

# THE TRIANGULAR THEOREM OF EIGHT AND A CERTAIN NON-FINITENESS THEOREM

WIEB BOSMA AND BEN KANE

ABSTRACT. We investigate here sums of triangular numbers  $f(x) := \sum_i b_i T_{x_i}$  where  $T_n$  is the  $n$ -th triangular number. We show that, fixing  $b_i \geq 0$ ,  $f(x)$  represents every nonnegative integer if and only if it represents 1, 2, 4, 5, and 8, with the standard application to sums of odd squares  $\sum_i b_i (2x_i + 1)^2$ . Moreover, we show that no finite subset will suffice if “cross terms” are included, in turn showing that there is no overarching finiteness theorem which generalizes from positive definite quadratic forms to totally positive quadratic polynomials.

## 1. INTRODUCTION

In 1638 Fermat claimed that every number is a sum of at most three triangular numbers, four square numbers, and in general  $n$  polygonal numbers of order  $n$ . Here the triangular numbers are  $T_x := \frac{x(x+1)}{2}$ , where we include  $x = 0$  for simplicity. For a more complete history of related questions about sums of figurate numbers and some new results, see Duke’s survey paper [5]. The claim for four squares was shown by Lagrange in 1772, while Gauss famously wrote

$$\text{Eureka, } \triangle + \triangle + \triangle = n$$

in his mathematical diary on July 10, 1796.

**Theorem** (Gauss, 1796). *Every positive integer is the sum of three triangular numbers.*

The first proof of the full assertion of Fermat was given by Cauchy in 1813.

In 1917, Ramanujan extended the question about four squares to consider which choices of  $b = (b_1, b_2, b_3, b_4)$  have the property that  $b_1 x_1^2 + b_2 x_2^2 + b_3 x_3^2 + b_4 x_4^2$  represents every positive integer. We shall refer to such forms as *universal diagonal forms*. He gives a list of 55 possible choices of  $b$  which he then claims are the complete list of universal quaternary diagonal forms (54 forms actually turned out to be universal).

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Recently, Conway and Schneeberger proved in unpublished work a nice classification for universal positive definite quadratic forms whose corresponding matrices have integer entries.

**Theorem** (Conway-Schneeberger). *A positive definite quadratic form  $Q(x) = x^t Ax$ , where  $A$  is a positive symmetric matrix with integer coefficients, represents every positive integer if and only if it represents the integers 1, 2, 3, 5, 6, 7, 10, 14, and 15.*

Bhargava gave a simpler proof of the Conway-Schneeberger 15 Theorem in [1], in addition to showing more generally that for any set  $S$  it is always sufficient to check a finite subset  $S_0$ , and showed the set  $S_0$  for  $S$  all odd integers and  $S$  all primes. More recently, Bhargava and Hanke have shown the 290-Theorem, stating the necessary set for universal forms when the corresponding matrix is half integral, the largest of which is 290 [2].

Similarly to Ramanujan's generalization of Lagrange's Four Squares Theorem, in 1862 Liouville proved the following generalization of Gauss's theorem.

**Theorem** (Liouville). *Let  $a, b, c$  be positive integers with  $a \leq b \leq c$ . Then every  $n \in \mathbb{N}$  can be written as  $aT_x + bT_y + cT_z$  if and only if  $(a, b, c)$  is one of the following:*

$$(1, 1, 1), (1, 1, 2), (1, 1, 4), (1, 1, 5), (1, 2, 2), (1, 2, 3), (1, 2, 4).$$

We will investigate whether finiteness theorems akin to the results of the Conway-Schneeberger 15 Theorem or the Bhargava-Hanke 290 theorem occur for sums of triangular numbers or more generally for totally positive quadratic polynomials.

We first determine every choice of  $b$  such that the sum

$$f(x) := f_b(x) := \sum_{i=1}^k b_i T_{x_i}$$

represents every nonnegative integer. We will see the following simple condition to determine whether  $f_b$  represents every integer.

**Theorem 1.1.** *Fix a sequence of (positive) integers  $b_1, \dots, b_k$ . Then*

1. *The triangular form  $f(x) = \sum_{i=1}^k b_i T_{x_i}$  represents every nonnegative integer if and only if it represents the integers 1, 2, 4, 5, and 8.*
2. *The corresponding diagonal quadratic form  $Q(x) = \sum_{i=1}^k b_i x_i^2$  with  $x_i$  all odd represents every integer of the form*

$$8n + \sum_{i=1}^k b_i$$

with  $n \geq 0$  if and only if it represents  $8 + \sum_{i=1}^k b_i$ ,  $16 + \sum_{i=1}^k b_i$ ,  $32 + \sum_{i=1}^k b_i$ ,  $40 + \sum_{i=1}^k b_i$ ,  
and  $64 + \sum_{i=1}^k b_i$ .

Moreover, if the integers 1, 2, 4, 5, and 8 are represented by the triangular form, then  $n$  is represented very many times unless  $n + 1$  has high 3-divisibility. For an integer  $n$ , we will set  $a_n := \frac{v_3(n+1)}{\log_3(n+1)}$ , so that  $3^{v_3(n+1)} = (n + 1)^{a_n}$  gives the 3-part of  $n + 1$  as a power of  $n + 1$ .

**Corollary 1.2.** *For  $\epsilon > 0$ , there is an absolute constant  $c_\epsilon$  such that if the triangular form  $f(x)$  represents 1, 2, 4, 5, and 8, then  $f(x)$  represents every nonnegative integer  $n$  at least  $\min\{c_\epsilon n^{\frac{1}{2}-\epsilon}, n^{1-a_n}\}$  times. In particular, if  $n$  is sufficiently large and  $a_n < \frac{1}{2}$  then  $f(x)$  represents  $n$  at least  $c_\epsilon n^{\frac{1}{2}-\epsilon}$  times.*

**Remark 1.3.** *The constant  $c_\epsilon$  is ineffective because it relies on Siegel's lower bound for the class number, but the bound of  $c_\epsilon n^{\frac{1}{2}-\epsilon}$  may be replaced with the minimum of finitely many choices of a constant times a Hurwitz class number of a certain imaginary quadratic order whose discriminant is linear in  $n$ .*

*We have the following example. Using the explicit bound in terms of the Hurwitz class number, we obtain for instance that if 1, 2, 4, 5, and 8 are represented, then the integer 195727301431 is represented at least 270390 times and the integer 48291403767737750 is necessarily represented at least 90542761 times (here  $a_n \approx 0.364$ ), while the integer  $50031545098999706 = 3^{35} - 1$  is only necessarily represented once. All of the bounds listed in these examples are sharp.*

Recall that a quadratic polynomial (over  $\mathbb{Z}$ ) is given by  $f(x_1, \dots, x_k) = Q(x) + \Lambda(x) + C$ , where  $Q(x)$  is a quadratic form in  $k$  variables,  $\Lambda(x)$  is a linear operator, and  $C$  is a constant. We will only consider quadratic polynomials with rational coefficients such that  $f(x) \in \mathbb{Z}$  for every  $x \in \mathbb{Z}^k$ . Notice that the quadratic form  $Q(x)$  is positive definite if and only if  $f(x)$  is bounded from below. We will say that a polynomial  $f(x_1, x_2, \dots, x_k)$  is a (normalized) totally positive quadratic polynomial if  $f$  is a quadratic polynomial such that  $f(x) \geq 0$  for every  $x \in \mathbb{Z}^k$ , and  $f(x) = 0$  for some  $x \in \mathbb{Z}^k$  (We will suppress "normalized" throughout.). We note that for each pair of a positive definite quadratic form  $Q(x)$  and a linear operator  $\Lambda(x)$  there is a unique choice  $C \in \mathbb{Z}$  such that  $f(x) = 0$  has a solution and  $f(x) \geq 0$ . It will turn out that no finiteness theorem will hold for totally positive quadratic polynomials, and moreover checking no proper subset will suffice.

**Proposition 1.4.** *Let  $S \subseteq \mathbb{N}$  be given. For every proper subset  $S_0 \subsetneq S$  there exists a totally positive quadratic polynomial which represents  $S_0$  but does not represent  $S$ .*

In order to prove Proposition 1.4, for each positive integer  $n$  we will construct a totally positive quadratic polynomial  $f_n(x)$  which represents precisely every natural number

other than  $n$ , so that in particular if  $n \in S \setminus S_0$  then  $f_n$  will represent  $S_0 \subset S \setminus \{n\}$  but not  $S$ . All of the  $f_n$  will come from a much more restricted set of totally positive quadratic polynomials related to triangular numbers with “cross terms,” and hence we will obtain a stronger result for this more restricted class of polynomials. We will now describe this class of quadratic polynomials. In order to describe our construction, we will say for simplicity that two quadratic polynomials  $f_1$  and  $f_2$  are (*arithmetically equivalent*) if the number of solutions to  $f_1(x) = n$  equals the number of solutions to  $f_2(x) = n$  for every integer  $n \geq 0$ .

There is a natural mapping from representations by sums of triangular numbers to (diagonal) quadratic forms, namely the mapping which takes  $x_i$  to  $2x_i + 1$  after multiplying by 8. We now relax the image to include all positive definite quadratic forms whose corresponding matrices have even integral cross terms and consider quadratic polynomials in the preimage of this map. Since the preimage is unique (modulo equivalence) up to a constant, there is precisely one totally positive quadratic polynomial in the preimage. If  $Q$  and  $\tilde{Q}$  are two equivalent quadratic forms such that the isomorphism preserves the condition that  $x_i$  is odd, then we shall refer to them as *equivalently odd*, and denote the equivalence class of such forms as  $[Q]_o$ . For a positive definite quadratic form  $Q$  whose associated matrix has even cross terms, we will define  $f_Q := f_{[Q]_o}$  to be the unique (up to equivalence) totally positive quadratic polynomial which restricts to  $Q$  under the mapping  $x_i \rightarrow 2x_i + 1$ . We will refer to  $f_Q$  as a *triangular sum with cross terms*. Proposition 1.4 will follow directly from the following stronger result.

**Theorem 1.5.** *Let  $S \subset \mathbb{N}$  be given. For every proper subset  $S_0$  there exists a triangular sum with cross terms which represents  $S_0$  but does not represent  $S$ .*

We will now construct a natural metric on  $f_Q$  such that restricting this metric will again give a finiteness result. Let a positive definite quadratic form with even cross terms in the corresponding matrix,

$$(1) \quad Q(x) := \sum_{i=1}^k b_i x_i^2 + \sum_{i < j} 4c_{ij} x_i x_j$$

be given. We define

$$\tilde{f}(x) := \widetilde{f_Q}(x) := \sum_{i=1}^k b_i T_{x_i} + \sum_{i < j, c_{ij} \geq 0} c_{ij} (2x_i x_j + x_i + x_j) + \sum_{i < j, c_{ij} < 0} c_{ij} (2x_i x_j + x_i + x_j + 1),$$

where  $T_x = \frac{x(x+1)}{2}$  extends to  $x$  negative.

**Remark 1.6.** *Our choice of adding the constant  $c_{ij}$  every time  $c_{ij} < 0$  may not seem canonical at first, but notice that if  $Q'$  is the equivalent quadratic form obtained by replacing  $x_1$  with  $-x_1$ , then we find that this choice leads to  $f_Q = f_{Q'}$ .*

We next define

$$\tilde{m}_{\tilde{f}} := -\min_{x \in \mathbb{Z}^k} \tilde{f}(x)$$

so that we have (up to equivalence)

$$f_Q = \widetilde{f}_Q + \tilde{m}_{\tilde{f}}.$$

Thus, we can define the metric

$$m_{f_Q} := m_{[Q]_o} := \min_{Q' \in [Q]_o} |\tilde{m}_{\widetilde{f}_{Q'}}|.$$

If  $m_f$  is bounded, then we will again find that checking a finite subset will suffice.

**Theorem 1.7.** *Fix an integer  $m$  and a subset  $S$  of the positive integers. Then there is a finite subset  $S_{0,m} \subset S$  depending only on  $m$  and  $S$  such that every triangular sum with cross terms  $f$  satisfying  $m_f \leq m$  represents  $S$  if and only if it represents  $S_{0,m}$ .*

*Moreover, for  $S = \mathbb{N}$ , we find that  $\max \mathbb{N}_{0,m} \gg m^2$ .*

**Remark 1.8.** *It may be of interest to investigate the growth of  $\max \mathbb{N}_{0,m}$  in terms of  $m$ . The  $m = 0$  case is precisely Theorem 1.1. Following the bounds given in the proof of Theorem 1.7, computational evidence suggests that*

$$\begin{aligned} \mathbb{N}_{0,1} = \{ & 1, 2, 3, 4, 5, 6, 8, 9, 10, 11, 12, 13, 14, 16, 17, 19, 20, 23, 24, 25, 26, 29, 32, 33, 34, \\ & 35, 38, 41, 46, 47, 48, 50, 53, 54, 58, 62, 63, 75, 86, 96, 101, 102, 113, 117, 129, \\ & 162, 195, 204, 233 \}. \end{aligned}$$

*A proof of the above identity using the techniques of Bhargava and Hanke [2] developed in the proof of the 290 Theorem may require a careful analysis of a possible Siegel zero. To exhibit this difficulty, consider the sum  $f(x, y, z) = T_x + 2T_y + 6T_z$ . In the construction of  $\mathbb{N}_{0,1}$  the computations imply that there are infinitely many  $Q$  with  $m_{f_Q} = 1$  for which  $f \oplus f_Q$  represents every positive integer. Hence we cannot merely check each case individually and must know information about the integers represented by  $f$  independently.*

*Although it appears that  $f$  represents all odd integers, a proof of this appears to be beyond current techniques due to ineffective lower bounds for the class number (see [7]). However, since a possible Siegel zero for  $L(\chi_d, s)$  would give a lower bound for the class number when  $d' \neq d$  (both fundamental), one may be able to show that  $f$  represents at least one of  $n$  or  $n - 1$  for every positive integer  $n$ , which would suffice for showing the above identity.*

The paper is organized as follows. In Section 2 we will use the theory of quadratic forms to show Theorem 1.1 and the growth obtained in Corollary 1.2. In Section 3 we will construct the totally positive quadratic polynomials  $f_n$  which represent every nonnegative integer other than  $n$ . We will then conclude by showing that bounding  $m_f$  will again give a finiteness result.

## 2. THEOREM OF EIGHT

We will assume throughout that the reader is familiar with genus theory for quadratic forms. For background information on quadratic forms, a good source is [6]. Here we prove Theorem 1.1 and Corollary 1.2. We will proceed by showing by a standard argument that the theorem is equivalent to a statement about (diagonal) quadratic forms, and then prove the corresponding result for quadratic forms. We will only need some elementary results about quadratic forms and a theorem of Siegel to show the desired result.

*Proof.* Consider the generating function

$$F(q) := F_b(q) := \sum_x q^{f(x)} = \sum_{n=0}^{\infty} t(n)q^n,$$

where  $t(n)$  is the number of solutions to  $f(x) = n$ . Then we see that

$$q^{\sum_{i=1}^k b_i} F(q^8) = \sum_x q^{\sum_{i=1}^k b_i(2x_i+1)^2},$$

so that  $t(n) = r_o\left(8n - \sum_{i=1}^k b_i\right)$ , where  $r_o(n)$  is the number of representations of  $n$  by the corresponding (diagonal) quadratic form with  $x_i$  odd. We proceed as with *escalator lattices* in [1]. Without loss of generality we have  $b_1 \leq b_2 \leq \dots \leq b_k$ . Fixing  $b = [b_1, \dots, b_{k-1}]$ , we will *escalate* to  $[b_1, \dots, b_k]$  by choosing all possible choices of  $b_k \geq b_{k-1}$  for which it is possible to represent the next largest integer not already represented. We will then develop an *escalator tree* by forming an edge between  $b$  and  $[b_1, \dots, b_k]$ , with  $\emptyset$  as the root. If  $b$  represents every integer, then  $b$  will be a leaf of our tree.

Since  $t(1) > 0$ , it follows that  $b_1 = 1$ . We need  $t(2) > 0$ , so  $b_2 = 1$  or  $b_2 = 2$ . If  $b_2 = 1$ , then we need  $t(5) > 0$ , so  $1 \leq b_3 \leq 5$ . For  $b_3 = 3$ , we need  $t(8) > 0$ , so  $3 \leq b_4 \leq 8$ . Likewise, if  $b_2 = 2$ , then  $2 \leq b_3 \leq 4$ . Therefore, if  $t(n) > 0$  for every  $n$ , then we must have one of the above choices of  $b_i$  as a sublattice. By showing that each of these choices of  $b_i$  satisfies  $t(n) > 0$  for every  $n$ , we will see that this condition is both necessary and sufficient.

For ease of notation, we will denote the triangular form corresponding to  $b$  with  $[b_1, b_2, \dots, b_k]$  and the corresponding quadratic form by  $(b_1, \dots, b_k)$ . All of the cases other than  $[1, 1, 3, k]$  with  $3 \leq k \leq 8$  are covered by Liouville's Theorem. However, to obtain the more precise version given in Corollary 1.2, we will use quadratic form genus theory.

For the forms  $[1, 1, 1]$ ,  $[1, 1, 4]$ ,  $[1, 1, 5]$ ,  $[1, 2, 2]$ , and  $[1, 2, 4]$ ,  $r_o(n) = r(n)$ , where  $r(n)$  is the number of representations of  $n$  without the restriction of  $x_i$  odd. For each of these choices of  $b$ ,  $(b_1, b_2, b_3)$  is a genus 1 quadratic form. Therefore, extending the classification of Jones [6, Theorem 86] to primitive representations when the integer is not square free,

$t_{[1,1,1]}(n) = 3H(-8n+3)$ ,  $r_{[1,1,4]}(n) = \frac{1}{2}H(-4(8n+6))$ ,  $t_{[1,2,2]}(n) = \frac{1}{2}H(-4(8n+5))$ , and  $t_{[1,2,4]}(n) = \frac{1}{4}H(-8(8n+7))$ , where  $H(D)$  is the Hurwitz class number for the order of discriminant  $D < 0$ .

For  $[1, 1, 5]$  we must be slightly more careful since 5 divides the discriminant. We will explain in some detail how to deal with this complication and then will henceforth ignore this difficulty when it arises. For  $5 \nmid 8n+7$  we have  $t_{[1,1,5]}(n) = \frac{1}{2}H(-5(8n+7))$ . Hence the only difficulty occurs with high divisibility by 5. For  $p \neq 5$  the local densities are equal to those for bounded divisibility. Thus, entirely analogously to the result of Jones we have  $t_{[1,1,5]}(n) = c_n H(-5(8n+7))$  for some constant  $c_n > 0$  which only depends 5-adically on  $8n+7$ . We calculate the cases  $v_5(8n+7) \leq 3$  by hand. Denote 5-primitive representations (i.e.  $5 \nmid \gcd(x, y, z)$ ) by  $r^*(n)$ . Checking locally, for  $5^2 \mid m := 8n+7$ , we will obtain the result inductively by showing  $\frac{r^*(25m)}{r^*(m)} = \frac{h(25m)/u(25m)}{h(m)/u(m)}$  and then summing to get  $r(m) \geq \frac{1}{2}H(-5m)$ . But, since  $5 \mid m$ , we have  $\frac{h(25m)/u(25m)}{h(m)/u(m)} = 5$  by the class number formula (see [3, Corollary 7.28, page 148]) so that this is a quick local check at the prime 5.

Our proofs for  $[1, 1, 2]$ ,  $[1, 2, 3]$ , and  $[1, 1, 3]$  will be essentially the same. For  $[1, 1, 2]$ , we note that if

$$x^2 + y^2 + 2z^2 = 8n + 4$$

has a solution with  $x, y$ , and  $z$  not all odd, then taking each side modulo 8 leads us to the conclusion that  $x, y$ , and  $z$  must all be even. Therefore, the solutions without  $x, y$ , and  $z$  odd correspond to solutions of

$$4x^2 + 4y^2 + 8z^2 = 8n + 4,$$

or,

$$x^2 + y^2 + 2z^2 = 2n + 1.$$

Using Siegel's theorem to compare the local density at 2, we see that the average of the number of representations over the genus is twice as large for  $8n+4$  as  $2n+1$ . However,  $(1, 1, 2)$  is again a genus 1 quadratic form, so  $r(8n+4) = 2r(2n+1)$ , and hence  $t_{[1,1,2]}(n) = r_o(8n+4) = r(8n+4) - r(2n+1) = r(2n+1)$ . Thus by Theorem 86 of Jones [6] we have  $t_{[1,1,2]} = H(-8(2n+1))$ . Similar arguments show that

$$\begin{aligned} t_{[1,2,3]}(n) &= r_{o,(1,2,3)}(8n+6) = r_{(1,2,3)}(8n+6) - r_{(4,2,12)}(8n+6) \\ &= r_{(1,2,3)}(8n+6) - r_{(1,2,6)}(4n+3) = r_{(1,2,6)}(4n+3). \end{aligned}$$

Similarly to the case  $[1, 1, 5]$ , we have  $t_{[1,2,3]}(n) \geq \frac{1}{4}H(-12(4n+3))$ .

For  $[1, 1, 3]$  we see analogously that

$$t_{[1,1,3]}(n) = r_{o,(1,1,3)}(8n+5) = r_{(1,1,3)}(8n+5) - r_{(1,1,12)}(8n+5) = r_{(1,1,12)}(8n+5),$$

and again  $(1, 1, 12)$  is genus 1. We conclude in the case  $3 \nmid (8n+5)$  that we have  $t_{[1,1,3]}(n) = \frac{1}{2}H(-3(8n+5))$ , and we may henceforth assume that  $3 \mid 8n+5$  (i.e.  $n \equiv 2 \pmod{3}$ ). Local conditions imply that  $3^{2k+1}(3n+2)$  is not represented by  $(1, 1, 12)$ , so

we have escalated to  $[1, 1, 3, k]$  for  $k \in 3..8$ . For  $3 \nmid k$ , by choosing  $x_4 = 1$  we have  $t_{[1,1,3,k]}(n) \geq \frac{1}{2}H(-3(8(n-k) + 5))$  since  $3 \nmid 8(n-k) + 5$ .

For  $k = 3$  we have

$$t_{[1,1,3,3]}(n) = r_{(1,1,3,3)}(8(n+1)) + r_{(4,4,12,12)}(8(n+1)) - 2r_{(1,3,3,4)}(8(n+1)).$$

Writing the difference of the  $\theta$ -series  $\sum_n r(8n)q^n$  for these quadratic forms as

$$\theta_{(1,1,3,3)}|U(8) + \theta_{(1,1,3,3)}|V(4)|U(8) - 2\theta_{(1,3,3,4)}|U(8),$$

it is easy to conclude that the generating function  $qF(\tau) = \sum_n t_{[1,1,3,3]}(n)q^{n+1}$  (with  $q = e^{2\pi i\tau}$ ) is a weight 2 modular form of level 48. Using Sturm's bound [10] and checking the first 16 coefficients reveals that  $qF(\tau) = \frac{\eta(2\tau)^4\eta(6\tau)^4}{\eta(\tau)^2\eta(3\tau)^2}$ . The coefficients are multiplicative, so that if we have the factorization  $n+1 = 2^e 3^f \prod_{p>3} p^{e_p}$ , then

$$t_{[1,1,3,3]}(n) = 2^e \prod_{p>3} \frac{p^{e_p+1} - 1}{p - 1} \geq \frac{n+1}{3^f} = (n+1)^{1-a_n}$$

Finally, for  $k = 6$  we check  $n < 10$  by hand and then note that  $t_{[1,3,6]}(n) = r_{(1,3,6)}(8n+10) - r_{(2,3,6)}(4n+5)$ , while both  $(1, 3, 6)$  and  $(2, 3, 6)$  are genus 1. Hence for  $n \not\equiv 2 \pmod{3}$  we have  $t_{[1,3,6]}(n) \geq \frac{1}{4}H(-4(4n+5))$ . We then take the remaining variable  $x_4 = 1$  to obtain for  $n \equiv 2 \pmod{3}$  that  $t_{[1,1,3,6]}(n) \geq \frac{1}{4}H(-4(4(n-1)+5))$ , since  $n-1 \not\equiv 2 \pmod{3}$ .

Having seen that each of our choices of  $b$  is indeed a leaf to the tree, we conclude that representing the integers 1, 2, 4, 5, and 8 suffices.  $\square$

### 3. CROSS TERMS

We will first show that triangular sums with cross terms do not satisfy any finiteness theorem, and hence there is no overarching finiteness theorem from quadratic polynomials. To do so, for every positive integer  $n$  we will construct a triangular sum with cross terms  $f_n$  which represents precisely every integer other than  $n$ , and hence for any  $n \in S \setminus S_0$ ,  $f_n$  represents  $S_0$  but  $f_n$  does not represent  $S$ .

*Proof of Theorem 1.5.* Let a set  $S \subseteq \mathbb{N}$  and a proper subset  $S_0$  be given. Since  $S_0$  is a proper subset, we may choose a positive integer  $n \in S \setminus S_0$ . We will proceed by explicit construction of the triangular sum with cross terms  $f_n$  which represents every integer other than  $n$ . First note that if the smallest positive integer represented by a form  $f$  is  $n$ , then since the sum of three triangular numbers represents every integer we have  $f \oplus (n+1)(T_x \oplus T_y \oplus T_z)$  representing all  $m \not\equiv n \pmod{n+1}$ . Hence we can choose  $f_n := f \oplus (n+1)(T_x \oplus T_y \oplus T_z) \oplus (n+2)T_w$ . It is therefore equivalent to construct  $f$  for which  $n$  is the smallest integer not represented by  $f$ .

Consider the quadratic form

$$Q^{(N)}(x, y) := Nx^2 + Ny^2 + 4xy,$$

and denote the corresponding triangular sum with cross terms by  $f^{(N)}$ . We first show that it is sufficient to determine that the generating function for  $f^{(N)}$  is

$$(2) \quad 2 + 2q + O(q^{N-12}).$$

Assuming equation (2), then the generating function for

$$f := \bigoplus_{i=1}^n f^{(N)}$$

is

$$2^n \left( 1 + \binom{n}{1}q + \cdots + \binom{n}{n}q^n \right) + O(q^{N-12}),$$

since  $m_f$  is additive across direct sums. If we choose  $N > n + 13$ , then the first integer not represented by  $f$  is  $n + 1$ . Therefore, since  $n$  was arbitrary, there is a form which represents every integer other than  $n + 1$  (this also suffices for  $n = 0$ ).

We now show that the generating function satisfies (2). Since

$$f^{(N)}(x, y) = NT_x + NT_y + (2xy + x + y) + 1,$$

taking  $x = 0, y = -1$  or  $x = -1, y = 0$  represents 0, while  $x = 0, y = 0$  or  $x = -1, y = -1$  represents 1. Now, without loss of generality assume that  $|x| \geq |y|$  and  $x \notin \{0, -1\}$ . Then,

$$|2xy + x + y| \leq 2|x|^2 + 2|x| = 4T_{|x|},$$

so that

$$f^{(N)}(x, y) \geq NT_x - 4T_{|x|} + NT_y.$$

When  $x \leq -2$  it is easy to check that  $4T_{|x|} \leq 12T_x$  so that

$$NT_x - 4T_{|x|} \geq (N - 12)T_x \geq N - 12.$$

and when  $x > 0$

$$NT_x - 4T_{|x|} = (N - 4)T_x \geq N - 4,$$

since  $T_x \geq 1$  for  $x \notin \{0, -1\}$ . Since  $T_y \geq 0$ , our assertion is verified.  $\square$

It is important here to note how the above counterexamples differ from the proof when we only have diagonal terms, since this observation will lead us to the proof of Theorem 1.7 when  $m_f$  is bounded.

We will call a triangular sum with cross terms  $f_Q$  (and also any corresponding  $\widetilde{f_Q}$ ) a *block* if the corresponding quadratic form  $Q$  has an irreducible matrix. We will build an escalator lattice by escalating (as a direct sum) by a block at each step. In Section 2, the breadth each time we escalated was finite, so that the overall tree was finite. In the above proof, however, there were infinitely many inequivalent blocks which represent 1, so that the breadth is infinite. What was expressed in the above proof was that the supremum of these depths went to infinity as we chose  $N$  increasing in terms of  $n$  in the proof.

We will refer to the cross terms as a (*cross term*) *configuration*. So for

$$f(x) = \sum_{i=1}^k b_i T_{x_i} + \sum_{1 \leq i < j \leq k} c_{ij} (2x_i x_j + x_i + x_j) + C$$

we will say that  $f$  has configuration  $c = (c_{ij})$ . Since the matrix of  $f$  is irreducible and hence the corresponding adjacency matrix is connected, we can assume throughout (by a change of variables) that for each  $j > 1$  there exists  $i < j$  with  $c_{ij} \neq 0$ .

*Proof of Theorem 1.7.* Fix a positive integer  $m$ . We will start with a small overview of the proof. As in the above remark, we will escalate with blocks. We will first show that when  $m_f \leq m$ , the number of blocks that are not dimension 1 in any branch of the escalator tree is bounded, and that there are only finitely many choices for the configuration of each block. We will then proceed by defining

$$N(M_1, M_2, \dots, M_k, c)$$

to be the smallest integer not represented by the totally positive quadratic polynomial corresponding to

$$\tilde{f}(x) := \sum_{i=1}^k M_i T_{x_i} + \sum_{i < j, c_{ij} \geq 0} c_{ij} (2x_i x_j + x_i + x_j) + \sum_{i < j, c_{ij} < 0} c_{ij} (2x_i x_j + x_i + x_j + 1).$$

Our claim is then equivalent to showing that in the escalator tree

$$\sup_{M_1, \dots, M_k, c} N(M_1, M_2, \dots, M_k, c)$$

is finite. To do so, we will effectively show that with the configurations of blocks of dimension greater than one fixed, the supremum with  $M_i$  sufficiently large is finite and independent of the choice of  $M_i$ , and then fix  $M_1 \leq m_1$ , and again show that the resulting supremum is independent of  $M_2, \dots, M_k$ , and so forth. Since there are only finitely many such choices of  $c$ , the result comes from taking the maximum of each of these supremums.

We begin with a lemma that will show that there are only finitely many choices of the cross term configuration.

**Lemma 3.1.** *If  $m_f \leq m$ , then there are only finitely many choices of the cross term configurations  $c_{ij}$  of all blocks of dimension greater than one, up to equivalent forms.*

*Proof.* First note that  $m_{f \oplus g} = m_f + m_g$ , so that we can only have at most  $m$  blocks  $f$  with  $m_f > 0$ , while we will see that  $m_f > 0$  unless  $f$  is one dimensional (and hence the block is a constant times  $T_x$ ). It therefore suffices to show that each block  $f$  of dimension greater than one has  $m_f > 0$  and those with the restriction  $m_f \leq m$  have bounded dimension and bounded coefficients in the configuration. Fix the configuration

$c$  of a block  $\tilde{f}$  with dimension  $k$  such that  $\tilde{m}_{\tilde{f}} = m_f$ , namely a minimal element. We will recursively show a particular choice of  $x_i$  such that

$$\tilde{f}(x) \leq -\max\{\max_{i,j} |c_{ij}|, k-1\},$$

so that the max of the  $c_{ij}$  is bounded by  $m$ , and the dimension is bounded by  $m+1$ .

First set  $x_1 = 0$ . Since  $\tilde{f}$  is a block, we know at step  $j$  that there is some  $i < j$  such that  $c_{ij} \neq 0$ . Choose  $i < j$  such that  $|c_{ij}|$  is maximal. If  $x_i = 0$ , then we set  $x_j = -1$  if  $c_{ij} > 0$  and  $x_j = 0$  otherwise. If  $x_i = -1$  then we set  $x_j = 0$  if  $c_{ij} > 0$  and  $x_j = -1$  otherwise.

Since all of our choices of  $x_i$  are 0 or  $-1$  and  $T_{-1} = T_0 = 0$ , the integer represented is independent of the diagonal terms  $M_i$ . Now we note that for  $x_i, x_j \in \{0, -1\}$  we have  $2x_i x_j + x_i + x_j = 0$  if  $x_i = x_j$  and  $2x_i x_j + x_i + x_j = -1$  otherwise. Therefore, if  $x_i = x_j$ , then from our definition of  $\tilde{f}$ , the cross term corresponding to  $c_{ij}$  adds 0 if  $c_{ij} \geq 0$  and adds  $-|c_{ij}|$  otherwise. If  $x_i = 0$  and  $x_j = -1$ , then the cross term adds  $-|c_{ij}|$  if  $c_{ij} \geq 0$  and adds 0 otherwise. Therefore by our construction above, we know that for  $|c_{ij}|$  maximal, we have added  $-|c_{ij}|$  to our sum, and we never add a positive integer, so the sum is at most  $-|c_{ij}|$ . Moreover, since the block is connected, we have added at most  $-1$  at each inductive step, so that the sum is at most  $-(k-1)$ .  $\square$

For simplicity, in our escalator tree, we will “push” up all of the blocks to the top of the tree which are not dimension 1. To do so, we will first build the tree with all possible choices of blocks which are not dimension 1, and then escalate with only dimension 1 blocks from each of the nodes of the tree, including the root (the empty set). Thus, every possible form will show up in our representation. This tree (without the blocks of dimension 1) is depth at most  $m$  in the number of blocks, but is of infinite breadth. Henceforth, we can consider the configuration  $c$  to be fixed, and take the maximum over all choices of  $c$ .

We will now see that the subtree from each fixed node is of finite depth. Consider the corresponding quadratic form  $Q$ . First note that the generating function for  $Q$  when all  $x_i$  are odd is the generating function for  $Q$  minus the generating function with some  $x_i$  even, and the others arbitrary, which is simply another quadratic form without any restrictions, taking  $x_i \rightarrow 2x_i$ . Thus, we have the generating function of a difference of finitely many quadratic forms, and hence we have the Fourier expansion of a modular form. Now we simply note that any quadratic form can be decomposed into an Eisenstein series and a cusp form (cf. [8]). Using the bounds of Tartakowsky [11] and Deligne [4], as long as the Eisenstein series is non-zero, the growth of the coefficients of the Eisenstein series can be shown to grow more quickly than the coefficients of the cusp form whenever the dimension is greater than or equal to 5, other than finitely many congruence classes for which the coefficients of both the Eisenstein series and the cusp form are zero.

Therefore, as long as the Eisenstein series is non-zero, there are only finitely many congruence classes and finitely many “sporadic” integers which are not represented by

the quadratic form. Thus, after dimension 5, there are only finitely many congruence classes and finitely many sporadic integers not represented by the form  $f$ . If at any step of the escalation, any of the integers in these congruence classes is represented, then we have less congruence classes, and only finitely many more sporadic integers which are not represented, so that the resulting depth is bounded. For the dimension 1 blocks, it is clear that the breadth of each escalation is finite, so there are only finitely many escalators coming from this node. Therefore, it suffices to show that the Eisenstein series is non-zero.

Again using Siegel's theorem [9], the Eisenstein series is simply a difference of the local densities. At every prime other than  $p = 2$ , the local densities of the quadratic forms, of which we are taking the difference, are equal, so we only need to show that the difference of the local densities at  $p = 2$  is positive. However, the difference of the number of local representations at a fixed 2 power must be positive, since the integer is locally represented with  $x_i$  odd, except possibly for finitely many congruence classes if a high 2-power divides the discriminant.

Therefore, we can define

$$\tilde{N}(M_1, \dots, M_k, c)$$

to be the maximum of  $N(M_1, \dots, M_k, M_{k+1}, \dots, M_l, c)$ , where  $M_{k+1}$  to  $M_l$  are the dimension 1 blocks coming from the (finite) subtree of this node.

We will show that  $\tilde{N}(M_1, \dots, M_k, c)$  is independent of the choice of  $M_i$  whenever  $M_i$  is sufficiently large by showing that the resulting subtrees are identical. We need the following lemma to obtain this goal. We will need some notation before we proceed.

For a set  $T$ , define the formal power series in  $q$

$$q^T := \sum_{t \in T} q^t.$$

For fixed sets  $S, T \subseteq \mathbb{N}$ , we will say that a form  $f(x) := \sum_{i=1}^k b_i T_{x_i}$  represents  $S/T$  if for every  $s \in S$  the coefficient of  $q^s$  in  $q^T g(q)$  is positive, where  $g(q)$  is the generating function for  $f(x)$  given by  $g(q) := \sum_{x \in \mathbb{Z}^k} q^{f(x)}$ .

**Lemma 3.2.** *Let a (diagonal) triangular form  $f$  be given. Fix  $S, T_1, T_2 \subseteq \mathbb{N}$  and  $M \in \mathbb{N}$  such that  $\min_{n \in T_2} n \geq M$ . Define  $T := T_1 \cup T_2$ . Then there exists a bound  $M_{T_1, S}$  and a finite subset  $S_0 \subseteq S$ , depending only on  $T_1$  and  $S$  such that if  $M > M_{T_1, S}$ , then  $f$  represents  $S/T$  if and only if  $f$  represents  $S_0/T_1$ .*

*Proof.* We will escalate as in [1] with a slight deviation. At each escalation node, there is a least element  $s \in S$  such that  $S/T_1$  is not represented by the form  $f$  corresponding to this node. As in [1], we shall refer to  $s$  as the *truant* of  $f$ . To represent  $\{s\}/T_1$ , we must have some  $t_1 \in T_1$  such that  $s - t_1$  is represented by  $f + bT_x$ . Therefore, for each  $t_1 < s$  we escalate with finitely many choices of  $b$ , and there are only finitely many choices of  $t_1$ . Thus, the breadth at each escalation is finite, and our argument above

using modular forms shows that the depth is also finite, so there are only finitely many choices of  $s \in S$  which are truants in the escalation tree. Take  $S_0$  to be the set of truants in the escalation tree and define  $M_{T_1, S} := \max s \in S_0 s + 1$ . The argument above shows that representing  $S/T_1$  is equivalent to representing  $S_0/T_1$ . When following the above process with  $T$  instead of  $T_1$  whenever  $M > M_{T_1, S}$ , we will have the same subtree and the same truants at each step, so that representing  $S/T$  is equivalent to representing  $S/T_1$ , and hence representing  $S/T$  is equivalent to representing  $S_0/T_1$ .  $\square$

**Remark 3.3.** *It is of interest to note that if we replace “(diagonal) triangular form” with “quadratic form” (without the odd condition), then the proof follows verbatim, since the breadth is also finite, so that this can be considered a generalization of Bhargava’s result that there is always a finite subset  $S_0$  of  $S$  such that the quadratic form represents  $S$  if and only if it represents  $S_0$ , since this is obtained by taking  $T_1 = T = \{0\}$ .*

Now consider

$$X_j := \{x : x_i \text{ arbitrary for } i \leq j, x_i \in \{0, -1\} \text{ otherwise}\}$$

and define

$$T_{1,j} := \{f(x) : x \in X_j\} \quad \text{and} \quad T_{2,j} := \{f(x) : x \notin X_j\}.$$

We will use Lemma 3.2 with  $T_1 = T_{1,j}$  and  $T_2 = T_{2,j}$  for each  $0 \leq j \leq k$ . To use the lemma effectively, we will show the following lemma.

**Lemma 3.4.** *There exist bounds  $M_{X_j}^{(i)}$  depending only on  $M_1, \dots, M_j, c$  such that if  $M_i \geq M_{X_j}^{(i)}$  for every  $i > j$ , then the smallest element of  $T_{2,j}$  is greater than  $M_{T_1, \mathbb{N}}$ , where  $M_{T_1, \mathbb{N}}$  is as defined in lemma 3.2.*

*Proof.* We will proceed by induction. For  $j = 0$ , we will take

$$M_{X_0}^{(i)} = M_{T_1, 0, \mathbb{N}} + 6 \sum_j |c_{ij}|.$$

Noting that for  $|x_j| < |x_i|$  we have  $|2(x_i - \frac{x_j}{2})x_j| \leq x_i^2$ , we get the inequality

$$c_{ij}(2x_i x_j + x_i + x_j) \geq -|c_{ij}|(2T_{|x_i|} + 2T_{|x_j|}).$$

The case  $j = 0$  then follows from the fact that for  $x_i \notin \{0, -1\}$  we have  $T_{|x_i|} \leq 3T_{x_i}$ .

We now continue by induction on  $j$ . For the corresponding quadratic form, we note that plugging in  $x_1 = \frac{-\sum_{j>1} c_{1j} x_j}{2M_1}$  gives the minimal value over the reals. The quadratic form  $Q'$  obtained by specializing this value of  $x_1$  has rational coefficients with denominator dividing  $2M_1$ . We therefore can consider  $\tilde{Q} := 4M_1 \cdot Q'$ , which is a quadratic form of the desired type. Thus, we can use the inductive step for  $\tilde{Q}$ . But this gives a bound which minimizes  $\tilde{Q}$ , and hence  $Q'$ , but an arbitrary choice of  $x_1$  must give a value greater than or equal to this, so the result follows.  $\square$

Now, by our choice of  $X_j$ ,  $T_{1,j}$  is independent of  $M_i$  for  $i > j$ , since  $T_{x_i} = 0$ . Thus, fix  $c$  and take  $M_i \geq M_{X_0}^{(i)}$ . Then the corresponding subtrees are independent of the choice of  $M_i$ , so that  $\sup \tilde{N}(M_1, \dots, M_k, c)$  is the unique largest truant in the subtree (effectively we may replace  $M_i = \infty$ ). We may now fix  $M_1 \leq M_{X_0}^{(1)}$ , since there are only finitely many such choices. With this  $M_1$  fixed, we define  $T_{1,1}$  as above, and again find bounds for the other  $M_i$ . Continuing recursively gives the desired result, since we know that  $k \leq m$ , so there are only finitely many supremums that we take.

We finally would like to show that  $\max \mathbb{N}_{0,m} \gg m^2$ . To do so, we consider again the construction of our counterexamples. Consider  $f(x, y) := \bigoplus_{i=1}^m f^{(N)} \oplus T_y$ . Since  $T_r = \sum_{n=1}^r n$ , for  $N$  sufficiently large the smallest integer not represented by  $f$  is clearly  $T_{m+1} - 1 \gg m^2$ .  $\square$

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DEPARTMENT OF MATHEMATICS, RADBOD UNIVERSITEIT, TOERNOOIVELD 1, 6525 NIJMEGEN, NETHERLANDS

*E-mail address:* bosma@math.ru.nl

*E-mail address:* bkane@science.ru.nl