MATHEMATISCHES INSTITUT
DER UNIVERSITÄT ZU KÖLN
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Computational Finance - 7th Assignment

Deadline: June, 6th

Exercise 25 (Integration by Parts for Itô Integrals) (2+3 points)

a) Show

$$\int_{t_0}^{t} s \, \mathrm{d}W_s = tW_t - t_0 W_{t_0} - \int_{t_0}^{t} W_s \, \mathrm{d}s.$$

Hint: Start with the Wiener process $X_t = W_t$ and apply the Itô Lemma with the transformation $y = g(x,t) := tx$.

b) Denote $\Delta Y := \int_{t_0}^{t} \int_{t_0}^{s} \mathrm{d}W_z \mathrm{d}s$, $\Delta W := W_t - W_{t_0}$ and $\Delta t := t - t_0$. Show by using a) that

$$\int_{t_0}^{t} \int_{t_0}^{s} \mathrm{d}z \, \mathrm{d}W_s = \Delta W \Delta t - \Delta Y.$$

Exercise 26 (Integral Representation) (8 points)

For a European put with time to maturity $\tau := T - t$ prove that

$$\begin{aligned}[V(S_t,t) =]e^{-r\tau} \int_0^\infty (K - S_T)^+ \frac{1}{S_T \sigma \sqrt{2\pi\tau}} \exp\left\{ -\frac{[\ln(S_T/S_t) - (r - \frac{\sigma^2}{2})\tau]^2}{2\sigma^2\tau} \right\} \mathrm{d}S_T \\ = e^{-r\tau} K F(-d_2) - S_t F(-d_1),\end{aligned}$$

where F, d_1 and d_2 were defined in exercise 6.
Hint: Use $(K - S_T)^+ = 0$ for $S_T > K$, and get two integrals.

(please turn over)

Exercise 27 (Moments of Itô Integrals) (3+7+3 points)

a) Use the Itô isometry

$$\mathrm{E}\left(\left[\int_a^b f(t,\omega)\,\mathrm{d}W_t\right]^2\right) = \int_a^b \mathrm{E}\left(f^2(t,\omega)\right)\,\mathrm{d}t$$

to show its generalization

$$\mathrm{E}(I(f)I(g)) = \int_a^b \mathrm{E}(fg)\,\mathrm{d}t, \quad \text{where} \quad I(f) = \int_a^b f(t,\omega)\,\mathrm{d}W_t.$$

Hint: $4fg = (f+g)^2 - (f-g)^2$

b) Show for ΔY, ΔW and Δt defined in exercise 25 by using a) and $\mathrm{E}\left(\int_a^b f(t,\omega)\,\mathrm{d}W_t\right) = 0$ the following assertions for the moments:

$$\mathrm{E}(\Delta Y) = 0, \quad \mathrm{E}(\Delta Y^2) = \frac{\Delta t^3}{3}, \quad \mathrm{E}(\Delta Y \Delta W) = \frac{\Delta t^2}{2}, \quad \mathrm{E}(\Delta Y \Delta W^2) = 0.$$

c) By transformation of two independent standard normally distributed random variables $Z_i \sim \mathcal{N}(0,1), i = 1,2$, two new random variables are obtained by

$$\Delta\widehat{W} := Z_1\sqrt{\Delta t}, \quad \Delta\widehat{Y} := \frac{1}{2}(\Delta t)^{3/2}(Z_1 + \frac{1}{\sqrt{3}}Z_2).$$

Show that $\Delta\widehat{W}$, $\Delta\widehat{Y}$ and their corresponding products have the same moments like ΔW and ΔY in b).