MATHEMATISCHES INSTITUT
DER UNIVERSITÄT ZU KÖLN
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Computational Finance - 6th Assignment

Deadline: May, 23th (written exercises)
June, 4th (programming exercise)

Exercise 20 (Analysis of a Random Number Generator) (5 points)

Consider the linear congruential generator

$$N_i = (455 N_{i-1} + 23) \mod 4096, \quad U_i = \frac{N_i}{4096}.$$

Construct a family of parallel straight lines containing all the points (U_{i-1}, U_i) so that only few of them cut the square $[0,1)^2$. What is the distance between them?
Hint: Examine the condition $c \in \mathbb{Z}$ for the linear equation. For this purpose consider the quotient $\frac{M}{a}$.

Exercise 21 (Deficient Random Number Generator) (5 points)

For some time the generator

$$N_i = a N_{i-1} \mod M \quad \text{with } a = 2^{16} + 3,\ M = 2^{31}$$

was in wide use. Show for the sequence $U_i := N_i / M$:

$$U_{i+2} - 6U_{i+1} + 9U_i \text{ is integer!}$$

What does this imply for the distribution of the triples (U_i, U_{i+1}, U_{i+2}) in the unit cube?

Exercise 22 (Inverting the Normal Distribution) (3+1+2 points)

Suppose $F(x)$ is the standard normal distribution function. Construct a rough approximation $G(u)$ to $F^{-1}(u)$ for $0.5 \le u < 1$ as follows:

a) Construct a rational function $G(u)$ with correct asymptotic behavior, point symmetry with respect to $(u, x) = (0.5, 0)$, using only one parameter.

b) Fix the parameter by interpolating a given point $(x_1, F(x_1))$.

c) What is a simple criterion for the error of the approximation?

Exercise 23 (Uniform Distribution) (6 points)

For the uniformly distributed random variable (V_1, V_2) on $V_1^2 + V_2^2 < 1$ consider the transformation

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} V_1^2 + V_2^2 \\ \frac{1}{2\pi} \arg(V_1, V_2) \end{pmatrix}.$$

Show that (X_1, X_2) is distributed uniformly.

Exercise 24 (Implied Volatility) (20P points)

For European options we take the valuation formula of Black and Scholes of the type $V = v(S, \tau, K, r, \sigma)$, where τ denotes the time to maturity, $\tau := T - t$. For the definition of the function v see exercise 6. If actual market data of the price V are known, then one of the parameters considered known so far can be viewed as unknown and fixed via the implicit equation

$$V - v(S, \tau, K, r, \sigma) = 0. \tag{$*$}$$

In this *calibration* approach the unknown parameter is calculated iteratively as solution of equation $(*)$. Consider σ to be in the role of the unknown parameter. The volatility σ determined in this way is called *implied volatility* and is a zero of $f(\sigma) := V - v(S, \tau, K, r, \sigma)$.

Assignment:

a) Design, implement and test an algorithm to calculate the implied volatility of a call. Use Newton's method to construct a sequence $x_k \to \sigma$. The derivative $f'(x_k)$ can be approximated by the difference quotient

$$\frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}}\,.$$

For the resulting *secant iteration* invent a stopping criterion that requires smallness of both $|f(x_k)|$ and $|x_k - x_{k-1}|$.

b) Consider the following market data for call options on the same underlying:

$$T - t = 0.32787,\ S_0 = 7133.06,\ r = 0.0487,$$

K	6400	6700	7000	7300	7600	7900	8200	8500	8800
V	934.0	690.0	469.0	283.0	145.0	62.0	22.0	7.5	2.1

Calculate the implied volatilities for these data. For each calculated value of σ enter the point (K, σ) into a figure and join the points with straight lines. (You will notice a convex shape of the curve. This shape has led to call this phenomenon *volatility smile*.)

(programming exercise)