use numerical quadrature. The integral can be approximated by sparse grid quadrature with sufficient accuracy even in higher dimensions, [11].

Example 4.2. The generator of Clayton copula is $\phi_\theta(t) = t^{-\theta} - 1$ with free parameter $\theta \in (0,\infty)$. Hence, we have $\phi_\theta^{-1} = (t+1)^{-1/\theta}$, $\phi_\theta' = -\theta t^{-\theta-1}$ and $\frac{d^n[\phi_\theta^{-1}]}{dt^n} = (-1)^n \prod_{i=1}^n \left(\frac{1}{\theta} + (i-1)\right) \cdot (t+1)^{-1/\theta - n}$. Then, Eq. (4.17) yields

$$c_\phi^{Clay}\,(u_1,...,u_n) = \theta^n \cdot \prod_{i=1}^n \left(\left(\frac{1}{\theta} + (i-1) \right) u_i^{-\theta-1} \right) \cdot \left(\sum_{i=1}^n u_i^{-\theta} - n + 1 \right)^{-\frac{1}{\theta}-n}. \tag{4.20}$$

For the optimization we restrict R to be an homogeneous matrix, i.e. ones on the diagonal and a constant entry ρ on all other entries so that Eq. (4.19) becomes a one-dimensional optimization problem. For fixed parameter θ we compute Kendall's τ^K by Eq. (3.4) of the Clayton copula and find the optimal R by numerical optimization. The results are plotted in Fig. 2 for several dimensions. We see that the

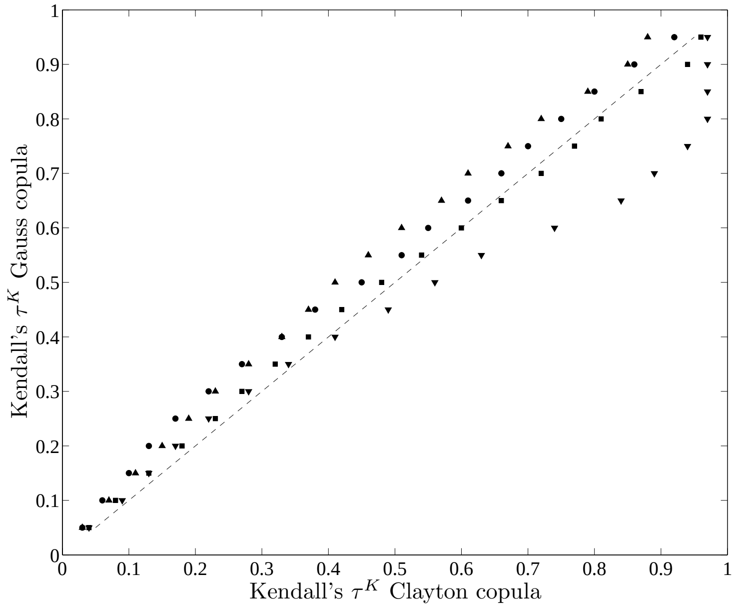


Figure 2

concordance of the Archimedean and Gaussian copula should match, which suggests by Eq. (3.2)

$$\rho = \sin\left(\frac{\pi \tau^K}{2}\right)$$

as sufficiently good approximation for the optimal value.

4.3 Numerical results

The performance of the variance reduction depends mainly on a good choice of the approximating Gaussian copula and we have seen that this is independent from the marginal distributions. For a good performance it is necessary that the default event triggering the contingent protection payment is a rare event and occurs only for very few Monte-Carlo simulations. In this case the introduction of synthetic probabilities in Eq. (4.8) leads to larger number of Monte-Carlo paths triggering a payment and the variance is reduced. On the contrary, if the basket is very risky and a triggering default event is highly probable the use of synthetic probabilities is redundant and might worsen the variance.

There are several model factors which influence the riskiness of the basket. These are

- the hazard rates – larger values $h_i(t)$ correspond to a more probable default of asset S_i,
- the maturity – for longer maturities it is more probable that a triggering default event occurs,
- the concordance structure – simultaneous default events are more probable in a highly concordant basket leading to a more probable triggering event,
- the number of assets – in a larger basket it is more probable that reference assets default implying a larger probability of a triggering event.

We study the performance of the algorithm with respect to these four factors in the following numerical example.

Example 4.3. We consider again the basic setup of Ex. 4.1, a homogeneous concordant basket of $n = 5$ obligors with homogeneous flat hazard rates $h :\equiv h_i(t)$. We assume that the concordance structure of the basket is determined by a Clayton copula with $\tau^K = 0.2$. For all numerical experiments we have chosen a second to default swap, hence $m = 2$, and compute the variance of the discounted expected protection payment without and with our variance reduction.

For the numerical experiments we fix three of the above model factors (h, T, τ^K or n) and vary the remaining factors. The numerical results are collected in Tab. 1. The first column shows the number of Monte-Carlo simulations, then we compare the variance without and with variance reduction. The exact value of the expected protection payment of the corresponding StD is printed in the top line.

Summarizing, the variance is significantly reduced for typical market situations by our approach. However, there are settings for which variance reduction is not applicable or performs unsatisfactorily. This is the case if the basket is very risky and a triggering default event is highly probable.

5 Quantifying model risk

We have seen that the model risk has its origin from three different sources of model uncertainty leading to uncertainty in the computed derivative price. We provided efficient numerical methods to compute prices for different copula models in Sect. 4 which can be used to quantify efficiently the uncertainty due to the choice of copula model. We discussed this in Example 4.1.

N	$h = 0.02$ ex. value: 0.042448 plain	with VR	$h = 0.06$ ex. value: 0.132742 plain	with VR
10^3	5.25E-05	3.20E-06	5.94E-05	8.63E-06
10^4	3.05E-06	1.14E-07	4.33E-06	3.62E-07
10^5	3.70E-07	3.53E-08	2.65E-07	1.31E-07
10^6	4.59E-08	4.62E-09	3.18E-08	1.33E-08

(a) Variable hazard rate

N	$T = 0.8$ ex. value: 0.092658 plain	with VR	$T = 1.6$ ex. value: 0.181625 plain	with VR
10^3	8.11E-05	6.54E-06	1.97E-04	1.02E-05
10^4	6.38E-06	2.73E-07	6.84E-06	4.86E-07
10^5	4.82E-07	8.64E-08	1.64E-06	1.70E-07
10^6	1.40E-08	9.33E-09	2.05E-07	1.69E-08

(b) Variable maturity

N	$\tau^K = 0.1$ ex. value: 0.219189 plain	with VR	$\tau^K = 0.5$ ex. value: 0.203938 plain	with VR
10^3	9.75E-05	3.56E-06	6.11E-05	6.24E-05
10^4	2.46E-05	3.28E-07	4.76E-06	2.44E-05
10^5	1.26E-06	5.33E-08	2.06E-06	4.02E-07
10^6	9.43E-08	7.70E-09	1.68E-07	2.37E-07

(c) Variable concordance

N	$n = 4$ ex. value: 0.17904 plain	with VR	$n = 8$ ex. value: 0.320403 plain	with VR
10^3	1.42E-04	3.22E-06	1.13E-04	7.22E-05
10^4	1.12E-05	4.14E-07	3.52E-05	9.99E-06
10^5	1.67E-06	1.63E-08	1.05E-06	1.03E-06
10^6	1.25E-07	3.28E-09	1.12E-07	2.15E-07

(d) Variable asset number

Table 1: Performance of the variance reduction for different model factor situations and a StD. In (a) we keep $T = 2, \tau^K = 0.2, n = 5$ while changing h. In (b) we keep $h = 0.1, \tau^K = 0.2, n = 5$ while changing T. In (c) we have $h = 0.1, T = 2, n = 5$ and change τ^K and finally, in (d) we keep $h = 0.1, T = 2, \tau^K = 0.2$ and change n.

In the following we will discuss the application of variance reduction to obtain efficient algorithms for determining the sensitivity of the basket derivative with respect to the underlying concordance structure and the hazard rates of the marginal distributions.

5.1 Changes in concordance structure M

Let $M = (m_{ij})_{i,j=1,\dots,n}$ be the concordance structure of the basket and let $V(M) := \mathbb{E}(\mathcal{Z}; M)$ the expected value of a discounted payoff as in Eq. (2.3) or Eq. (2.5), for example, where we have denoted the dependence of the expectation on M explicitly.

The uncertainty due to the entry m_{ij} can be quantified as the derivative of $V(M)$ with respect to entry m_{ij}. Denote by $E_{ij} \in \mathbb{R}^{n \times n}$ the matrix which has only zeros except for a one at entry (i, j). Then,

$$\frac{\partial V}{\partial m_{ij}} = \lim_{h \to 0} \frac{\mathbb{E}(\mathcal{Z}; M + h \cdot E_{ij}) - \mathbb{E}(\mathcal{Z}; M - h \cdot E_{ij})}{2h}. \tag{5.1}$$

To exploit this formula for numerical approximation we have to make sure that the variance in the

	$m=1$		$m=2$		$m=3$		$m=4$	
	ex. value: -0.661250		ex. value: -0.006590		ex. value: 0.260219		ex. value: 0.266428	
N	plain	with VR	plain	with VR	plain	with VR	plain	with VR
10^4	2.30E-04	2.10E-04	2.24E-04	6.45E-05	9.41E-05	4.83E-05	9.92E-05	2.47E-05
10^5	8.40E-05	2.42E-05	5.46E-05	1.33E-05	2.66E-05	7.89E-06	1.57E-05	4.04E-06
10^6	1.65E-05	2.89E-06	1.34E-05	1.79E-06	9.87E-06	1.01E-06	3.36E-06	4.19E-07

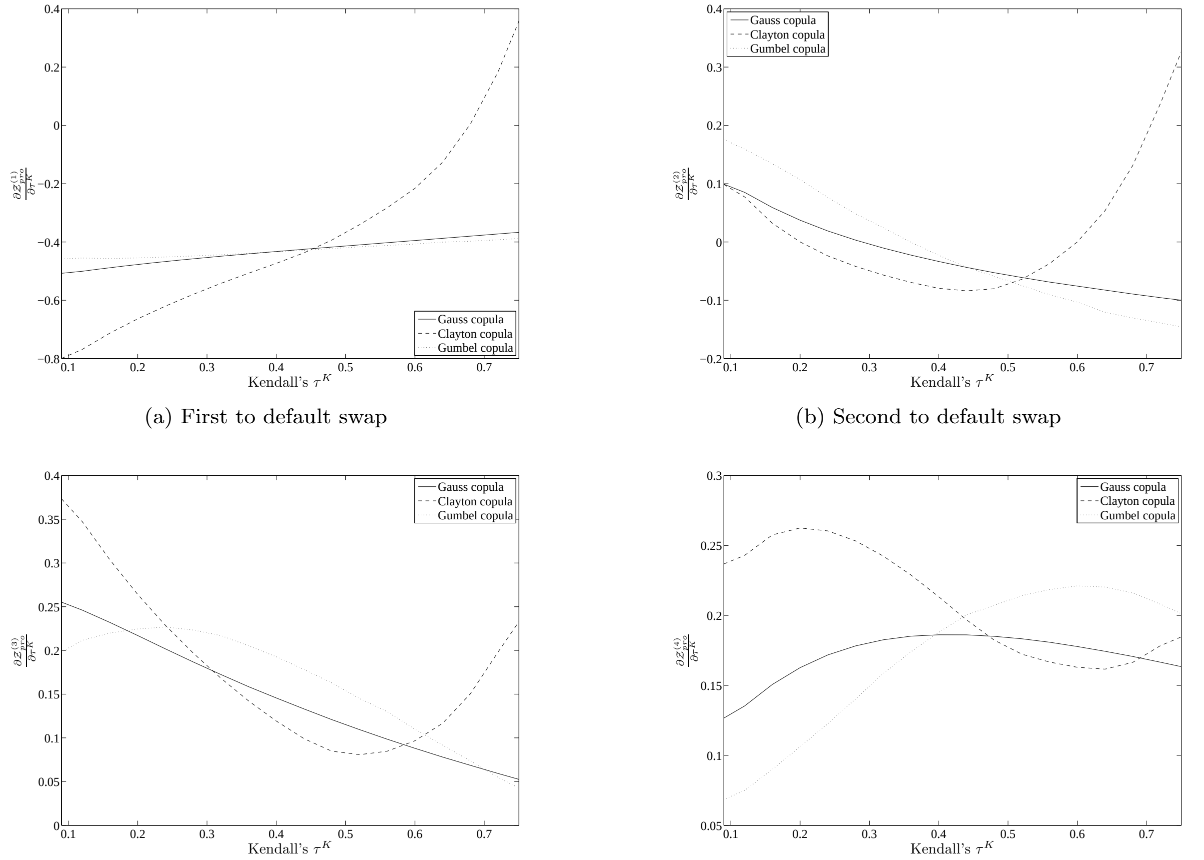
Table 2: Analysis of the variance of $\frac{\partial \mathbb{E}(\mathcal{Z}_{pro}^{(m)})}{\partial \tau^K}$ for different m. Variance reduction performs significantly better than plain Monte-Carlo simulation for large m.

computation of $\mathbb{E}(\mathcal{Z}; M + h \cdot E_{ij})$ respectively $\mathbb{E}(\mathcal{Z}; M - h \cdot E_{ij})$ does not spoil the finite difference approximation which happens if the variance is substantially larger than the chosen h, which is discussed generally for Monte-Carlo simulations in [12]. Neglecting leading coefficients it is shown that the variance is of order $\mathcal{O}\left(N^{-1}\right)$ whereas the optimal step size h_N is of order $\mathcal{O}\left(N^{-1/5}\right)$ with N denoting the number of simulations used in the calculation of $\mathbb{E}\left(\mathcal{Z}; M \pm h \cdot E_{ij}\right)$. Applying common random numbers in simulating $\mathbb{E}\left(\mathcal{Z}; M + h \cdot E_{ij}\right)$ and $\mathbb{E}\left(\mathcal{Z}; M - h \cdot E_{ij}\right)$ the finite difference are appropriate here and we can approximate $\frac{\partial V}{\partial m_{ij}}$ by Eq. (5.1) numerically setting $h = N^{-1/5}$.

To study the model sensitivity we return to Example 4.1.

Example 5.1. Hence, we assume a homogeneous concordance structure M with constant parameter τ^K, see Eq. (4.2), and compute the expected protection payment $V(M) = V(\tau^K) = \mathbb{E}(\mathcal{Z}_{pro}^{(m)}; \tau^K)$ for different m-th to default swaps assuming a Clayton copula implied concordance structure. Now, we use variance reduction together with the finite difference approximation (5.1) to approximate the sensitivity $\frac{\partial V}{\partial \tau^K} = \lim\limits_{h \to 0} \frac{\mathbb{E}(\mathcal{Z}_{pro}^{(m)}; \tau^K + h) - \mathbb{E}(\mathcal{Z}_{pro}^{(m)}; \tau^K - h)}{2h}$ as measure for the model risk due to uncertainty in the given concordance structure. Fig. 3 shows the sensitivity $\frac{\partial V}{\partial \tau^K}$ for a first, a second, a third and a fourth to default swap.

To test the performance of variance reduction we carried out the following numerical experiment. The basic setup was the above portfolio of 5 assets with given homogeneous concordance structure $\tau^K = 0.2$, maturity $T = 2$ and flat hazard rates $h_i = 0.1$. In Tab. 2 we summarize the performance results for a first, second, third and fourth to default swap written on this basket. The first column contains the number of Monte-Carlo simulations and the other columns compare the variance of the expected protection payment for each basket. For increasing m, i. e. a xess probable protection payment, variance reduction performs significantly better than plain simulation as expected by the general theory outlined in Sect. 4.1.



(a) First to default swap

(b) Second to default swap

(c) Third to default swap

(d) Fourth to default swap

Figure 3: The sensitivity $\frac{\partial \mathcal{Z}_{pro}^{(m)}}{\partial \tau^K}$ of the expected protection leg with respect to the concordance structure for a first, second, third and fourth to default swap in Ex. 5.1.

5.2 Changes in the hazard rate

Last, we discuss model risk due to uncertainty in the hazard rates $h_i(t)$. For that we use a suitable parameterization of $h_i(t)$ to avoid cumbersome mathematical technicalities. Thus, assume that each $h_i(t)$ can be written as

$$h_i(t) = \sum_{j=1}^{J} h_i^{(j)} \psi_j(t) \tag{5.2}$$

with basis functions $\psi_j(t) : [0, T] \to \mathbb{R}, j = 1, \ldots, J$ and defining parameters $h_i^{(j)} \in \mathbb{R}$ for each hazard rate $h_i(t)$, $i = 1, \ldots, n$. The integral in Eq. (2.7) is then parameterized by

$$\int_0^t h_i(u)du = \sum_{j=1}^{J} h_i^{(j)} \int_0^t \psi_j(u)du =: \sum_{j=1}^{J} h_i^{(j)} \Psi_j(t). \tag{5.3}$$

Let $V(h_1, \ldots, h_n) := \mathbb{E}(\mathcal{Z}; h_1, \ldots, h_n)$ be the expected value of a discounted payoff, where we have denoted the dependence of the expectation on h_i explicitly. More precisely, due to the parameterization we can write $V(h_i^{(j)}|i = 1, \ldots, n, j = 1, \ldots, J) = V(\boldsymbol{h})$ to denote the dependence on the defining parameters $h_i^{(j)}$.

To compute the sensitivities $\frac{\partial V}{\partial h_i^{(j)}}$ we apply the likelihood ratio method, see [12], and interchange differation and integration. Let $f(t_1, \ldots, t_n; \boldsymbol{h})$ be the joint default probability density, then we have for $i = 1, \ldots, n, j = 1, \ldots, J$

$$\begin{aligned}
\frac{\partial V}{\partial h_i^{(j)}} &= \frac{\partial}{\partial h_i^{(j)}} \mathbb{E}(\mathcal{Z}) = \frac{\partial}{\partial h_i^{(j)}} \int_{\mathbb{R}^n} \mathcal{Z}(t_1, \ldots, t_n) f(t_1, \ldots, t_n; \boldsymbol{h}) \mathrm{d}\boldsymbol{t} \\
&= \int_{\mathbb{R}^n} \mathcal{Z}(t_1, \ldots, t_n) \frac{\partial}{\partial h_i^{(j)}} f(t_1, \ldots, t_n; \boldsymbol{h}) \frac{f(t_1, \ldots, t_n; \boldsymbol{h})}{f(t_1, \ldots, t_n; \boldsymbol{h})} \mathrm{d}\boldsymbol{t} \\
&= \int_{\mathbb{R}^n} \mathcal{Z}(t_1, \ldots, t_n) \frac{\partial(\log f)}{\partial h_i^{(j)}} f(t_1, \ldots, t_n; \boldsymbol{h}) \mathrm{d}\boldsymbol{t}
\end{aligned} \tag{5.4}$$

From Eqs. (4.16) and (4.18) we know that the density has the functional form

$$f = c(F_1(t_1), \ldots, F_n(u_n)) \cdot \prod_{i=1}^{n} \frac{\mathrm{d}F_i\,(t_i)}{\mathrm{d}t_i},$$

where $c(u_1, \ldots, u_n) := \frac{\partial^n C(u_1, \ldots, u_n)}{\partial u_1 \cdots \partial u_n}$ is the density of the chosen copula. This yields,

$$\log f = \log c(F_1(t_1), \ldots, F_n(u_n)) + \sum_{i=1}^{n} \log \frac{\mathrm{d}F_i\,(t_i)}{\mathrm{d}t_i}, \tag{5.5}$$

and further,

$$\frac{\partial(\log f)}{\partial h_i^{(j)}} = \frac{1}{c} \cdot \frac{\partial c}{\partial F_i} \frac{\partial F_i}{\partial h_i^{(j)}} + \frac{\partial^2 F_i}{\partial t_i \partial h_i^{(j)}} \Big/ \frac{\partial F_i}{\partial t_i}. \tag{5.6}$$

Assuming the functional form $F_i(t_i) = 1 - \exp\left(-\int_0^{t_i} h_i(u)du\right)$ of the marginals and the parameterization

(5.2) we can explicitly express,

$$\begin{aligned}
\frac{\partial F_i}{\partial h_i^{(j)}} &= \Psi_j(t) \exp\left(-\sum_{j=1}^{J} h_i^{(j)} \Psi_j(t)\right), \\
\frac{\partial F_i}{\partial t_i} &= \left(\sum_{j=1}^{J} h_i^{(j)} \psi_j(t)\right) \cdot \exp\left(-\sum_{j=1}^{J} h_i^{(j)} \Psi_j(t)\right), \\
\frac{\partial^2 F_i}{\partial t_i \partial h_i^{(j)}} &= \left(\psi_j(t) - h_i^{(j)} \Psi_j(t)\right) \cdot \exp\left(-\sum_{j=1}^{J} h_i^{(j)} \Psi_j(t)\right)
\end{aligned} \tag{5.7}$$

in terms of the basis functions, their integrals and the defining parameters. Further, Eq. (5.4) allows for an application of variance reduction method so that the sensitivities $\frac{\partial V}{\partial h_i^{(j)}}$ can be evaluated efficiently.

Next, we discuss sets of basis functions which are most suitable for typical practical relevant situations including flat, piecewise-constant and linear piecewise continuous hazard rates. Let

$$\mathcal{T} = \{0 = T_0 < T_1 < \ldots < T_J = T\}$$

be a partition of the time interval $[0, T]$.

Flat hazard rates. In this situation one assumes that the hazard rates $h_i(t) = h_i^{(1)}$ are constant till maturity. We have $J = 1$ and the single basis function $\psi_1(t) \equiv 1$ with $\Psi_1(t) = t$.

Piecewise-constant hazard rates. In this situation the hazard rates $h_i(t)$ are constant on the intervals $[T_{j-1}, T_j)$ with value $h_i(t) = h_i^{(j)}$ for $t \in [T_j, T_{j+1})$ and possible jumps at $t = T_j$. As basis functions we take the indicator functions of the subintervals $[T_{j-1}, T_j)$,

$$\psi_j(t) = \begin{cases} 1 & : \quad T_{j-1} \le t < T_j \\ 0 & : \quad \text{otherwise} \end{cases}, \qquad \Psi_j(t) = \begin{cases} 0 & : \quad t < T_{j-1} \\ t - T_{j-1} & : \quad T_{j-1} \le t < T_j \\ T_j - T_{j-1} & : \quad T_j \le t \end{cases}. \tag{5.8}$$

Linear, piecewise-continuous hazard rates. In this situations we parameterize the hazard rates $h_i(t)$ by

$$h_i(t) = \sum_{j=0}^{J} h_i^{(j)} \psi_j(t)$$

which are continuous on $[0,T]$ and linear on the intervals $[T_{j-1},T_j)$ with values $h_i(T_j)=h_i^{(j)}$. As basis functions we take the so-called *hat-functions*, [24]. These are for $j=1,\dots,J-1$

$$\psi_j(t)=\begin{cases}\frac{t-T_{j-1}}{T_j-T_{j-1}} & : \ T_{j-1}\le t<T_j\\ \frac{T_{j+1}-t}{T_{j+1}-T_j} & : \ T_j\le t<T_{j+1}\\ 0 & : \ \text{otherwise}\end{cases},\ \Psi_j(t)=\begin{cases}0 & : \ t<T_{j-1}\\ \frac{(t-T_{j-1})^2}{2(T_j-T_{j-1})} & : \ T_{j-1}\le t<T_j\\ \frac{T_j-T_{j-1}}{2}+\frac{(t-T_j)(2T_{j+1}-(T_j-t))}{2(T_{j+1}-T_j)} & : \ T_j\le t<T_{j+1}\\ \frac{T_{j+1}-T_{j-1}}{2} & : \ T_{j+1}\le t\end{cases}\tag{5.9}$$

and at the left boundary

$$\psi_0(t)=\begin{cases}\frac{T_1-t}{T_1} & : \ 0\le t<T_1\\ 0 & : \ \text{elsewhere}\end{cases},\qquad \Psi_0(t)=\begin{cases}\frac{2T_1t-t^2}{2T_1} & : \ 0\le t<T_1\\ \frac{T_1}{2} & : \ T_1\le t\end{cases}\tag{5.10}$$

respectively at the right boundary,

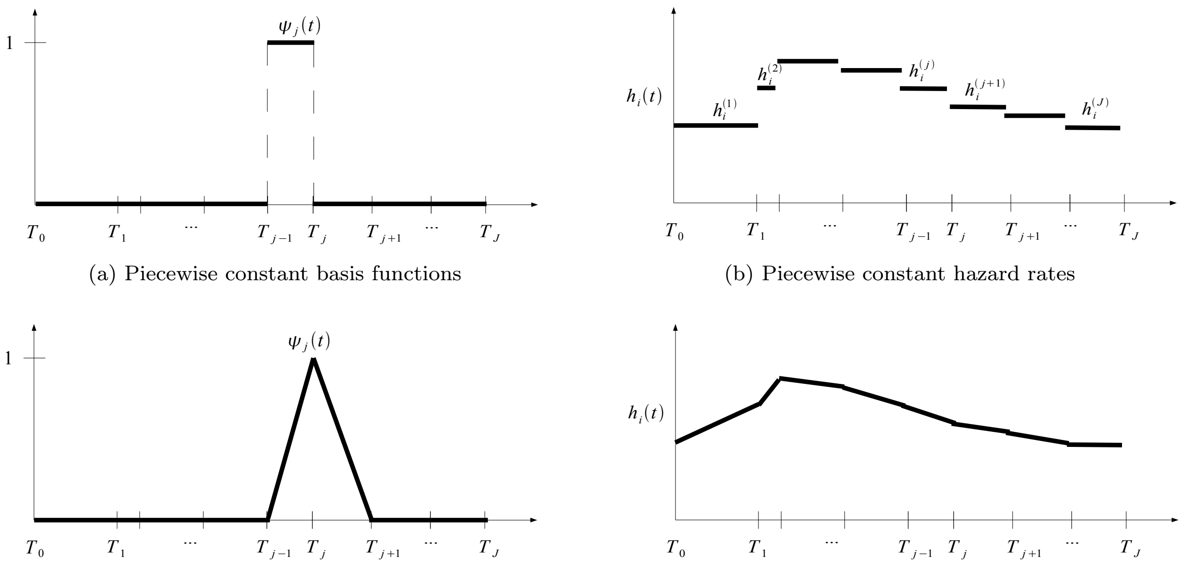
$$\psi_J(t)=\begin{cases}\frac{t-T_J}{T_J-T_{J-1}} & : \ T_{J-1}\le t\le T_J\\ 0 & : \ \text{elsewhere}\end{cases},\qquad \Psi_J(t)=\begin{cases}\frac{t^2-2T_Jt}{2(T_J-T_{J-1})} & : \ T_{J-1}\le t\le T_J\\ 0 & : \ \text{elsewhere}\end{cases}.\tag{5.11}$$

In Fig. 4 we summarized the basis functions and show possible hazard rates. Our parameterization can be generalized to even more smooth basis functions by B-splines, [7] and consequently smoother hazard rates. However, formulas get much more involved and the gain of smoothness is small compared to the induced model error.

We demonstrate the efficiency of our approach and compute the hazard rate sensitivities for Example 4.1.

Example 5.2. We assume a Clayton copula, a homogeneous concordance structure $\tau^K=0.2$ and flat hazard rates $h_i(t)=h_i^{(1)}=0.1$. From Eqs. (4.18) and (4.20) we first compute

$$\begin{aligned}\log f_{Clay}&=n\cdot\log\theta+\sum_{k=1}^{n}\left\{\log\left(\frac{1}{\theta}+(k-1)\right)-(\theta+1)\cdot\log F_k(t_k;h_k^{(1)})\right\}\\&\quad-\left(\frac{1}{\theta}+n\right)\cdot\log\left(\sum_{k=1}^{n}F_k(t_k;h_k^{(1)})^{-\theta}-n+1\right)+\sum_{k=1}^{n}\log\frac{\partial F_k(t_k;h_k^{(1)})}{\partial t_k}\end{aligned}$$



(a) Piecewise constant basis functions

(b) Piecewise constant hazard rates

(c) Piecewise linear continuous basis functions

(d) Piecewise linear continuous hazard rates

Figure 4: Basis functions and parameterized hazard curves, (a), (b) for piecewise constant hazard rates and (c), (d) for piecewise linear continuous hazard rates.

Differentiation yields,

$$\frac{\partial(\log f_{Clay})}{\partial h_i^{(1)}} = -\frac{\theta+1}{F_i(t_i;h_i^{(1)})}\frac{\partial F_i}{\partial h_i^{(1)}} + \left(\frac{1}{\theta}+n\right)\frac{\theta\cdot F_i(t_i;h_i^{(1)})^{-\theta-1}}{\sum_{k=1}^{n}F_k(t_k;h_k^{(1)})^{-\theta}-n+1}\cdot\frac{\partial F_i}{\partial h_i^{(1)}} + \frac{\partial^2 F_i(t_i;h_i^{(1)})}{\partial t_i \partial h_i^{(1)}} \Big/ \frac{\partial F_i(t_i;h_i^{(1)})}{\partial t_i} \tag{5.12}$$

and from Eq. (5.7) we get explicitly,

$$F_i(t) = 1-\exp\left(-h_i^{(1)}t\right), \qquad \frac{\partial F_i}{\partial h_i^{(1)}} = t\exp\left(-h_i^{(1)}t\right),$$

$$\frac{\partial F_i(t_i;h_i^{(1)})}{\partial t_i} = h_i^{(1)}\exp\left(-h_i^{(1)}t\right), \qquad \frac{\partial^2 F_i(t_i;h_i^{(1)})}{\partial t_i\partial h_i^{(1)}} = (1-h_i^{(1)}t)\exp\left(-h_i^{(1)}t\right).$$

Similar to Ex. 5.1 we compare the performance of variance reduction to plain Monte-Carlo simulation for different swaps written on the above basket. We apply variance reduction to Eq. (5.4) together with Eq. (5.12). As expected, variance reduction performs significantly better than the plain simulation for large m, see Tab. 3.

	$m=1$ ex. value: 0.622242		$m=2$ ex. value: 0.443100		$m=3$ ex. value: 0.262252		$m=4$ ex. value: 0.126674	
N	plain	with VR	plain	with VR	plain	with VR	plain	with VR
10^3	1.79E-02	7.01E-03	1.05E-02	3.08E-03	3.12E-03	6.75E-04	8.30E-04	1.65E-04
10^4	8.01E-04	2.53E-04	8.69E-04	8.98E-05	4.43E-04	1.46E-05	1.33E-04	3.31E-06
10^5	1.07E-04	1.65E-04	1.30E-04	8.58E-05	2.33E-05	2.79E-05	1.80E-05	1.93E-06
10^6	1.84E-05	2.46E-06	2.56E-06	9.89E-07	3.97E-06	3.85E-07	1.82E-06	4.46E-08

Table 3: Analysis of the variance of $\frac{\partial \mathbb{E}(\mathcal{Z}_{pro}^{(m)})}{\partial h_i^{(1)}}$ using fixed parameters $\tau^K = 0.2$, $h_i = 0.1$, $T = 2$ and $n = 5$.

6 Conclusion

In this paper we presented numerical techniques to quantify the model risk involved in valuing credit derivatives written on a portfolio of multiple obligors. Following works of Joshi and Kainth, Chen and Glasserman our approach applies variance reduction principles to compute the expected payoff of a contingent claim triggered by a default event. Here, we also allow for joint probabilities of default times linked by an Archimedean copula model. This improvement is of interest for the valuation of basket default swaps because Archimedean copula models are promising alternatives to the standard approach with a Gaussian copula model. Hence, a vast range of new pricing alternatives is opened up by the application of variance reduction to this class of copula models.

Further, we suggest to use the concordance structure of the underlying portfolio in contrast to the correlation coefficients to describe the interrelations among the obligors. This guarantees the comparability of different copula models and allows to quantify the model risk induced by the choice of the model copula. It became clear that the choice of the copula has a significant influence on the evaluation of a basket default swap even if the copula implied concordance structures are the same. The copula is a model parameter in this sense.

Beside the choice of the copula model we also analyzed the sensitivities with respect to hazard rates and concordance structure as part of the model risk. We introduced techniques to calculate these sensitivities efficiently and analyzed them with regard to the application of variance reduction in context of Archimedean copula models. Especially for these calculations variance reduction is absolutely necessary to obtain accurate, robust and reliable results as our numerical experiments showed.

References

[1] T. R. Bielecki and M. Rutkowski. *Credit risk. modeling, valuation, and hedging.* Springer, 2002.

[2] M. Broadie and P. Glasserman. Estimating security price derivatives using simulation. *Management Science*, 42(2):269–285, 1996.

[3] R. Bruyere, R. Cont, and R. Copinot. *Credit Derivatives and Structured Credit.* Wiley Finance, 2006.

[4] X. Burtschell, J. Gregory, and J. P. Laurent. A comparative analysis of cdo pricing models. *preprint*, 2005.

[5] Z. Chen, , and P. Glasserman. Fast pricing of basket default swaps. *Operations Research*, 56(2):286–303, 2008.

[6] U. Cherubini, E. Luciano, W. Vecchiato, and G. Cherubini. *Copula methods in finance.* Wiley Finance, 2003.

[7] C. de Boor. *A practical guide to splines.* Springer-Verlag, 1994.

[8] D. Duffie and K. J. Singleton. *Credit Risk: Pricing, Measurement, and Management.* Princeton University Press, 2003.

[9] P. Embrechts, F. Lindskog, and A. McNeil. Modelling dependence with copulas and applications to risk management. In S. T. Rachev, editor, *Handbook of heavy tailed distributions in finance.* Elsevier/North-Holland, 2003.

[10] P. Embrechts, A. McNeil, and D. Straumann. Correlation: Pitfalls and alternatives. risk magazine. *RISK*, 12(5):69–71, 1999.

[11] T. Gerstner and M. Griebel. Numerical integration using sparse grids. *Numerical Algorithms*, 18(3-4):209–232, 1998.

[12] P. Glasserman. *Monte Carlo methods in financial engineering.* Springer-Verlag, 2004.

[13] J. Gregory and J.-P. Laurent. Basket default swaps, cdo's and factor copulas. *Journal of Risk*, 7(4):103–122, 2005.

[14] M. Hofert. Sampling archimedean copulas. *Computational Statistics & Data Analysis*, 52(12):5163–5174, 2008.

[15] J. Hull and A. White. Valuation of a cdo and an n-th to default cds without monte carlo simulation. *Journal of Derivatives*, 12(2):8–23, 2004.

[16] I. Iscoe and A. Kreinin. Recursive valuation of basket default swaps. *Journal of Computational Finance*, 9(3):95–116, 2006.

[17] M. S. Joshi and D. Kainth. Rapid and accurate development of prices and Greeks for nth to default credit swaps in the Li model. *Quantitative Finance*, 4(3):266–275, 2004.

[18] D. X. Li. On default correlation: A copula function approach. *Journal of Fixed Income*, 9(4):43–54, 2000.

[19] D. B. Madan, M. Konikov, and M. Marinescu. Credit and basket default swaps. *Journal of Credit Risk*, 2(1), 2006.

[20] A. W. Marshall and I. Olkin. Families of multivariate distributions. *Journal of the American Statistical Association*, 83(403):834–841, 1988.

[21] R. B. Nelsen. *An introduction to copulas.* Springer-Verlag, 2006.

[22] P. J. Schönbucher. *Credit Derivatives Pricing Models: Models, Pricing, Implementation.* Wiley Finance, 2003.

[23] P. J. Schönbucher and D. Schubert. Copula-dependent defaults in intensity models. Technical report, Department of Statistics, Bonn University, 2001.

[24] R. Seydel. *Tools for Computational Finance.* Springer-Verlag, Berlin, 2009.

[25] A. Sklar. Random variables, joint distribution functions, and copulas. *Kybernetika*, 9(6):449–460, 1973.