THE TRIANGULAR THEOREM OF EIGHT

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Abstract. We investigate here sums of triangular numbers $f(x) := \sum_i b_i T_{x_i}$ where T_n is the n-th triangular number. We show that, fixing $b_i \geq 0$, $f(x)$ represents every (nonnegative) integer if and only if it represents 1, 2, 4, 5, and 8, with the standard application to sums of odd squares $\sum_i (2x_i+1)^2$. Moreover, we show that no finite subset will suffice if cross terms are included.

1. Introduction

In 1638 Fermat wrote that every number is a sum of at most three triangular numbers, four square numbers, and in general n polygonal numbers of order n. Here the triangular numbers are $T_x := \frac{x(x+1)}{2}$, where we include $x = 0$ for simplicity. The claim for four squares was shown by Lagrange in 1772, while Gauss famously wrote "Eureka, $\triangle + \triangle + \triangle = n$" in his mathematical diary on July 10, 1796.

Theorem (Gauss, 1796)**.** Every positive integer is the sum of three triangular numbers.

The first proof of the full assertion of Fermat was given by Cauchy in 1813.

In 1917, Ramanujan extended the question about four squares to consider which choices of $b = (b_1, b_2, b_3, b_4)$ have the property that $b_1x_1^2 + b_2x_2^2 + b_3x_3^2 + b_4x_4^2$ represents every positive integer. We shall refer to such forms as *universal diagonal forms*. He gives a list of 55 possible choices of b which he then claims are the complete list of universal quarternary diagonal forms (54 forms actually turned out to be universal).

In light of Ramanujan's generalization of Lagrange's Theorem, we will consider the following generalization of Gauss's "Eureka" Theorem. We will investigate determining every choice of b such that the sum

$$f(x) := f_b(x) := \sum_{i=1}^{k} b_i T_{x_i}$$

represents every integer. We will see the following simple condition to determine whether f_b represents every integer.

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Theorem 1.1. *Fix a sequence of integers $b_1, \dots, b_k$. Then*

1 *The triangular form $f(x) = \sum_{i=1}^{k} b_i T_{x_i}$ represents every integer if and only if it represents the integers* 1, 2, 4, 5, *and* 8.

2 *The corresponding diagonal quadratic form $Q(x) = \sum_{i=1}^{k} b_i x_i^2$ with x_i all odd represents every integer of the form*

$$8n + \sum_{i=1}^{k} b_i$$

if and only if it represents $8+\sum_{i=1}^{k} b_i$, $16+\sum_{i=1}^{k} b_i$, $32+\sum_{i=1}^{k} b_i$, $40+\sum_{i=1}^{k} b_i$, and $64+\sum_{i=1}^{k} b_i$.

Recently, Conway and Schneeberger proved a similar condition for positive definite quadratic forms whose corresponding matrix has integer entries, but without publishing their results.

Theorem (Conway-Schneeberger)**.** A positive definite quadratic form $Q(x) = x^t Ax$, where A is a positive symmetric matrix with integer coefficients, represents every positive integer if and only if it represents the integers $1, 2, 3, 5, 6, 7, 10, 14$, and 15.

Bhargava gave a simpler proof of the Conway-Schneeberger 15 theorem in [1], in addition to showing more generally that for any set S it is always sufficient to check a finite subset S_0, and showed the set S_0 for S all odd integers and S all primes. More recently, Bhargava and Hanke have shown the 290 theorem, stating the necessary set when the corresponding matrix is half integral, the largest of which is 290 [2]. For a more complete history of related questions about sums of figurate numbers and some new results, please see [4]. For further information and background about quadratic forms, a good source is [5].

There is a natural mapping from representations by sums of triangular numbers to quadratic forms, namely the mapping which takes x_i to $2x_i + 1$ after multiplying by 8. In light of the Conway-Schneeberger 15 theorem and the Bhargava-Hanke 290 theorem, one next considers whether a similar result holds for the preimage of the above map. This preimage is unique up to a constant, so we will first make a cannonical mapping which will allow us to define a certain metric and then later add this metric to our function to obtain a form for which we can address an analogous question to the Conway-Schneeberger 15 theorem.

To this end, we define the canonical preimage of a positive definite quadratic form Q whose corresponding matrix A has even non-diagonal coefficients as

$$\tilde{f}(x) := \tilde{f}_Q(x) := \sum_{i=1}^{k} b_i T_{x_i} + \sum_{i<j, c_{ij}\geq 0} c_{ij}(2x_ix_j + x_i + x_j) + \sum_{i<j, c_{ij}<0} c_{ij}(2x_ix_j + x_i + x_j + 1),$$

where here $T_x = \frac{x(x+1)}{2}$ extends to x negative and Q will be defined below.

Remark 1.2. *Our choice of adding the constant c_{ij} every time $c_{ij} < 0$ may not seem canonical at first, but notice that if Q' is the equivalent quadratic form obtained by replacing x_1 with $-x_1$, then we find that this choice leads to $\tilde{f_Q} = \tilde{f_{Q'}}$.*

Note that $\tilde{f}(x)$ represents the integer n if and only if the quadratic form

$$Q(x) := \sum_{i=1}^{k} b_i x_i^2 + \sum_{i<j} 4c_{ij}x_ix_j$$

represents the integer $8n + \sum_{i=1}^{k} b_i + \sum_{i<j} 4|c_{ij}|$ with x_i all odd.

However, note that if we have $\tilde{Q}$ and $\tilde{Q}'$ are two equivalent quadratic forms with the mapping between them preserving x_i odd (we will refer to such forms as *equivalently odd*, then it is not necessarily the case that $\tilde{f_Q} = \tilde{f_{Q'}}$, and that $\tilde{f}$ may represent (finitely many) negative integers. Thus we define the natural metric

$$\tilde{m}_{\tilde{f}} := |\min_x \tilde{f}(x)|$$

and define

$$f(x) := f_{[Q]_o}(x) := \tilde{f}(x) + \tilde{m}_{\tilde{f}}, \tag{1.1}$$

where $[Q]_o$ denotes the equivalently odd equivalence class of Q. Clearly, f is independent of the choice of Q' in the equivalently odd equivalence class of Q, and f only represents nonnegative integers.

This brings one to consider whether a condition similar to the 15 theorem or the 290 theorem holds for determining whether f_Q represents every positive integer. However, no such subset exists.

Theorem 1.3. *For a set S, there is no proper subset S_0 such that it holds for every Q with even non-diagonal coefficients that f_Q represents S_0 if and only if f_Q represents S. In particular, there is no proper subset of $\mathbb{N}$.*

Our proof is by construction of forms f which represent every integer other than any arbitrary integer n.

Remark 1.4. *The corresponding statement about the quadratic form Q is that there is no finite subset to determine if Q represents every integer from its minimum value which is congruent modulo 8 to the minimum value. Due to the congruence conditions modulo 8 implied by having all x_i odd, these are the only possible values that Q may represent.*

One may consider the question of whether the form Q represents every integer congruent to 1 mod 8, or 2 mod 8, etc., with x_i odd. Using a simple mapping to forms with x_1 odd and x_i even otherwise, one can continue with escalator lattices as in Bhargava's work, escalating after the first dimension only to choices of vectors which would occur *with x_i even. There should only be finitely many escalators in the resulting tree, and hence there would be a finite subset in this case. We will not work out the details of this observation here, however.*

However, under a certain restriction, we will find again that a finite subset will suffice. Consider $\tilde{f}$ equivalent to $\tilde{f}'$, written $\tilde{f} \sim \tilde{f}'$, if the corresponding $f = f'$. We will denote the equivalence class by $[f]$. Then one can define the following metric, depending only on the equivalence class $[f]$, or, correspondingly, the equivalently odd equivalence class $[Q]_o$,

$$m_{[Q]_o} := m_{[f]} := \min_{\tilde{f}\in[f]} \tilde{m}_{\tilde{f}}.$$

Then we will be able to find a finite subset of the positive integers will suffice when $m_{[f]}$ is bounded.

Theorem 1.5. *Fix an integer m and a subset S of the positive integers. Then there is a finite subset $S_{0,m} \subset S$ depending only on m and S such that if $m_{[f]} \leq m$ then f represents S if and only if f represents $S_{0,m}$.*

Moreover, for $S = \mathbb{N}$, we find that $\mathbb{N}_{0,m} \gg m^2$.

It may be of interest to investigate the growth of $\mathbb{N}_{0,m}$ in terms of m.

Acknowledgements

2. Theorem of Eight

Here we prove Theorem 1.1. We will proceed by showing by a standard argument that the theorem is equivalent to a statement about (diagonal) quadratic forms, and then prove the corresponding result for quadratic forms. We will only need some elementary results about quadratic forms and a theorem of Siegel to show the desired result.

Proof. Consider the generating function

$$F(q) := F_b(q) := \sum_x q^{f(x)} = \sum_{n=0}^{\infty} t(n) q^n,$$

where $t(n)$ is the number of solutions to $f(x) = n$. Then we see that

$$q^{\sum_{i=1}^k b_i} F(q^8) = \sum_x q^{\sum_{i=1}^k b_i(2x_i+1)^2},$$

so that $t(n) = r_o(8n - \sum_{i=1}^k b_i)$, where $r_o(n)$ is the number of representations of n by the corresponding (diagonal) quadratic form with x_i odd. We proceed as with *escalator lattices* in [1]. Without loss of generality, $b_1 \leq b_2 \leq \cdots \leq b_k$. Fixing $b = [b_1, \ldots, b_{k-1}]$, we will *escalate* to $[b_1, \ldots, b_k]$ by choosing all possible choices of $b_k \geq b_{k-1}$ for which it is possible to represent the next largest integer not already represented. We will then develop an *escalator tree* by forming an edge between b and $[b_1, \ldots, b_k]$, with $\emptyset$ as the root. If b represents every integer, then b will be a leaf of our tree.

Since $t(1) > 0$, it follows that $b_1 = 1$. We need $t(2) > 0$, so $b_2 = 1$ or $b_2 = 2$. If $b_2 = 1$, then we need $t(5) > 0$, so $1 \leq b_3 \leq 5$. For $b_3 = 3$, we need $t(8) > 0$, so $3 \leq b_4 \leq 8$. Likewise, if $b_2 = 2$, then $2 \leq b_3 \leq 4$. Therefore, if $t(n) > 0$ for every n, then we must have one of the above choices of b_i as a sublattice. By showing that each of these choices of b_i satisfies $t(n) > 0$ for every n, we will see that this condition is both necessary and sufficient.

For ease of notation, we will denote the triangular form corresponding to b with $[b_1, b_2, \ldots, b_k]$ and the corresponding quadratic form by $(b_1, \ldots, b_k)$. For the forms $[1, 1, 1]$, $[1, 1, 4]$, $[1, 1, 5]$, $[1, 2, 2]$, and $[1, 2, 4]$, $r_o(n) = r(n)$, where $r(n)$ is the number of representations of n without the restriction of x_i odd. For each of these choices of b, (b_1, b_2, b_3) is a Genus 1 quadratic form. Therefore, n is represented globally if and only if n is represented locally, and for these choices of b, a quick check of the local density shows that each integer $8n + b_1 + b_2 + b_3$ is locally represented for every n.

Our proofs for $[1, 1, 2]$, $[1, 2, 3]$, and $[1, 1, 3]$ will be essentially the same. For $[1, 1, 2]$, we note that if

$$x^2 + y^2 + 2z^2 = 8n + 4$$

has a solution with x, y, and z not all odd, then taking each side modulo 8 leads us to the conclusion that x, y, and z must all be even. Therefore, the solutions without x, y, and z odd correspond to solutions of

$$4x^2 + 4y^2 + 8z^2 = 8n + 4,$$

or,

$$x^2 + y^2 + 2z^2 = 2n + 1.$$

Using Siegel's theorem to compare the local density at 2, we see that the average of the number of representations over the genus is twice as large for $8n + 4$ as $2n + 1$. However, $(1, 1, 2)$ is again a Genus 1 quadratic form, so $r(8n + 4) = 2r(2n + 1)$, and hence $t(n) = r_o(8n + 4) = r(8n + 4) - r(2n + 1) = r(2n + 1)$. Since $(1, 1, 2)$ is Genus 1, we need again only check local conditions. Similar arguments show that

$$\begin{aligned} t_{[1,2,3]}(n) = r_{o,(1,2,3)}(8n + 6) &= r_{(1,2,3)}(8n + 6) - r_{(4,2,12)}(8n + 6) \\ &= r_{(1,2,3)}(8n + 6) - r_{(1,2,6)}(4n + 3) = r_{(1,2,6)}(4n + 3). \end{aligned}$$

Again $(1, 2, 6)$ is Genus 1, so checking local conditions shows that every integer n is represented. For $[1, 1, 3]$ we see analogously that

$$t_{[1,1,3]}(n) = r_{0,(1,1,3)} = r_{(1,1,3)} - r_{(1,1,12)} = r_{(1,1,12)},$$

and again $(1, 1, 12)$ is Genus 1. However, in this case, local conditions indicate that integers congruent to 8 modulo 9 are note represented by $[1, 1, 3]$. Considering $[1, 1, 3, k]$ for $k \in 3..8$, we see that for $n \equiv 8 \mod 9$, taking the fourth variable to be 1 will give us $n-k \not\equiv 8 \mod 9$, and hence $n-k$ is represented by $[1,1,3]$, so that n is represented by $[1,1,3,k]$.

Having seen that each of our choices of b is indeed a leaf to the tree, we conclude that the representing the integers $1,2,4,5$, and 8 suffices. $\square$

3. Cross Terms

We will now consider the "triangular forms" f defined by equation (1.1) corresponding to quadratic forms whose corresponding matrix has all non-diagonal entries even. We shall begin by showing that no finite subset as in section 2 will suffice for forms of this type.

Proof of Theorem 1.3. Let a set $S \subseteq \mathbb{N}$ be given. We will proceed by explicit construction of a triangular form f which represents exactly every integer other than an arbitrary fixed $m \in S$, so that no checking no proper subset of S will suffice. First note that if the smallest positive integer represented by a form f is m, then since the sum of three triangular numbers represents every integer, we can add $m+1$ times the sum of three triangular numbers plus $m+2$ times the sum of three triangular numbers so that there is a form which represents exactly every integer other than m.

Consider the quadratic form

$$Q_N(x,y) := Nx^2 + Ny^2 + 4xy,$$

and denote to the corresponding triangular form as f_N. We first show that it is sufficient to show that the generating function for f_N is

$$2 + 2q + O(q^{N-6}). \tag{3.1}$$

Assuming equation (3.1), then the generating function for

$$f := \bigoplus_{i=1}^{m} f_N$$

is

$$2^m\left(1 + \binom{m}{1}q + \cdots + \binom{m}{m}q^m\right) + O(q^{N-6}),$$

since $m_{[f]}$ is additive across independent sums. If we choose N large enough, then the first integer not represented by f is $m+1$. Therefore, since m was arbitrary, there is a form which represents every integer other than $m+1$ (this also sufficies for $m=0$, by the argument above).

We now show that the generating function satisfies (3.1). Since

$$f_N = NT_x + NT_y + (2xy + x + y) + 1,$$

taking $x=0$, $y=-1$ or $x=-1$, $y=0$ represents 0, and $x=0$, $y=0$ or $x=-1$, $y=-1$ represents 1. Now, without loss of generality assume that $|x| \geq |y|$ and one of them is not 0 or -1. Then

$$|2xy + x + y| \leq 2|x|^2 + 2|x| = 4T_{|x|},$$

so that

$$f_N \geq NT_x - 4T_{|x|} + NT_y.$$

When $x \leq -2$ it is easy to check that

$$NT_x - 4T_{|x|} \geq (N-6)T_{|x|-1} \geq N-6.$$

and when $x > 0$

$$NT_x - 4T_{|x|} \geq (N-4)T_x \geq N-4,$$

since $|x| \geq |y|$ and one of them is not 0 or -1, so that the terms $T_x \geq 1$ or $T_{|x|-1}$ are positive. Since $T_y \geq 0$, or assertion is verified. □

Remark 3.1. *It is important here to note how the above counterexamples differ from the proof when we only have diagonal terms, since this observation will lead us to the following proof when $m_{[f]}$ is bounded.*

If, at each step of the escalator lattice we escalate by what will here be referred to as a block, a form whose corresponding quadratic form has an irreducible matrix, which is independent of the other variables, then, after the depth is at least 4, we can use known bounds for modular forms to show that the depth from a given branch off of this depth 4 escalator will be finite. In Section 2, the breadth each time we escalate will be finite, so that the overall tree is finite, and hence there is a largest truant, so there is a finite subset S_0 which will suffice. In the above proof, however, there are infinitely many inequivalent forms f which represent the integer 1, so that the breadth is infinite.

Proof of Theorem 1.5. Fix a positive integer m. We will start with a small overview of the proof. As in the above remark, we will escalate with blocks. We will first show that when $m_{[f]} \leq m$, the number of blocks that are not dimension 1 in any branch of the escalator tree is bounded, and that there are only finitely many choices for the cross terms c. We will then proceed by defining

$$N(M_1, M_2, \ldots, M_k, c)$$

to be the smallest integer not represented by the form corresponding to

$$\tilde{f}(x) := \sum_{i=1}^{k} M_i T_{x_i} + \sum_{i<j, c_{ij}\geq 0} c_{ij}(2x_ix_j + x_i + x_j) + \sum_{i<j, c_{ij}<0} c_{ij}(2x_ix_j + x_i + x_j + 1).$$

Our claim is then equivalent to showing that in the escalator tree

$$\sup_{M_1,\ldots M_k, c} N(M_1, M_2, \ldots, M_k, c)$$

is finite. To do so, we will effectively show that with c fixed, the supremum with M_i sufficiently large is finite and independent of the choice of M_i, and then fix $M_1 \leq m_1$, and again show that the resulting supremum is independent of $M_2, \ldots, M_k$, and so forth.

Since there are only finitely many such choices of c, the result comes from taking the maximum of each of these supremums.

We begin with a lemma that will show that there are only finitely many choices of the cross terms.

Lemma 3.2. *If $m_{[f]} \leq m$, then there are only finitely many choices of the cross terms c_{ij}, up to equivalent forms.*

Proof. First note that $m_{[f_1 \oplus f_2]} = m_{[f_1]} + m_{[f_2]}$, so that we can only have at most m blocks. It therefore sufficies to show that each block has bounded dimension and bounded entries. Fix the cross terms c of a block $\tilde{f}$ with dimension k such that $\tilde{m}_{\tilde{f}} = m_{[f]}$, namely a minimal element. We will recursively show a particular choice of x_i such that

$$\tilde{f}(x) \leq -\max\{\max_{i,j} |c_{ij}|, k-1\},$$

so that the max of the c_{ij} is bounded by m, and the dimension is bounded by $m+1$.

First set $x_1 = 0$. Since $\tilde{f}$ is a block, then we know at step j that there is some $i < j$ such that $c_{ij} \neq 0$. Choose $i < j$ such that $|c_{ij}|$ is maximal. If $x_i = 0$, then we set $x_j = -1$ if $c_{ij} > 0$ and $x_j = 0$ otherwise. If $x_i = -1$ then we set $x_j = 0$ if $c_{ij} > 0$ and $x_j = -1$ otherwise.

Since all of our choices of x_i are 0 or -1, we note that the integer represented is independent of M_i, since $T_{-1} = T_0 = 0$. Now we note that if $x_i = x_j = 0$ or $x_i = x_j = -1$ then $2x_ix_j + x_i + x_j = 0$ and $2x_ix_j + x_i + x_j = -1$ otherwise. Therefore, if $x_i = x_j$, then from our definition of $\tilde{f}$, the cross term corresponding to c_{ij} adds 0 if $c_{ij} \geq 0$ and adds $-|c_{ij}|$ otherwise. If $x_i = 0$ and $x_j = -1$, then the cross term adds $-|c_{ij}|$ if $c_{ij} \geq 0$ and adds 0 otherwise. Therefore by our construction above, we know that for $|c_{ij}|$ maximal, we have added $-|c_{ij}|$ to our sum, and we never add a positive integer, so the sum is at most $-|c_{ij}|$. Moreover, since the block is connected, we have added at most -1 at each inductive step, so that the sum is at most $-(k-1)$. □

For simplicity, in our escalator tree, we will "push" up all of the blocks tothe top of the tree which are not dimension 1. To do so, we will first build the tree with all possible choices of blocks which are not dimension 1, and then escalate with only dimension 1 blocks from each of the nodes of the tree, including the root (the empty set). Thus, every possible form will show up in our representation. This tree is depth at most m in the number of blocks, but is of infinite breadth. Henceforth, we can consider c to be fixed, and take the maximum over all choices of c.

We will now see that the subtree from each fixed node is of finite depth. Consider the corresponding quadratic form Q. First note that the generating function for Q when all x_i are odd is the generating function for Q minus the generating functino with some x_i even, and the others arbitrary, which is simply another quadratic form without any restrictions, taking $x_i \to 2x_i$. Thus, we have the generating function of a difference of finitely many quadratic forms, and hence we have the Fourier expansion of a modular form. Now we simply note that any quadratic form can be decomposed into an Eisenstein series and a cusp form (cf. [6]). Using the bounds of Tartakowsky [8] and Deligne [3], as long as the Eisenstein series is non-zero, the growth of the coefficients of the Eisenstein series can be shown to grow more quickly than the coefficients of the cusp form whenever the dimension is greater than or equal to 4, other than finitely many congruences classes for which the coefficients of both the Eisenstein series and the cusp form are zero.

Therefore, as long as the Eisenstein series is non-zero, there are only finitely many congruence classes and finitely many "sporadic" integers which are not represented by the quadratic form. Thus, after dimension 4, there are only finitely many congruence classes and finitely many sporadic integers not represented by the form f. If at any step of the escalation, any of the integers in these congruence classes is represented, then we have less congruence classes, and only finitely many more sporadic integers which are not represented, so that the resulting depth is bounded. For the dimension 1 blocks, it is clear that the breadth of each escalation is finite, so there are only finitely many escalators coming from this node. Therefore, it suffices to show that the Eisenstein series is non-zero.

Again using Siegel's theorem [7], the Eisenstein series is simply a difference of the local densities. At every prime other than $p = 2$, the local densities are equal, so we only need to show that the difference of the local densities at $p = 2$ is positive. However, the difference of the number of local representations at a fixed 2 power must be positive, since the integer is locally represented with x_i odd, except for finitely many congruence classes.

Therefore, we can define

$$\tilde{N}(M_1, \ldots, M_k, c)$$

to be the maximum of $N(M_1, \ldots, M_k, M_{k+1}, \ldots M_l, c)$, where M_{k+1} to M_l are the dimension 1 blocks coming from this node.

We will show that $\tilde{N}(M_1, \ldots, M_k, c)$ is independent of the choice of M_i whenever M_i is sufficiently large by showing that the resulting subtrees are identical. We need the following lemma to obtain this goal. We will need some notation before we proceed.

For a set T, define

$$q^T := \sum_{t\in T} q^t.$$

For fixed sets $S, T \subseteq \mathbb{N}$, we will say that a form $f(x) := \sum_{i=1}^{k} b_i T_{x_i}$ *represents* S/T if the generating function $g(x)$ for $f(x)$ satisfies $q^T g(x)$ having for each $s \in S$, the coefficient of q^s is positive.

Lemma 3.3. *Let a (diagonal) triangular form f be given. Fix $S, T_1, T_2 \subseteq \mathbb{N}$ and $M \in \mathbb{N}$ such that $\min_{n\in T_2} \geq M$. Define $T := T_1 \cup T_2$. Then there exists a bound M_{T_1} and a finite subset $S_0 \subseteq S$, depending only on T_1 and S such that if $M > M_{T_1,S}$, then f represents S/T if and only if f represents S_0/T_1.*

Proof. We will escalate as in [1] with a slight deviation. At each escalation node, there is a least element $s \in S$ such that S/T_1 is not represented by the form f corresponding to this node. As in [1], we shall refer to s as the *truant* of f. To represent $\{s\}/T_1$, we must have some $t_1 \in T_1$ such that $s - t_1$ is represented by $f + bT_x$. Therefore, for each $t_1 < s$ we escalate with finitely many choices of b, and there are only finitely many choices of t_1. Thus, the breadth at each escalation is finite, and our argument above using modular forms shows that the depth is also finite, so there are only finitely many choices of $s \in S$ which are truants in the escalation tree. Take S_0 to be the set of truants in the escalation tree and define $M_{T_1,S} := \max s \in S_0 s + 1$. The argument above shows that representing S/T_1 is equivalent to representing S_0/T_1. When following the above process with T instead of T_1 whenever $M > M_{T_1,S}$, we will have the same truants at each step, so that representing S/T is equivalent to representing S/T_1, and hence representing S/T is equivalent to representing S_0/T_1. □

Remark 3.4. *It is of interest to note that if we replace "(diagonal) triangular form" with "quadratic form" (without the odd condition), then the proof follows verbatim, since the breadth is also finite, so that this can be considered a generalization of Bhargava's result that there is always a finite subset S_0 of S such that the quadratic form represents S if and only if it represents S, since this is obtained by taking $T_1 = T = \{0\}$.*

Now consider

$$X_j := \{x : x_i \text{ arbitrary for } i \leq j,\ x_i \in \{0, -1\} \text{ otherwise}\}$$

and define

$$T_{1,j} := \{f(x) : x \in X_j\} \text{ and } T_{2,j} := \{f(x) : x \notin X_j\} \text{ and}$$

We will use Lemma 3.3 with $T_1 = T_{1,j}$ and $T_2 = T_{2,j}$. To use the lemma effectively, we will show the following lemma.

Lemma 3.5. *There exist bounds $M_{X_j}^{(i)}$ depending only on $M_1, \ldots, M_j, c$ such that if $M_i \geq m_{X_j}^{(i)}$ for every $i > j$, then the smallest element of $T_{2,j}$ is greater than $M_{T_1,\mathbb{N}}$, where $M_{T_1,\mathbb{N}}$ is as defined in lemma 3.3.*

Proof. We will proceed by induction. For $j = 0$, we will take

$$M_{X_0}^{(i)} = M_{T_1,\mathbb{N}} + 6\sum_j |c_{ij}|.$$

Noting that

$$c_{ij}(2x_i x_j + x_i + x_j) \geq -|c_{ij}|(2T_{|x_i|} + 2T_{|x_j|})$$

Again, our bounds for $T_{|x_i|}$ in terms of $T_{|x_i|}$ give the desired result, noting that $x_i \notin \{0, -1\}$.

We now continue by induction on j. For the corresponding quadratic form, we note that plugging in $x_1 = \frac{-\sum_j c_{1j}}{2M_1}$ gives the minimal value over the reals, and a quadratic form Q' with rational coefficients and denominator dividing by $2M_1$. We therefore can consider $\tilde{Q} := 4M_1 \cdot Q'$, which is a quadratic form of the desired type. Thus, we can use the inductive step for $\tilde{Q}$. But this gives a bound which minimizes $\tilde{Q}$, and hence Q', but our choice of x_1 must give a value greater than or equal to this, so the result follows. □

Now, by our choice of X_j, $T_{1,j}$ is independent of M_i for $i > j$, since $T_{x_i} = 0$. Thus, fix c and take $M_i \geq M_{X_0}^{(i)}$. Then we have the corresponding subtrees identical, so that $\sup \tilde{N}(M_1, \ldots, M_k, c)$ is this unique element. We may now fix $M_1 \leq M_{X_0}^{(1)}$, since there are only finitely many such choices. With this M_1 fixed, we define $T_{1,1}$ as above, and again find bounds for the other M_i. Continuing recursively gives the desired result, since we know that $k \leq m$, so there are only finitely many supremums that we take.

We finally would like to show that $N_{\mathbb{N},m} \gg m^2$. To do so, we consider again the construction of our counterexamples. Consider $f(x) := \bigoplus_{i=1}^{m} f_N \oplus T_x$. For N sufficiently large, the smallest integer not represented by f is clearly $T_{m+1} - 1 \gg m^2$. □

References

[1] M. Bhargava, On the Conway-Schneeberger fifteen theorem, Quadratic Forms and their applications (Dublin, 1999), 27-37, Contemp. Math 272, Amer. Math. Soc., Providence, RI, 2000.

[2] M. Bhargava, J. Hanke, Universal Quadratic Forms and the 290-Theorem, preprint.

[3] P. Deligne, La Conjecture de Weil I, Inst. Hautes Études Sci. Publ. Math., 43, 273-307, 1974.

[4] B. Duke, Some Old Problems and New Results about Quadratic Forms, Notices Amer. Math. Soc. 44, no. 2, 190–196, 1997.

[5] B. Jones, The Regularity of a Genus of Postiive Ternary Quadratic Forms, Trans. Amer. Math. Soc., 111-124, 1931.

[6] K. Ono, Web of Modularity: Arithmetic of the Coefficients of Modular Forms and Q-series, Amer. Math. Society, 2003.

[7] C. Siegel, ber die analytische Theorie der quadratischen Formen. Ann. Math. 36, 527-606, 1935.

[8] W. Tartakowsky, Die Gesamtheit der Zahlen, die durch eine quadratische form $F(x_1, x_2, \ldots x_s)$, $(s \geq 4)$ darstellbar sind., Izv. Akad. Nauk SSSR, 7, 111-122, 165-196, 1929.

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