MATHEMATISCHES INSTITUT
DER UNIVERSITÄT ZU KÖLN
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Computational Finance - 10th Assignment

Deadline: June, 27th (written exercises)
July, 4th (programming exercise)

Exercise 36 (Perpetual Put Option) (4+2+4 points)

For $T \to \infty$ it is sufficient to analyze the ODE

$$\frac{\sigma^2}{2} S^2 \frac{\mathrm{d}^2 V}{\mathrm{d}S^2} + (r-\delta) S \frac{\mathrm{d}V}{\mathrm{d}S} - rV = 0.$$

Consider an American put with high contact to the payoff $V = (K-S)^+$ at $S = S_\mathrm{f}$. Show:

a) Upon substituting the boundary condition for $S \to \infty$ one obtains

$$V(S) = c \left(\frac{S}{K}\right)^{\lambda_2},$$

where $\lambda_2 = \frac{1}{2}\left(1 - q_\delta - \sqrt{(q_\delta - 1)^2 + 4q}\right)$, $\quad q = \frac{2r}{\sigma^2}$, $\quad q_\delta = \frac{2(r-\delta)}{\sigma^2}$
and c is a positive constant.
Hint: Apply the transformation $S = Ke^x$. (The other root λ_1 drops out.)

b) V is convex.

For $S < S_\mathrm{f}$ the option is exercised; then its intrinsic value is $K - S$. For $S > S_\mathrm{f}$ the option is not exercised and has a value $V(S) > K - S$. The holder of the option decides when to exercise. This means, the holder makes a decision on the high contact S_f such that the value of the option becomes maximal.

c) Show: $V'(S_\mathrm{f}) = -1$, if S_f maximizes the value of the option.
Hint: Determine the constant c such that $V(S)$ is continuous in the contact point.

Exercise 37 (Discrete Dividend Payment) (3 points)

Assume that a stock pays a dividend D at ex–dividend date t_D, with $0 < t_D < T$. Define for an American put with strike K

$$\tilde{t} := t_D - \frac{1}{r}\log\left(\frac{D}{K} + 1\right).$$

Assume $S = 0$, $r > 0$, $D > 0$ and a time instant t in $\tilde{t} < t < t_D$. Argue that instead of exercising early it is reasonable to wait for the dividend. Note: For $\tilde{t} > 0$, depending on S, early exercise may reasonable for $0 \le t < \tilde{t}$.

Exercise 38 (UL decomposition) (3 points)

Assume a system of linear equations $Ax = b$ with irreducible diagonally dominant tridiagonal matrix A. Formulate the UL decomposition as an algorithm.

Exercise 39 (Brennan-Schwartz Algorithm) (3+2 points)

Let A be a irreducible diagonally dominant tridiagonal matrix and b and g vectors. The system of equations $Aw = b$ is to be solved such that the side condition $w \geq g$ is obeyed componentwise. Assume for a put $w_i = g_i$ for $1 \leq i \leq i_{\mathrm{f}}$ and $w_i > g_i$ for $i_{\mathrm{f}} < i \leq n$, with unknown i_{f}.

a) Formulate an algorithm that solves $Aw = b$ in the *backward/forward* approach. In the final forward loop, for each i the calculated candidate w_i is tested for $w_i \geq g_i$: In case $w_i < g_i$ the calculated value w_i is corrected to $w_i = g_i$.

b) Apply the algorithm to the case of a put with relevant A, b, g.

Exercise 40 (Computation of American Options) (20P points)

Implement an algorithm for the calculation of American-style options, following the prototype algorithm below. Use exercises 38 and 39. For this assignment, it is sufficient to implement the case of a put.

Test your program with the following example: $K = 10$, $r = 0.25$, $\sigma = 0.6$, $T = 1$, $\delta = 0.2$. Calculate approximations to $V_{\mathrm{P}}(10, 0)$.

Algorithm (prototype algorithm)

Set up the function $g(x, \tau)$ listed in the summary below.
Choose θ ($\theta = 1/2$ for Crank-Nicolson).
Fix the discretization by choosing $x_{\min},\ x_{\max},\ m,\ \nu_{\max}$
(for example: $x_{\min} = -5,\ x_{\max} = 5,\ \nu_{\max} = m = 100$).
Calculate $\Delta x := (x_{\max} - x_{\min})/m$,

$$\Delta\tau := \tfrac{1}{2}\sigma^2 T/\nu_{\max},$$
$$x_i := x_{\min} + i\Delta x \quad \text{for} \quad i = 0, \ldots, m,$$
$$\lambda := \Delta\tau/\Delta x^2 \quad \text{and} \quad \alpha := \lambda\theta.$$

Initialize the iteration vector w with
$g^{(0)} = (g(x_1, 0), \ldots, g(x_{m-1}, 0))^{tr}$.
Arrange for the matrix A.

τ-loop: for $\nu = 0, 1, ..., \nu_{\max} - 1$:
$\tau_\nu := \nu\Delta\tau$
initialize the vector b with
$b_i := w_i + \lambda(1-\theta)(w_{i+1} - 2w_i + w_{i-1})$ for $2 \leq i \leq m-2$,
$b_1 := w_1 + \lambda(1-\theta)(w_2 - 2w_1 + g_{0\nu}) + \alpha g_{0,\nu+1}$,
$b_{m-1} := w_{m-1} + \lambda(1-\theta)(g_{m\nu} - 2w_{m-1} + w_{m-2}) + \alpha g_{m,\nu+1}$
subroutine for the LCP solution w, directly as in exercises 38 and 39
$w^{(\nu+1)} = w$

(please turn over)

Summary of American options, for a put ($r > 0$) or a call ($\delta > 0$), after transformation into (x, τ, y)-variables:

$$q := \frac{2r}{\sigma^2}; \quad q_\delta := \frac{2(r-\delta)}{\sigma^2}$$

put: $g(x,\tau) := \exp\{\frac{1}{4}((q_\delta - 1)^2 + 4q)\tau\} \max\{e^{\frac{1}{2}(q_\delta - 1)x} - e^{\frac{1}{2}(q_\delta + 1)x}, 0\}$

call: $g(x,\tau) := \exp\{\frac{1}{4}((q_\delta - 1)^2 + 4q)\tau\} \max\{e^{\frac{1}{2}(q_\delta + 1)x} - e^{\frac{1}{2}(q_\delta - 1)x}, 0\}$

$$\left(\frac{\partial y}{\partial \tau} - \frac{\partial^2 y}{\partial x^2}\right)(y - g) = 0$$

$$\frac{\partial y}{\partial \tau} - \frac{\partial^2 y}{\partial x^2} \geq 0, \quad y - g \geq 0$$

$$y(x,0) = g(x,0), \quad 0 \leq \tau \leq \tfrac{1}{2}\sigma^2 T$$

$$\lim_{x \to \pm\infty} y(x,\tau) = \lim_{x \to \pm\infty} g(x,\tau)$$

(programming exercise)